Modelling Environmental Extremes

Lee Fawcett, Jill Johnson and Dave Walshaw

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Topics for the day

1. Classical models and threshold models
2. Dependence and non–stationarity
3. R session: weather extremes
4. Multivariate extremes
5. Bayesian inference for Extremes
6. R session: multivariate analysis and Bayesian inference

4.1 Introduction

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- to model the joint extremes of two or more different variables at a particular location (e.g. wind and rain at a site);
- to model the joint behaviour of extremes which occur as consecutive observations in a time–series (e.g. consecutive hourly maximum wind gusts during a storm).

All of these problems suggest fitting an appropriate limiting multivariate distribution to the relevant data.

However, as we shall see, the derivation of such a multivariate distribution is not as easy as we might hope. The analogy with the Normal distribution as a model for means breaks down as we move into n dimensions!

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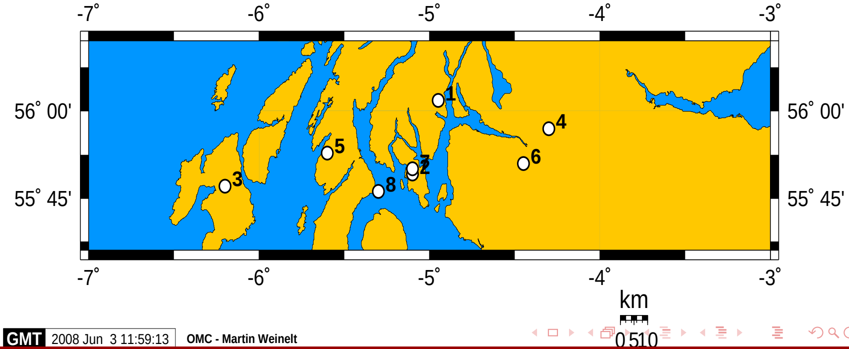
Most of the increased complexity is apparent in the move from 1 to 2 dimensions, so we will focus largely on bivariate problems.

4.2 Componentwise maxima models

Suppose we want to study the joint extremes of daily rainfall accumulations at the network of 8 sites shown in Figure 14.



Such issues are of great interest, especially currently, e.g. given the severe flooding experienced in the UK recently.

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The raw mutlivariate observations are 8–dimensional vectors of the daily rainfall over the eight sites.

Now suppose we wish to take a block–maxima approach, with ‘blocks’ being years. For any given year, the 8–dimensional vector of annual maxima is unlikely to be one of the raw multivariate observations.

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Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be i.i.d. vectors with distribution function $F(x, y)$.

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$\mathbf{M_n}$ is not necessarily one of the original observations (X_i, Y_i). Nevertheless, we are interested in the limiting behaviour of $\mathbf{M_n}$ as $n \to \infty$.

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We assume that the X_i and Y_i variables have a known marginal distribution. It is convenient to assume this is the GEV(0,1,1) distribution, also known as the unit Fréchet distribution, which has c.d.f.

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This gives rise to a very simple normalization of maxima:

$$\Pr(X_i < x) = \Pr(M_{x,n}/n < x) = \exp(-1/x), \qquad x > 0,$$

(and similarly for Y_i).

So if we consider the re–scaled vector

$$\mathbf{M}_{\mathbf{n}}^{*} = \left(\max_{i=1,\ldots,n} \{X_i\}/n, \max_{i=1,\ldots,n} \{Y_i\}/n \right),$$

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Unfortunately no limiting parametric family exists! (for bivariate extremes, or multivariate extremes in general).

Theorem: limiting distributions for bivariate extremes

Let $\mathbf{M}_\mathbf{n}^* = (M_{x,n}^*, M_{y,n}^*)$ be the normalized maxima as above, where the (X_i, Y_i) are i.i.d. with standard Fréchet marginal distibutions. Then if

$$\Pr(M_{x,n}^*, M_{y,n}^*) \to G(x, y),$$

where G is non–degenerate, then G has the form

$$G\,(x, y) = \exp\left\{-V\,(x, y)\right\}; \quad x > 0, \;\; y > 0 \tag{13}$$

where:

$$V\,(x, y) = 2\int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) dH\,(\omega) \tag{14}$$

and H is a distribution function on $[0, 1]$ satisfying the mean constraint:

$$\int_0^1 \omega\; dH\,(\omega) = 0.5. \tag{15}$$

Hence the class of bivariate extreme value distributions is in one–to–one correspondence with distribution functions of the form (13) satisfying the constraint (15). If H is differentiable with density h, then (14) becomes

$$V(x, y) = 2\int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) h(\omega) d\omega.$$

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However some simple models arise when H is not differentiable. E.g. if H places mass 0.5 on each of $\omega = 0$ and $\omega = 1$, then we get

$$G(x,y) = \exp\{-(x^{-1} + y^{-1})\}, \qquad x > 0, y > 0,$$

corresponding to independent x and y.

Since the GEV provides the complete class of marginal limit distributions, then the complete class of bivariate extreme value distributions is obtained as follows. If we suppose X and Y are GEV with parameters (μ_x, σ_x, ξ_x) and (μ_y, σ_y, ξ_y) respectively, then the transformations

$$\tilde{x} = \left[1 + \xi_x \left(\frac{x - \mu_x}{\sigma_x}\right)\right]^{1/\xi_x} \quad \text{and} \quad \tilde{y} = \left[1 + \xi_y \left(\frac{y - \mu_y}{\sigma_y}\right)\right]^{1/\xi_y}$$

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obtain unit Fréchet margins. Hence

$$G(x, y) = \exp\{-V(\tilde{x}, \tilde{y})\}$$

is a bivariate extreme value distribution with the appropriate margins for valid $V(.)$, and provided $[1 + \xi_x(x - \mu_x)/\sigma_x] > 0]$ and $[1 + \xi_y(x - \mu_y)/\sigma_y] > 0]$.

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All of these procedures, including the choice of models, are handled in a very similar way when dealing with threshold exceedances. We consider the details in the next section.

4.3 Threshold excess models

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For our bivariate observation (X, Y), let's focus on X. We have already seen that the distribution function for the exceedances of a threshold u by a variable X, conditional on $X > u$ for large enough u, is given by:

$$G(x) = 1 - \lambda \left\{ 1 + \frac{\xi (x - u)}{\sigma} \right\}^{-1/\xi}$$

defined on $\{x - u : x - u > 0 \text{ and } (1 + \xi (x - u)/\sigma) > 0\}$, where $\xi \neq 0$, $\sigma > 0$, and $\lambda = Pr(X > u)$.

Now we can obtain a unit Fréchet margin with the transformation:

$$\tilde{X} = -\left(\log\left\{1 - \lambda_x \left[1 + \frac{\xi_x (X - u_x)}{\sigma_x}\right]^{-1/\xi_x}\right\}\right)^{-1}.$$

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If we apply the analagous transformation to in the Y margin, we obtain

$$\tilde{F}\left(\tilde{x}, \tilde{y}\right) = \exp\left\{-V\left(\tilde{x}, \tilde{y}\right)\right\}; \quad x > u_x, \;\; y > u_y,$$

where:

$$V\left(x, y\right) = 2\int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) dH\left(\omega\right)$$

and H is a distribution function on $[0, 1]$ satisfying the mean constraint:

$$\int_0^1 \omega \, dH\left(\omega\right) = 0.5.$$

Example: wave–surge data

Here we choose a different type of example of dependence to the rainfall problem considered in Section 4.2. Here we consider two variables recorded concurrently at the same site.

A series of 3-hourly measurements on sea–surge were obtained from Newlyn, southwest England.

For suitably high thresholds, we can identify which observations are extreme.

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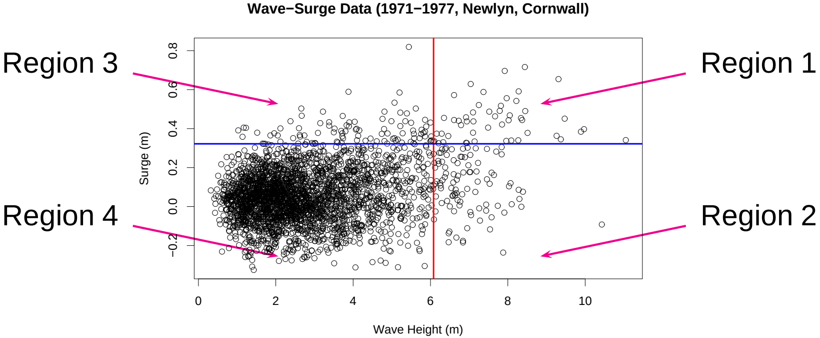

Modelling the Dependence Structure

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Possible choices are:

- Logistic Model – symmetric
- Negative Logistic Model
- Bilogistic Model – asymmetric
- Dirichlet Model

Here we will focus on the logistic model and the bilogistic model as two commonly used but contrasting choices.

The Logistic Model

Logistic Model

$$G(x, y) = \exp\left\{-\left(x^{-1/\alpha} + y^{-1/\alpha}\right)^{\alpha}\right\}$$

Where $x > 0$, $y > 0$ and $\alpha \in (0, 1)$.

- $\alpha \to 1$ corresponds to independent variables.
- $\alpha \to 0$ corresponds to perfectly dependent variables.
- This model is symmetric - the variables are exchangeable.

The Bilogistic Model

Bilogistic Model

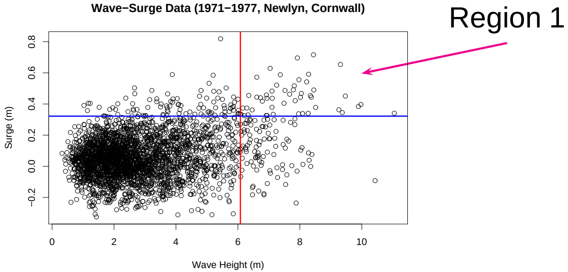
$$G(x,y) = \exp\left\{x\gamma^{1-\alpha} + y(1-\gamma)^{1-\beta}\right\}$$

where $0 < \alpha < 1$, $0 < \beta < 1$ and $\gamma = \gamma(x, y; \alpha, \beta)$ is the solution of:

$$(1-\alpha)x(1-\gamma)^{\beta} = (1-\beta)y\gamma^{\alpha}$$

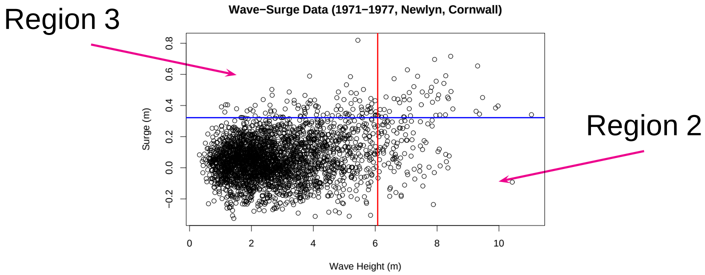
- Independence is obtained when $\alpha = \beta \to 1$ and when one of α or β is fixed and the other approaches 1.
- When $\alpha = \beta$ the model reduces to the logistic model.
- The value of $\alpha - \beta$ determines the extent of asymmetry in the dependence structure.

Threshold Representation



- For points in Region 1, the bivariate model structure shown applies, and the density of $\tilde{F}(\tilde{x}, \tilde{y})$ gives the appropriate likelihood component.

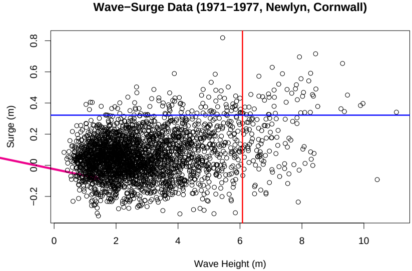
Threshold Representation



- In other regions, the likelihood component for the points must be censored.

Threshold Representation

Region 4



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Likelihood Function

The likelihood function can be written as:

$$L\left(\theta;(x_1,y_1),\ldots,(x_n,y_n)\right)=\prod_{i=1}^{n}\psi\left(\theta;(x_i,y_i)\right)$$

where θ gives the parameters of F and

$$\psi\left(\theta;(x,y)\right)=\begin{cases}\left.\frac{\partial^2 F}{\partial x\partial y}\right|_{(x,y)} & \text{if } (x,y)\in \text{Region 1}\\ \left.\frac{\partial F}{\partial x}\right|_{(x,u_y)} & \text{if } (x,y)\in \text{Region 2}\\ \left.\frac{\partial F}{\partial y}\right|_{(u_x,y)} & \text{if } (x,y)\in \text{Region 3}\\ F\left(u_x,u_y\right) & \text{if } (x,y)\in \text{Region 4}\end{cases}$$

The various models can be fitted to data by maximum likelihood estimation using routines available in the R package `evd`. We will explore this in the second R practical.

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Let $(x_1, y_1), (x_2, y_2), \ldots$ be a sequence of independent bivariate observations form a distribution with standard Fréchet margins such that

$$\Pr\{M^*_{x,n} \leq x, M^*_{y,n} \leq y\} \to G(x, y).$$

Let N_n be a sequence of point processes defined by

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Then

$$N_n \to N$$

on regions bounded away from $(0, 0)$, where N is a non–homogeneous Poisson process on $(0, \infty) \times (0, \infty)$.

Moreover, if we change our coordinates to an angular-radial form ('pseudo-polar') by setting

$$r = x \quad \text{and} \quad \omega = \frac{x}{x+y},$$

then the intensity function of N is

$$\lambda(r, \omega) = 2\frac{dH(\omega)}{r^2},$$

where H is related to G in the usual way [(13) — (15)].

This is helpful because r and ω are measures of distance (from the origin) and angle (from the x-axis) respectively, and the dependence function H determines the angular spread of points of N, *and is independent of radial distance*.

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It is fairly easy now to picture what different densities $h(.)$ will look like it terms of the scatter of points in the limiting point process N.

The point process representation in practice

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Convergence is guaranteed on any region bounded from the origin, and things are especially simple if we choose a region of from $A = \{(x, y) : x/n + y/n > r_0\}$ for suitably large r_0, since then

$$\Lambda(A) = 2\int_A \frac{dr}{r^2} dH(\omega) = 2\int_{r=r_0}^{\infty} \frac{dr}{r^2} \int_{\omega=0}^{1} dH(\omega) = 2/r_0,$$

which is constant with respect to the parameters of H.

If we assume H has density h, then the likelihood is given by

$$\begin{aligned} L(\theta;(x_1,y_1),\ldots,(x_n,y_n)) &= \exp\{\Lambda(A)\}\prod_{i=1}^{N_A}\lambda(x_{(i)}/n,y_{(i)}/n) \\ &\propto \prod_{i=1}^{N_A}h(\omega_i), \end{aligned}$$

where $\omega_i = x_{(i)}/(x_{(i)}+y_{(i)})$ for the N_A points $(x_{(i)},y_{(i)})$ which are in A.

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Now we can fit the model using maximum–likelihood estimation.

Point process model for wave–surge data

A point process model was fitted to the wave–surge data after transformation to unit Fréchet margins, and using a threshold of the form $X + Y = r_0$, where r_0 was chosen so that the marginal thresholds are both at the 95th percentile.

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Fitting the two dependence models (logistic and bilogistic) to the wave–surge data we obtain the following results:

Model	log–lik.	α	β
Logistic	227.2	0.659 (0.013)	
Bilogistic	230.2	0.704 (0.024)	0.603 (0.032)

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These results suggest a fairly weak, while clearly significant, dependence. The logistic and bilogistic models can be compared using a likelihood ratio test, and significant asymmetry is suggested.

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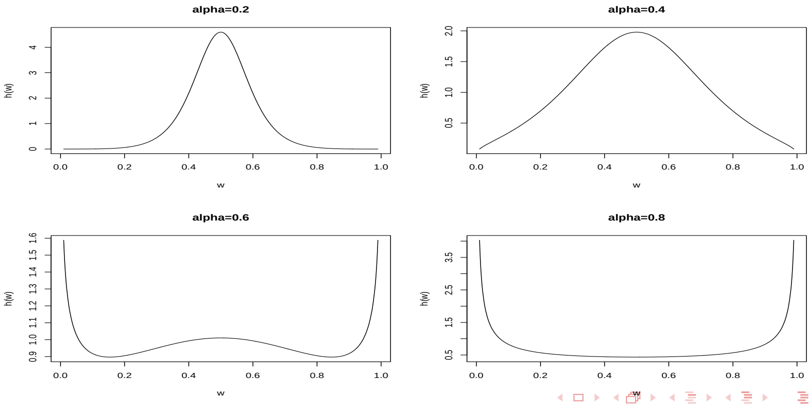
Here we just show the $h(\omega)$ functions for some members of the logistic family.



4.5 Asymptotic dependence and independence

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In practice this imposition is not helpful . . . it is often the case that asymptotic independence is suggested by the data, and yet quite strong dependence is present, even at high levels.

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Consider the region $A = \left\{\frac{X}{n} > u, \frac{Y}{n} > v\right\}$. Then:

$$Pr\left[\left(\frac{X}{n}, \frac{Y}{n}\right) \in A\right] = \begin{cases} C/n, & \textit{Asymptotic Dependence} \\ C/n^2, & \textit{Exact Independence} \end{cases}$$

where C is a constant term that does not depend on n.

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where u is a threshold above which the data are regarded as extreme and K is a (almost) constant term with respect to t.

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δ gives a measure of extremal dependence between X and Y and is known as the "**coefficient of tail dependence**".

Inference for δ

The likelihood function for T is:

$$L(K,\delta;t) = \left(1 - \frac{K}{u^{1/\delta}}\right)^{n-n_u} \left(\frac{K}{\delta}\right)^{n_u} \prod_{i=1}^{n_u} t_i^{-(1+1/\delta)}$$

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Maximum likelihood sstimation gives the estimate:

$$\hat{\delta} = \frac{1}{n_u} \sum_{i=1}^{n_u} \log\left(\frac{t_i}{u}\right)$$

evaluated for the n_u points in the data set above u.

Inference for δ

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- $\delta = 1$ implies asymptotic dependence.
- $\delta = \frac{1}{2}$ implies near independence.

Inference for δ

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- $\frac{1}{2} < \delta < 1$ implies positive association.
- $0 < \delta < \frac{1}{2}$ implies negative association.

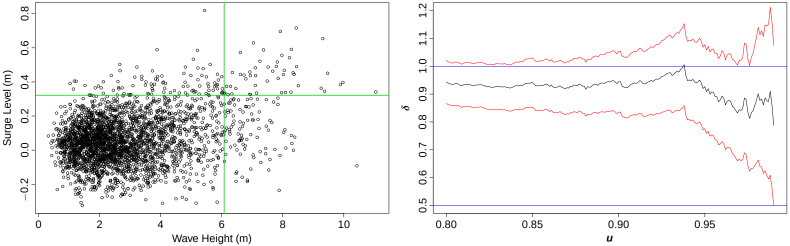
Inference for δ

δ describes the limiting dependence structure:

- $\delta = 1$ implies asymptotic dependence.
- $\frac{1}{2} < \delta < 1$ implies positive association.
- $\delta = \frac{1}{2}$ implies near independence.
- $0 < \delta < \frac{1}{2}$ implies negative association.

Plots of $\hat{\delta}$ against increasing u give an indication of the level of dependence present between two processes in the limiting distribution.

Wave–surge data



Wave-Surge data with 95% quantiles; δ–plot with 95% confidence bounds.

$\delta = 1$ is within the 95% confidence bounds for all *u* as *u* increases, suggesting the data to be **asymptotically dependent**.

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This approach can be quite messy in implementation, combining as it does a range of different estimation procedures, and some *ad hoc* assumptions concerning the parametric forms of the key normalizing constants.

Here we briefly consider another approach, suggested by (Bortot *et al.*, 2000), and currently the subject of ongoing work by Atyeo and Walshaw.

The multivariate Gaussian tail model

The multivariate Gaussian tail model for the multivariate distribution function F is defined on the **joint tail region** (Bortot *et al.*, 2000):

$$R(\mathbf{u}) = (u_1, \infty) \times \ldots \times (u_p, \infty)$$

where $\mathbf{u} = (u_1, \ldots, u_p)$. (e.g. Region 1 in Figure 15)

For each observation in the joint tail region $R(\mathbf{u})$ we transform each marginal observation to have a standard Normal marginal distribution, and then apply the p–dimensional standard Normal distribution function. We then transform *back* to extreme value margins.

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We have been able to fit such models to the 8-dimensional rainfall problem associated with Figure 14, however inference for this problem was much simplified by adopting a Bayesian approach . . .