Modelling Environmental Extremes

Lee Fawcett and Dave Walshaw

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Topics for the day

1. **Classical models and threshold models**

2. **Dependence and non–stationarity**

3. **R session: weather extremes**

4. **Multivariate extremes**

5. **Bayesian inference for Extremes**

6. **R session: multivariate analysis and Bayesian inference**

1.1 Introduction

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1.1 Introduction

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reliability — anything we build needs to have a good chance of surviving the weather/environment for the whole of its working life.

This has obvious implications for civil engineers and planners. They need to know:

- how strong to make buildings;
- how high to build sea walls;
- how tall to build reservoire dams;
- how much fuel to stockpile;

etc.

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- strongest wind;
- highest tide;
- heaviest rainfall;
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The models themselves are motivated by asymptotic theory, and this is our starting point . . .

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1.2 Classical models

Extreme value modelling has a central theoretical result, analagous to the Central Limit Theorem . . .

Suppose $X_1, X_2, \ldots,$ is an independent and identically distributed sequence of random variables. Define

$$M_n = \max\{X_1, \ldots, X_n\}.$$

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We are interested in the limiting distribution of M_n as $n \to \infty$.

As with the mean, $\bar{X}$, of $\{X_1, \ldots, X_n\}$, the limiting distribution of M_n as $n \to \infty$ is *degenerate*, and we need to work with a normalized version . . .

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The Extremal Types Theorem (Fisher and Tippett, 1928)

If there exist sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\Pr\{(M_n - b_n)/a_n \le z\} \to G(z) \quad \text{as} \quad n \to \infty,$$

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$$
\begin{aligned}
I : G(z) &= \exp\left\{-\exp\left[-\left(\frac{z-\beta}{\gamma}\right)\right]\right\}, \quad -\infty < z < \infty; \\
II : G(z) &= \exp\left\{-\left(\frac{z-\beta}{\gamma}\right)^{-\alpha}\right\}, \quad z > \beta; \quad [G(z) = 0, z \leq \beta]; \\
III : G(z) &= \exp\left\{-\left[-\left(\frac{z-\beta}{\gamma}\right)^{\alpha}\right]\right\}, \quad z < \beta; \quad [G(z) = 1, z \geq \beta],
\end{aligned}
$$

for parameters $\gamma > 0, \beta$, and $\alpha > 0$.

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The Generalized Extreme Value Distribution (GEV)

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Fortunately they can be combined into a single family, known as the Generalized Extreme Value Distribution (GEV), with c.d.f.

$$G(z) = \exp\left\{-\left[1 + \xi\left(\frac{z-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}, \qquad (1)$$

defined on the set $\{z : 1 + \xi(z-\mu)/\sigma > 0\}$, and where μ, $\sigma > 0$ and ξ are *location*, *scale* and *shape* parameters respectively.

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So the Extremal Types Theorem can be restated with (1) as the limiting form, and this provides the basis for our first modelling approach ...

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Note that the Extreme Value Types I, II and III correspond to the cases $\xi = 0$, $\xi > 0$ and $\xi < 0$ respectively.

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Note that the Extreme Value Types I, II and III correspond to the cases $\xi = 0$, $\xi > 0$ and $\xi < 0$ respectively.

For Type I, we need to take the limiting form of Equation (1) as $\xi \to 0$, which gives

$$G(z) = \exp\left\{-\exp\left[-\left(\frac{z-\mu}{\sigma}\right)\right]\right\}, \qquad (2)$$

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Approach 1: "Block maxima"

Break up our sequence $X_1, X_2, \ldots$ into blocks of size n (with n reasonably large), and extract only the maximum observation from each block.

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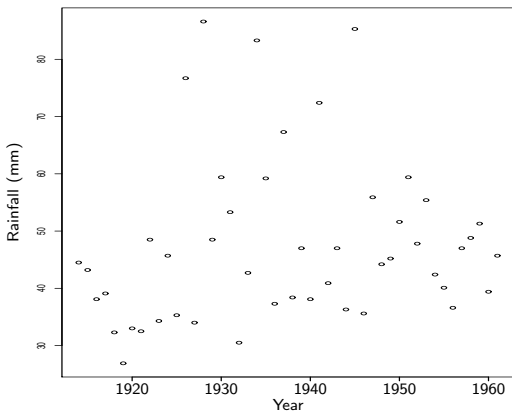
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This rough and ready approach has shown itself to be surprisingly robust!

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Approach 1: Example

Consider the annual maxima of daily rainfall accumulations (*mm*) at a location in SW England, from 1914 to 1961.

Figure 1. Annual Maxima for Rain Data

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Approach 1: Inferences

Here our blocks have $n = 365$, which is reasonably large, so we fit Model (1) to the $N = 49$ annual maxima (e.g. using *maximum likelihood estimation*).

We obtain fitted parameter values (standard errors in parentheses):

$$\mu = 40.7(1.5) \qquad \sigma = 9.4(1.2) \qquad \xi = 0.14(0.12).$$

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For example, the 99th percentile in the distribution of annual maxima is known as the *100 year return level*.

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For example, the 99th percentile in the distribution of annual maxima is known as the *100 year return level*.

The fitted value of this is easily obtained on inversion of Model (1):

$$q_{100} = 101.3(18.9).$$

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- The assumption of n independent and identically distributed variables in each block is cavalier, but inferences are surprisingly robust.

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Approach 1: Remarks

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- The assumption of n independent and identically distributed variables in each block is cavalier, but inferences are surprisingly robust.

- The inferences on return levels are crucial for designers and engineers, to the extent they are built into legally binding codes of practice.

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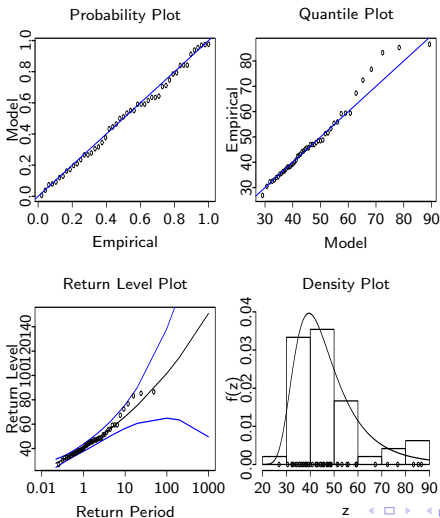
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- In actual fact, the existing codes of practice are usually based on a very primitive version of the methods just described. Fits are often based on restricting to one of the Fisher–Tippett types, ignoring estimation uncertainty, and using an *ad hoc* interpolation of return levels across a network of sites.

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- In any case the block–maxima approach is often *very* wasteful of data, leading to large uncertainties on return level estimates. This motivates a different approach (see later).

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Approach 1: Diagnostics

The goodness–of–fit of the GEV model is most easily assessed using various diagnostic plots. Here we consider four plots:

1. **Probability plot:** the fitted value of the c.d.f. is plotted against the empirical value of the c.d.f. for each data point.
2. **Quantile plot:** the empirical quantile is plotted against the fitted quantile for each data point.
3. **Return level plot:** the return level (with error bars) is plotted against the return period. Each data point defines a sample point.
4. **Density plot:** the fitted p.d.f. is supereimposed on a histogram of the data.

For our rainfall example, the diagnostic plots look like this . . .

Approach 1: Confidence intervals for return levels

Although we could construct a symmetrical confidence interval for the r–year return–level using classical likelihood theory ($\hat{q}_r \pm 1.96 \times$ standard error), this is not recommended.
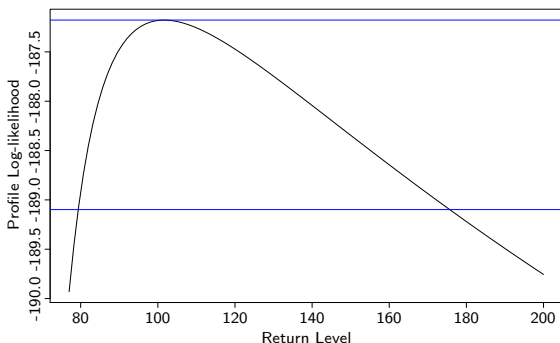
This practice assumes the limiting quadratic behaviour of the likelihood surface near the maximum, whereas in fact the surface is usually very asymmetrical.

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We recommend using the method of *profile likelihood* to take this into account: by reparametrization of Equation (1) to replace one of the parameters by q_r, we can maximize the likelihood *conditional* on q_r taking each possible value.

We plot this constrained value against q_r ...

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Approach 1: Profile likelihood confidence interval for q_{100}

For the rainfall example we get . . .

The likelihood–ratio test can be applied directly to this likelihood surface by using a cut–off equal to $0.5 \times \chi_1^2(.)$. Here we see that the 95% confidence interval is approximately (78,176).

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1.3 Threshold methods

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We must first go back and consider the asymptotic theory appropriate for this new situation.

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$$H(y) \;=\; 1 - \left(1 + \frac{\xi y}{\sigma}\right)_+^{-1/\xi}, \qquad (3)$$

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$$H(y) = 1 - \left(1 + \frac{\xi y}{\sigma}\right)_+^{-1/\xi}, \tag{3}$$

where $a_+ = \max(0, a)$ and σ ($\sigma > 0$) and ξ ($-\infty < \xi < \infty$) are scale and shape parameters respectively.

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Once again the GPD exists for $\xi = 0$, and is given by taking the limit of (3) as $\xi \to 0$. This time we get

$$H(y) \;=\; 1 - \exp\left(\frac{-y}{\sigma}\right), \qquad (4)$$

defined for $y > 0$.

This shows that when $\xi = 0$, the GPD is in fact the Exponential Distribution with mean equal to the scale parameter σ ($\sigma > 0$).

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Return levels for the threshold excesses approach

If the GPD is a suitable model for exceedances of a threshold u by a random variable X, then for $x > u$,

$$\Pr\{X > x | X > u\} = \left[1 + \xi \left(\frac{x - u}{\sigma}\right)\right]^{-1/\xi}.$$

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$$\Pr\{X > x | X > u\} = \left[1 + \xi \left(\frac{x - u}{\sigma}\right)\right]^{-1/\xi}.$$

It follows that

$$\Pr\{X > x\} = \lambda_u \left[1 + \xi \left(\frac{x - u}{\sigma}\right)\right]^{-1/\xi}, \qquad (5)$$

where $\lambda_u = \Pr\{X > u\}$. So the level x_m that is exceeded once every m observations is the solution of

$$\lambda_u \left[1 + \xi \left(\frac{x - u}{\sigma}\right)\right]^{-1/\xi} = \frac{1}{m}.$$

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Rearranging this we obtain

$$x_m = u + \frac{\sigma}{\xi}[(m\lambda_u)^\xi - 1],$$

so long as m is large enough to ensure that $x_m > u$.

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Rearranging this we obtain

$$x_m = u + \frac{\sigma}{\xi}[(m\lambda_u)^\xi - 1],$$

so long as m is large enough to ensure that $x_m > u$. Now if there are n_y observations per year, then by setting $m = N \times n_y$, the N–year return level is obtained as

$$z_N = \mu + \frac{\sigma}{\xi}[(Nn_y\lambda_u)^\xi - 1] \quad (6)$$

or when $\xi = 0$,

$$z_N = u + \sigma \log(Nn_y\lambda_u),$$

and standard errors can be obtained using the delta method.

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Approach 2: "Exceedances over thresholds"

In practice, modelling might typically proceed as follows:
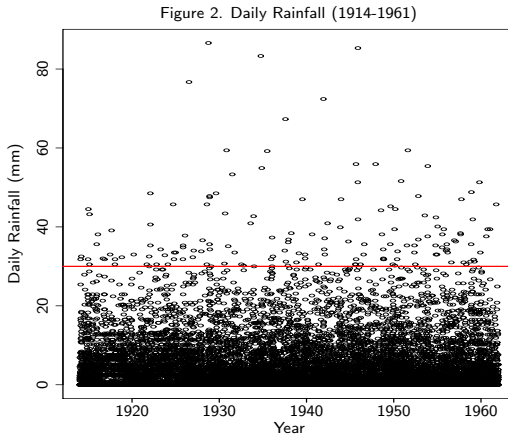
In practice, modelling might typically proceed as follows:

1. Choose some threshold u_0 which is high enough so that the GPD (3) is a good model for $(X - u_0 | X > u_0)$.

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In practice, modelling might typically proceed as follows:

1 Choose some threshold u_0 which is high enough so that the GPD (3) is a good model for $(X - u_0 | X > u_0)$.

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3. Use the fitted GPD, together with some model for the rate of exceedances $X > u_0$, to provide estimates for *return levels*.

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For the rainfall data we used before, now consider the daily totals themselves.

Figure 2. Daily Rainfall (1914-1961)

Approach 2: Threshold choice

Mean residual life plot

We make use of the fact that if the GPD is the correct model for all the exceedances x_i above some high threshold u_0, then the *mean excess*, i.e. the mean value of $(x_i - u)$, plotted against $u > u_0$, should give a linear plot (Davison and Smith, 1990).

[Because $E[X_i - u_0]$ is a linear function of $u : u > u_0$]

By producing such a plot for values of u starting at zero, we can select reasonable candidate values for u_0.
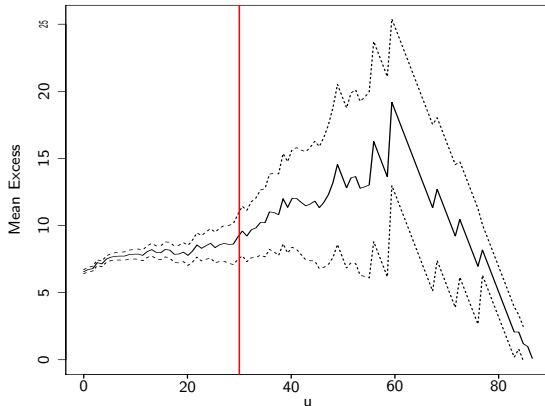
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replacements

Approach 2: Inferences

Model (3) turns out to work reasonably well for all the excesses above $u_0 = 30mm$.

This gives 152 exceedancs x_i; $i = 1, \ldots, 152$, and Model (3) is fitted to the values $(x_i - u)$, again using maximum likelihood. We get

$$\sigma = 7.44(0.96) \qquad \xi = 0.18(0.10).$$

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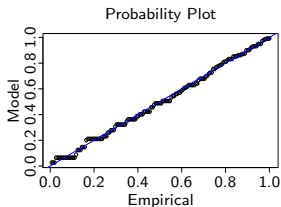
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Assuming a uniform rate of exceedances, we estimate the 100–year return level:
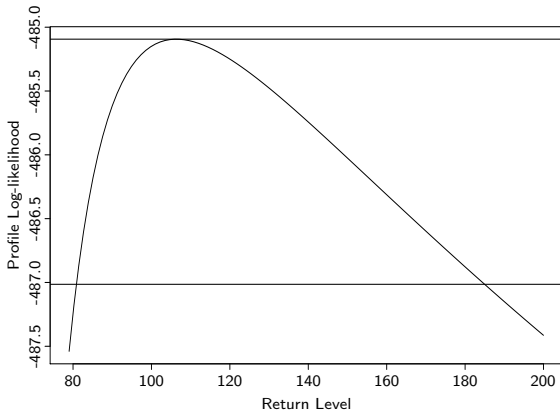
$$q_{100} = 106.3(20.8).$$

Approach 2: Diagnostics
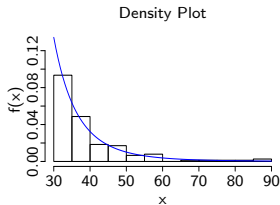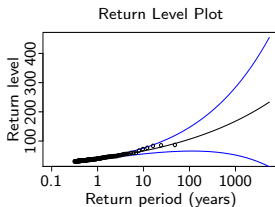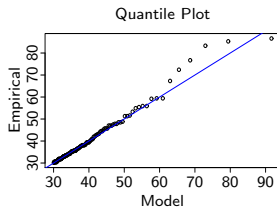
Approach 2: Profile likelilhood confidence interval for q_{100}

From the graph below, this is approximately (81,184).

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Approach 2: Threshold choice revisited

If the GPD with shape parameter ξ and scale parameter σ_{u_0} is the correct model for excesses over u_0, then for any threshold $u > u_0$, the excesses will be GPD with shape parameter ξ, and scale parameter

$$\sigma_u = \sigma_{u_0} + \xi(u - u_0).$$

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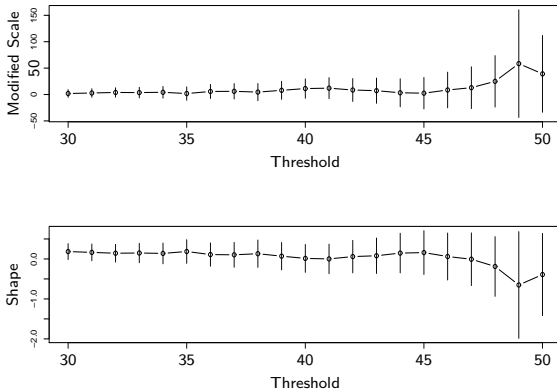If we now use a modified version of the scale parameter,

$$\sigma^* = \sigma_u - \xi u,$$

we can see that both σ^* and ξ should be constant over thresholds greater than u_0 if we model excesses $x_i - u$ for $u > u_0$ using the GPD.

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This provides us with a further tool for assessing our original choice of threshold u_0.

We refit the GPD for a range of thresholds upwards of u_0, and investigate the stability of our estimates of ξ and σ^*.

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Modelling Environmental Extremes

We can be reassured about our original choice of $u_0 = 30$!

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