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2008 J. Phys. A: Math. Theor. 41 355002

(http://iopscience.iop.org/1751-8121/41/35/355002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 355002 (20pp) doi:10.1088/1751-8113/41/35/355002

The evolution of a single-paired immigration death process

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Received 30 April 2008, in final form 2 July 2008
Published 28 July 2008
Online at stacks.iop.org/JPhysA/41/355002

Abstract
The general question of whether it is possible to determine the fundamental structure of a hidden stochastic process purely from counts of escaping individuals is of immense importance in fields such as quantum optics, where externally based radiation elucidates the nature of the electromagnetic radiation process. Although the general probability structure has been derived in an earlier paper in terms of the joint probability generating function of the (hidden) population size and (known) counts, its complex nature hides some particularly intriguing features of the underlying process. Our current objective is therefore to examine specific immigration regimes in order to highlight the underlying saw-tooth behaviour of the underlying probability and moment structures. The paper first explores paired- and triple-immigration schemes, and then introduces birth in order to show that the technique is equally successful in exposing hidden multiplicative effects. These analyses uncover novel and highly illuminating features, and emphasize the potential of this population-counting construct for expanding into more complex multi-type situations.

PACS numbers: l05.40.−a, 05.49.Df, 89.75.−k

1. Introduction

Over the past 40 years stochastic population processes have figured prominently in the description of phenomena across a wide variety of fields in Science and Engineering. An area of application that has been particularly successful has been quantum optics, where population models have been routinely used to describe the quantum nature of electromagnetic radiation (Srinivasan 1988). An especially interesting problem concerns the stochastic evolution of populations of photons within optical cavities. This field has been studied since the inception of the laser (Shimoda *et al* 1957), and understanding the counting statistics of photo-electron pulses registered by detectors such as photomultiplier tubes has been essential for the interpretation of experimental measurements (Saleh 1978). Jakeman and Shepherd (1984) and Shepherd (1984) monitor the cavity population of interest via the counting statistics of emigration, modelled as a simple death process. Properties of the number of emigrants leaving the population in a fixed time interval correspond precisely to the experimentally measured photon-counting statistics, and so provide an indirect measure of the evolution of the cavity population. This approach not only provides additional insight into the quantum formulation of the problem, but it also enables the interchange of models and techniques with those in the field of classical population statistics. In this context, Jakeman *et al* (1995) draw attention to the unusual properties of an exactly solvable population model which is generic to the recently developed area of quantum optics known as 'non-classical light'.

For many years following the invention of the laser, it was generally accepted that an adequate representation of photodetection was provided by the doubly stochastic Poisson process. In this classical situation the probability of registering c such pulses during a time interval of fixed length T (see Mandel (1959), Cox and Lewis (1966)) is given by

$$p_c(t) = \int_0^\infty \frac{I^c\,\mathrm{e}^{-I}}{c!} f(I)\,\mathrm{d}I, \tag{1.1}$$

where I is the instantaneous light intensity integrated over the interval $(0, t)$ and $f(I)$ is its probability density function. It follows from (1.1) that the variance to mean ratio, ρ, of c (i.e. the Fano factor) must always be greater than or equal to that of the Poisson distribution, for which $\rho = 1$. Light with counting statistics which can be represented in this way such as coherent (laser) light, which is Poisson distributed, or thermal light, which satisfies a geometric distribution, is termed *classical* light. However, it is easy to construct discrete models for the incident photon flux which cannot be derived through the representation (1.1). Whilst sub-Poissonian models ($\rho < 1$) clearly cannot be so represented, neither can a wide range of super-Poissonian models ($\rho > 1$). Such light is now termed *non-classical* (for references see Jakeman *et al* (1995) and Gillespie and Renshaw (2005)).

Intense activity in the development of experimental methods for generating non-classical light has produced a range of techniques which provide overwhelming evidence for its existence. Since most applications have required consideration of both wave and particle properties, theoretical treatments have generally avoided a classical population statistics approach. One of the few exceptions is Jakeman *et al*'s (1995) investigation of light, which involves the simultaneous emission of *pairs* of photons at rate α_2 and the subsequent death of each individual single photon at rate μ. This was itself stimulated by the burgeoning area of non-classical light in the late 1980s (Louden and Knight 1987). Although the equilibrium distribution is super-Poissonian, odd–even effects ensure the breakdown of (1.1), so the resulting light is non-classical. Not only does the simplicity of this model mean that it is analytically amenable, but it reflects one of the earliest mechanisms used to produce non-classical light through parametric down-conversion in a nonlinear crystal (Burnham and Weinberg 1970), and so is clearly realizable experimentally.

Gillespie and Renshaw (2005) expand the work of Jakeman *et al* (1995) by investigating the general problem where batches of photons enter the population with constant rate αq_i, where $\sum_{i=1}^{\infty} q_i = 1$. This formulation subsumes many previous results:

basic single-immigration-death process	$q_1 = 1$ and $q_i = 0$ for $i \neq 1$
paired immigrants (Jakeman *et al* 1995)	$q_2 = 1$ and $q_i = 0$ for $i \neq 2$
geometric immigration (Matthews *et al* 2003)	$q_i = (1-\xi)\xi^{i-1}$ for $0 < \xi < 1$
power-law process (Hopcraft *et al* 2002)	$q_i = \nu\Gamma(i-\nu)/[\Gamma(1-\nu)i!]$ for $0 < \nu < 1$.

Gillespie and Renshaw also examine two further immigration strategies. First, they consider the case in which each batch comprises exactly k immigrants, so that $q_k = 1$ and $q_j = 0$ for $j \neq k$, and develop attractive expressions for the population size probabilities and cumulants, together with a particularly neat approximation for the cumulants of the associated counting process. The population and counting behaviours exhibit similar manifestations, in both their probability and moment structures, thereby demonstrating that inference can be made on hidden population behaviour purely by examining the structure of successive counts. They then tackle a related question of uniqueness. For since an infinite number of probability measures can give rise to the same set of moments, there is no reason to presuppose that such similarity will hold universally. So for their second case they exploit a classic example of this phenomenon due to Schoenberg (1983); see Stoyanov (1988) for a comprehensive review of such probability classes. Essentially, they retain the individual death rate μ, but now allow batches of immigrants of size 2^k to enter the population. Specifically, for $|\epsilon| \leqslant 1$ the immigration rate takes the Schoenberg–Poisson form

$$q_{2^k} = \frac{\mathrm{e}^{-2}2^k}{k!}\{1 + \epsilon k!(-1)^k[(2-1)(2^2-1)\cdots(2^k-1)]^{-1}\} \tag{1.2}$$

with $q_i = 0$ otherwise. Expressions are then derived to show that although the moment structure remains independent of ϵ for both the population and counting processes, the corresponding probability structures do indeed change as ϵ varies.

However, the rather complex nature of these examples hides some particularly intriguing features of the underlying process, and so our current objective is to use simpler immigration regimes in order to illuminate the underlying stochastic behaviour.

2. The batch immigration-death-counting process

Suppose that a stochastic process is monitored purely *externally*, by counting (for example) the number of individuals emigrating during a fixed time interval $(0, t)$, with emigration being modelled by a pure death process with rate η. Note the fine distinction between death and emigration. For as well as allowing μ to denote the individual death rate and η the individual 'escape' rate from the system (with all escapees being subsequently counted); we could also have an individual death rate of $\mu+\eta$ and 'inefficient counting', with each individual who dies having probability $\eta/(\mu+\eta)$ of having their death recorded. The raison d'être which underlies this approach is of fundamental significance, since it provides a means of answering a general question which is of considerable importance across a wide range of disciplines, namely: 'If a stochastic process is developing within a hidden system, with the only information provided being the event times of 'escaping' individuals, can the properties of the hidden process be inferred purely from knowledge of these counting statistics?'. Though here we present an algebraically tractable example based on immigration and death (and later on birth as well), the counting concept is easily extended to all other types of processes. For example, in biological and epidemiological scenarios we might record death and notifiable illness, respectively.

Let $p_{nc}(t)$ denote the probability that at time $t > 0$ a population contains n individuals and that c emigrants have been counted. Moreover, suppose that batches of i immigrants enter the population at constant rate αq_i, where $\sum_{i=1}^{\infty} q_i = 1$, and let each individual die at rate μ. The population is monitored externally by counting the number of individuals who emigrate at rate η during a fixed time $(0, t)$. Then under the assumption that individuals develop independently from each other, the joint population size/count probabilities $\{p_{nc}(t)\}$ are defined through the

Kolmogorov forward equation

$$\frac{\mathrm{d}p_{nc}(t)}{\mathrm{d}t} = \mu(n+1)p_{n+1,c}(t) + \eta(n+1)p_{n+1,c-1}(t) + \alpha\sum_{i=1}^{\infty} q_i p_{n-i,c}(t) - [(\mu+\eta)n+\alpha]p_{nc}(t), \tag{2.1}$$

where $n, c = 0, 1, 2, \ldots$ and $p_{nc}(t) = 0$ for $n, c = -1, -2, \ldots$. On defining the joint probability generating function

$$Q(z, s; t) \equiv \sum_{n,c=0}^{\infty} z^n s^c p_{nc}(t), \tag{2.2}$$

and denoting $\delta = \mu + \eta$, equations (2.1) reduce to the single partial differential equation

$$\frac{\partial Q}{\partial t} + (\delta z - \eta s - \mu)\frac{\partial Q}{\partial z} = \alpha Q \sum_{i=1}^{\infty} q_i (z^i - 1). \tag{2.3}$$

Whence on assuming that the population is of size zero at time $t = 0$, Gillespie and Renshaw (2005) obtain the general solution

$$Q(z, s; t) = \exp\left(\sum_{i=1}^{\infty} \frac{\alpha t(\eta s+\mu)^i q_i}{\delta^i} - \alpha t\right) \times \exp\left(\sum_{i=1}^{\infty} \frac{\alpha q_i}{\delta^{i+1}} \sum_{r=1}^{i} \frac{1}{r}\binom{i}{i-r}(\eta s+\mu)^{i-r}(\delta z - \eta s - \mu)^r (1 - \mathrm{e}^{-r\delta t})\right). \tag{2.4}$$

It should be noted that the above derivations can also be framed in the context of a marked Poisson process (see Kingman 1993). In this scenario we fix a time τ and mark a particular batch immigration event formed before τ by its number of descendants at time τ. Each point in this marked space is a pair (T, D) denoting the random time, T, of the occurrence of the cluster and its eventual number of descendants, D. The total number of particles can now be obtained through Campbell's formula.

2.1. *Paired immigration*

Suppose that single and paired immigrants arrive at rate $\alpha q_1 = \alpha_1$ and $\alpha q_2 = \alpha_2$, respectively, so $q_i = 0\,(i > 2)$. Moreover, each individual either dies at rate μ or escapes (i.e. emigrates) and is subsequently counted at rate η. Then the general p.g.f. solution (2.4) reduces to

$$Q(z, s; t) = \exp\left(\frac{\alpha_1}{\delta^2}(\xi(1-\mathrm{e}^{-\delta t}) + \delta\eta t(s-1))\right)\exp\left(\frac{\alpha_2}{2\delta^3}(2\delta t(\eta s+\mu)^2 - 2\delta^3 t)\right) \times \exp\left(\frac{\alpha_2}{2\delta^3}(\xi^2(1-\mathrm{e}^{-2\delta t}) + 4\xi(1-\mathrm{e}^{-\delta t})(\eta s+\mu))\right), \tag{2.5}$$

where $\delta = \mu + \eta$ and $\xi = \delta z - \eta s - \mu$. Although Gillespie and Renshaw (2005) provide the general marginal probabilities for this process, these hide the underlying odd–even structure. To expose this we first note that the generating function representation for Hermite polynomials is

$$\sum_{n=0}^{\infty} H_n(x)z^n/n! \equiv \exp(2xz - z^2) \tag{2.6}$$

(Abramowitz and Stegun [A&S] (1970), result 22.9.17), and that we also have the relationships

$$H_{2n}(x) \equiv (-1)^n 2^{2n} n! L_n^{-1/2}(x^2) \text{ and } H_{2n+1}(x) \equiv (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2) \tag{2.7}$$

(A&S, 22.5.40/41) between Hermite and Laguerre polynomials. Then to recover the even and odd time-dependent probabilities, $p_n(t) = p_{n.}(t)$, from (2.5), extracting the coefficients of z^{2n} and z^{2n+1} at $s = 1$ yields for $k = 0, 1$,

$$p_{2n+k}(t) = \frac{n!2^{2n}}{(2n+k)!}\left(\frac{\alpha_2(1-\mathrm{e}^{-2\mu t})}{2\mu}\right)^n f_1(k)\exp\left(\frac{(2\alpha_1+\alpha_2(3-\mathrm{e}^{-\mu t}))(\mathrm{e}^{-\mu t}-1)}{2\mu}\right) \times L_n^{k-1/2}\left(\frac{(\alpha_1+\alpha_2(1-\mathrm{e}^{-\mu t}))^2(1-\mathrm{e}^{-\mu t})^2}{2\alpha_2\mu(\mathrm{e}^{-2\mu t}-1)}\right), \tag{2.8}$$

where $f_1(0) = 1$ and $f_1(1) = (\alpha_1 + \alpha_2(1 - \mathrm{e}^{-\mu t}))(1 - \mathrm{e}^{-\mu t})/\mu$. Likewise, placing $z = 1$ recovers the marginal counting probabilities $p^c_{2c}(t) = p_{.,2c}(t)$ and $p^c_{2c+1}(t) = p_{.,2c+1}(t)$, namely for $k = 0, 1$,

$$p^c_{2c+k}(t) = \frac{c!}{(2c+k)!}\left\{\frac{2\alpha_2\eta^2}{\delta^3}[2\delta t-(1-\mathrm{e}^{-\delta t})(3-\mathrm{e}^{-\delta t})]\right\}^c f_2(k)\exp\left\{\frac{\alpha_1\eta}{\delta^2}[1-\delta t-\mathrm{e}^{-\delta t}]\right\} \times\exp\left\{\frac{\alpha_2}{2\delta^3}[\eta(4\mu(1-\mathrm{e}^{-\delta t})+\eta(1-\mathrm{e}^{-2\delta t}))+2\delta t(\mu^2-\delta^2)]\right\} \times L_c^{k-1/2}\left\{-\frac{\theta}{2\delta^3\alpha_2[(1-\mathrm{e}^{-2\delta t})-4(1-\mathrm{e}^{-\delta t})+2\delta t]}\right\}, \tag{2.9}$$

where

$\theta = \{\alpha_1\delta[\delta t + \mathrm{e}^{-\delta t} - 1] + \alpha_2[2(\eta-\mu)(1-\mathrm{e}^{-\delta t}) - \eta(1-\mathrm{e}^{-2\delta t}) + 2\delta\mu t]\}^2$,

$f_2(0) = 1$,

and

$f_2(1) = \frac{\alpha_2\eta}{\delta^3}[2(\eta-\mu)(1-\mathrm{e}^{-\delta t}) - \eta(1-\mathrm{e}^{-2\delta t}) + 2\delta\mu t] + \frac{\alpha_1\eta}{\delta^2}(\delta t + \mathrm{e}^{-\delta t} - 1)$.

Although at first glance solutions (2.8) and (2.9) look rather opaque, closer inspection soon exposes the odd–even effect in the probability structure. For $p_{2n}(t)$ and $p_{2n+1}(t)$, and also $p^c_{2c}(t)$ and $p^c_{2c+1}(t)$, essentially differ only by the powers of the Laguerre polynomial, namely $L_c^{-1/2}(\cdot)$ and $L_c^{1/2}(\cdot)$, respectively. It is this 'saw-tooth' effect that causes the generation of non-classical light.

The factorial moments $\{M^c_r(t)\}$ of the counting distribution may also be obtained directly, by successively differentiating $Q(z, s; t)$ with respect to s and forming $Q^{(2r)}(1, s; t)$ and $Q^{(2r+1)}(1, s; t)$. This procedure yields for $k = 0, 1$,

$$M^c_{2r+k}(t) = r!2^{2r}\left\{\frac{\alpha_2\eta^2}{2\delta^3}[2\delta t-(1-\mathrm{e}^{-\delta t})(3-\mathrm{e}^{-\delta t})]\right\}^r \times L_r^{k-1/2}\left\{\frac{(\alpha_1+2\alpha_2)^2(\delta t+\mathrm{e}^{-\delta t}-1)^2}{2\alpha_2\delta[(1-\mathrm{e}^{-\delta t})(3-\mathrm{e}^{-\delta t})-2\delta t]}\right\} f_3(k), \tag{2.10}$$

where $f_3(0) = 1$ and $f_3(1) = (\eta/\delta^2)(\alpha_1 + 2\alpha_2)(\delta t + \mathrm{e}^{-\delta t} - 1)$. These expressions clearly highlight the odd–even effect in the same manner as with the counting probabilities; the only substantive difference between the counting moments $M^c_{2r}(t)$ and $M^c_{2r+1}(t)$ once again lies in the powers of the Laguerre polynomials, namely $L_r^{-1/2}(\cdot)$ and $L_r^{1/2}(\cdot)$.

To illustrate how this odd–even structure degrades when single immigrants are allowed, suppose we take time $t = 2$ and a total immigration rate of 10, i.e. $\alpha_1 + 2\alpha_2 = 10$. Figure 1 shows how changing the balance between α_1 and α_2 affects the counting probabilities $p_c(2)$, and it is clear that the extreme saw-tooth shape of the distribution is only (visually) present when $\alpha_1 \simeq 0$. This can be explained by considering

$$\begin{aligned}\tilde{p}_0(t) &= \Pr(\text{no deaths from single immigrants by time } t)\\ &= \exp\{(\alpha_1/\eta)(1-\mathrm{e}^{-\eta t}-\eta t)\}.\end{aligned} \tag{2.11}$$

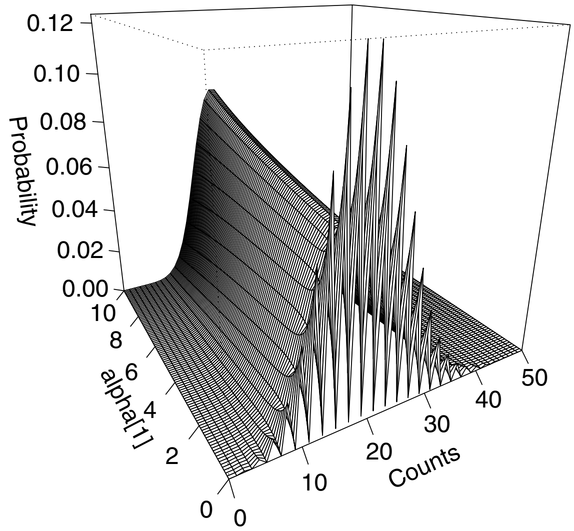

Figure 1. Count probabilities $p_c^c(t)$ for $\eta = 300$ and $\mu = 0$, with $\alpha_1 + 2\alpha_2 = 10$ for varying α_1 and fixed time $t = 2$.

For when $\alpha_1/\eta \simeq 0$, (2.11) simplifies to $\tilde{p}_0(t) \simeq \mathrm{e}^{-\alpha_1 t}$, and since this swiftly decays to zero with increasing $\alpha_1 t$ the intensity of the odd–even effect must decay with it. Indeed, the time required to reduce $\tilde{p}_0(t)$ to (say) 0.5, namely $\ln(2)/\alpha_1$, reduces inversely with α_1.

The overall effect that α_1 induces on the even and odd probabilities is best seen by recalling that Laguerre and Hermite polynomials are neatly related through (2.7). For on applying

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} H_{2n}(x) z^n \equiv \mathrm{e}^z \cos(2x\sqrt{z}) \tag{2.12}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} H_{2n+1}(x) z^n \equiv z^{-1/2}\, \mathrm{e}^z \sin(2x\sqrt{z}) \tag{2.13}$$

(A&S section 22.9.18/19) to the full probability expressions (2.9) with $\mu = 0$ and η large compared with $(\alpha_1 + \alpha_2)^t$, we obtain

$$\sum_{c=0}^{\infty} p_{2c}^c(t) = \mathrm{e}^{-\alpha_1 t} \sinh(\alpha_1 t) \text{ and } \sum_{c=0}^{\infty} p_{2c+1}^c(t) = \mathrm{e}^{-\alpha_1 t} \cosh(\alpha_1 t). \tag{2.14}$$

Since $\tanh(\alpha_1 t) \uparrow 1$ as $t \to \infty$, it follows that when $\sum_{c=0}^{\infty} p_{2c}^c(t) \simeq 0.5$ the odd–even effect is no longer 'visible' as the sums of the even and odd probabilities are equal. Rather surprisingly, expressions (2.14) do not depend on α_2, indicating that the degradation of the odd–even effect depends solely on α_1 and t. Figure 2 illustrates how the probabilities (2.9) change through time when $\alpha_1 = 0.075$ is substantially smaller than $\alpha_2 = 1.2$. Whilst a strong odd–even effect is clearly present at times $t = 10$ and $t = 15$, as t increases further the saw-tooth shape becomes much less pronounced. Quantitatively, we see from (2.11) that the probability of zero single immigrant deaths reduces from $\tilde{p}_0(10) = 0.472$ to $\tilde{p}_0(30) = 0.105$. So at time $t = 30$, the probability of no single-immigrant deaths is about five times smaller than that at $t = 10$, resulting in a much reduced odd–even effect.

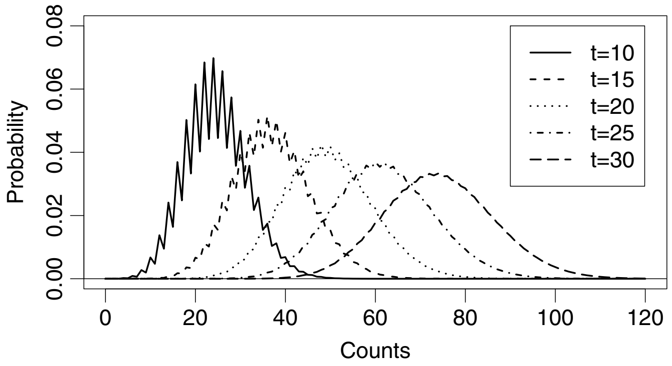

Figure 2. Count probabilities $p_c(t)$ for $\eta = 300$ and $\mu = 0$, with $\alpha_1 = 0.075$ and $\alpha_2 = 1.2$ for varying times $t = 10, 15, \ldots, 30$.

To determine the corresponding effects on the second-order moments, first note that the population variance

$$\mathrm{Var}[n(t)] = \{2\alpha_2(1 - \mathrm{e}^{-2\mu t}) + [\alpha_1 + \alpha_2(1 - \mathrm{e}^{-\mu t})](1 - \mathrm{e}^{-\mu t})\}/\mu \tag{2.15}$$

is linear in α_1 and α_2; whilst the autocovariance function

$$\mathrm{E}[n(t)n(t+s)] - \mathrm{E}[n(t)]\mathrm{E}[n(t+s)] = \mathrm{e}^{-\mu s}(1 - \mathrm{e}^{-\mu t})[\alpha_1 + \alpha_2(3 + \mathrm{e}^{-\mu t})]/\mu. \tag{2.16}$$

Hence as $t \to \infty$, the limiting autocorrelation function

$$\rho(\infty; s) = \mathrm{e}^{-\mu s}, \tag{2.17}$$

which, perhaps surprisingly, does not depend on either α_1 or α_2.

To obtain the associated counting statistics we parallel the approach taken by Shepherd and Jakeman (1987), who develop the correlation properties of the counting process from the joint probability $\Pr\{c_1; (t_1, t_1 + T) \text{ and } c_2; (t_2, t_2 + T)\}$ of counting c_1 individuals during the period $(t_1, t_1 + T)$ and c_2 individuals during the subsequent, and non-overlapping period, $(t_2, t_2 + T)$. Though our analysis is slightly different, since we use a specific initial condition (e.g. $p_{00}(0) = 1$) rather than assuming that the process starts in equilibrium, in order to examine both transient and persistent behaviour. First note that

$$\begin{aligned}
&\Pr\{c_1; (t_1, t_1 + T) \text{ and } c_2; (t_2, t_2 + T)\} \\
&\quad = \sum_{N_1, N_2, N_3, N_4 = 0}^{\infty} \Pr(\text{size } N_1 \text{ at time } t_1 \mid \text{size } n_0 \text{ at time } 0) \\
&\qquad \times \Pr(\text{counting } c_1 \text{ in } (t_1, t_1 + T) \text{ and size } N_2 \text{ at time } t_1 + T \mid \text{size } N_1 \text{ at time } t_1) \\
&\qquad \times \Pr(\text{size } N_3 \text{ at time } t_2 \mid \text{size } N_2 \text{ at time } t_1 + T) \\
&\qquad \times \Pr(\text{counting } c_2 \text{ in } (t_2, t_2 + T) \text{ and size } N_4 \text{ at time } t_2 + T \mid \text{size } N_3 \text{ at time } t_2) \\
&\quad = \sum_{N_1, N_2, N_3 = 0}^{\infty} \Pr(N_1; t_1 \mid n_0; 0) \Pr(c_1; (t_1, t_1 + T) \text{ and } N_2; t_1 + T \mid N_1; t_1) \\
&\qquad \times \Pr(N_3; t_2 \mid N_2; t_1 + T) \Pr(c_2; (t_2, t_2 + T) \mid N_3; t_2).
\end{aligned} \tag{2.18}$$

Then after some algebra, it may be shown that the counting autocorrelation function for the single-paired immigration process is given by

$$\rho^c(c_1(t_1,t_1+T),c_2(t_2,t_2+T)) = \frac{\alpha_2\eta^2\,\mathrm{e}^{-\delta(t_1+t_2)}(1-\mathrm{e}^{-\delta T})^2(\mathrm{e}^{\delta(2t_1+T)}-1)}{\delta^3\sqrt{\mathrm{Var}(t_1,t_1+T)\,\mathrm{Var}(t_2,t_2+T)}} - \frac{n_0\eta^2\,\mathrm{e}^{-\delta(t_1+t_2)}(1-\mathrm{e}^{-\delta T})^2}{\delta^2\sqrt{\mathrm{Var}(t_1,t_1+T)\,\mathrm{Var}(t_2,t_2+T)}}, \tag{2.19}$$

where, for $i = 1, 2$,

$$\mathrm{Var}(t_i,t_i+T) = \frac{\alpha_1\eta T}{\delta} - \frac{\alpha_1\eta\,\mathrm{e}^{-\delta t_i}(1-\mathrm{e}^{-\delta T})}{\delta^2} + \frac{2\alpha_2\eta T(\delta+\eta)}{\delta^2} - \frac{2\alpha_2\eta^2(1-\mathrm{e}^{-\delta T})}{\delta^3} - \frac{\eta^2\,\mathrm{e}^{-2\delta t_i}(\alpha_2+\delta n_0)(1-\mathrm{e}^{-\delta T})^2}{\delta^3} + \frac{\eta\,\mathrm{e}^{-\delta t_i}(\delta n_0-2\alpha_2)(1-\mathrm{e}^{-\delta T})}{\delta^2}. \tag{2.20}$$

Placing $t_2 = t_1 + T + \tau$ in the counting a.c.f. (2.19) and letting $t_1 \to \infty$ then yields

$$\rho^c(c_1(t_1,t_1+T),c_2(t_1+T+\tau,t_1+2T+\tau)) \to \frac{\alpha_2\eta(1-\mathrm{e}^{-\delta T})^2\,\mathrm{e}^{-\delta\tau}}{2\alpha_2(T(2\delta^2-\mu^2-\eta\mu)-\eta(1-\mathrm{e}^{-\delta T}))+\alpha_1\delta^2 T}. \tag{2.21}$$

This result reveals four interesting insights into the counting process. First, it shows how the correlation decreases as τ increases; as the two time periods being compared become further apart the corresponding (driving) population processes have fewer individuals in common. Second, it describes how as α_1 increases, the correlation decreases; for the process becomes dominated by single immigrants. Indeed, when $\alpha_2 = 0$ the process reverts back to a Poisson process for which $\rho \equiv 0$. Third, when $\alpha_1 = 0$ expression (2.21) is independent of α_2. Finally, as the counting rate $\eta \to \infty$, we see that $\rho^c \to 0$, since immigrants entering the system are immediately removed, and so the non-overlapping time intervals cannot contain individuals common to both.

2.2. Triple immigration

Having seen that the general solution (2.4) produces a highly definitive saw-tooth effect under paired immigration, provided $\alpha_1 \ll \alpha_2$, questions naturally arise as to the specific form of the solution under say triple immigration with $\alpha_3 = \alpha q_3 \gg \alpha_1, \alpha_2$, and what happens in a multiplicative environment in which the complexity of the underlying population process is increased by including birth. Tackling the former scenario first, we assume, as before, that individuals leave the system at rate η, die unnoticed at rate μ and are counted at rate η. So the general forward Kolmogorov equation (2.1) now takes the specific form

$$\mathrm{d}p_{nc}(t)/\mathrm{d}t = \mu(n+1)p_{n+1,c}(t) + \eta(n+1)p_{n+1,c-1}(t) + \alpha_3 p_{n-3,c}(t) - (\mu n + \eta n + \alpha_3)p_{nc}(t), \tag{2.22}$$

for $n, c = 0, 1, 2, \ldots,$ where we define $p_{-1,c}(t) = p_{-2,c}(t) = p_{-3,c}(t) = 0$. This equation may be solved in terms of the associated generating function

$$R(z,s;t) = \exp\left\{\frac{\alpha_3}{6\delta^4}18(\eta s+\mu)^2(\delta z-\eta s-\mu)(1-\mathrm{e}^{-\delta t})\right\} \times \exp\left\{\frac{\alpha_3}{6\delta^4}9(\eta s+\mu)(\delta z-\eta s-\mu)^2(1-\mathrm{e}^{-2\delta t})\right\} \times \exp\left\{\frac{\alpha_3}{6\delta^4}2(\delta z-\eta s-\mu)^3(1-\mathrm{e}^{-3\delta t})\right\} \times \exp\left\{\frac{\alpha_3}{6\delta^3}6t((\eta s+\mu)^3-\delta^3)\right\}, \tag{2.23}$$

where $\delta = \mu + \eta$. Whence it follows that the (marginal) population p.g.f. can be expressed as

$$R(z, 1; t) = \exp(\xi_0 + \xi_1 z + \xi_2 z^2 + \xi_3 z^3), \tag{2.24}$$

where

$$\begin{aligned} \xi_1 &= (\alpha_3/\mu)(1 - \mathrm{e}^{-\mu t})^3, & \xi_2 &= (\alpha_3/2\mu)(1 - \mathrm{e}^{-\mu t})^2(1 + 2\,\mathrm{e}^{-\mu t}), \\ \xi_3 &= \frac{\alpha_3}{3\mu}(1 - \mathrm{e}^{-3\mu t}) \qquad \text{and} & \xi_0 &= -\xi_3 - \xi_2 - \xi_1. \end{aligned} \tag{2.25}$$

Similarly, the (marginal) counting p.g.f. takes the form

$$R(1, s; t) = \exp(\zeta_0 + \zeta_1 s + \zeta_2 s^2 + \zeta_3 s^3), \tag{2.26}$$

where

$$\begin{aligned} \zeta_1 &= \frac{\alpha_3\eta}{6\delta^4}[18\mu(-\mu + 2\eta)(1 - \mathrm{e}^{-\delta t}) + 9\eta(-2\mu + \eta)(1 - \mathrm{e}^{-2\delta t})] \\ &\quad - \frac{\alpha_3\eta^3(1 - \mathrm{e}^{-3\delta t})}{\delta^4} + \frac{3\alpha_3\mu^2\eta t}{\delta^3}, \\ \zeta_2 &= \frac{\alpha_3\eta^2}{6\delta^4}[18(\eta - 2\mu)(1 - \mathrm{e}^{-\delta t}) + 9(\mu - 2\eta)(1 - \mathrm{e}^{-2\delta t}) + 6\eta(1 - \mathrm{e}^{-3\delta t})] + \frac{3\alpha_3\mu\eta^2 t}{\delta^3}, \\ \zeta_3 &= \frac{\alpha_3\eta^3}{6\delta^4}[9\,\mathrm{e}^{-\delta t}(2 - \mathrm{e}^{-\delta t}) + 2\,\mathrm{e}^{-3\delta t} - 11] + \frac{\alpha_3\eta^3 t}{\delta^3}, \end{aligned} \tag{2.27}$$

and $\zeta_0 = -\zeta_3 - \zeta_2 - \zeta_1$. On transforming the marginal p.g.f.s $R(z, 1; t)$ and $R(1, s; t)$ into their cumulant generating function (c.g.f.) forms, namely $\ln[R(\mathrm{e}^\theta, 1; t)]$ and $\ln[R(1, \mathrm{e}^\phi; t)]$, it immediately follows that the ith cumulants of the population and counting processes are respectively given by

$$\kappa_i(t) = \xi_1 + 2^i\xi_2 + 3^i\xi_3 \qquad \text{and} \qquad \kappa_i^c(t) = \zeta_1 + 2^i\zeta_2 + 3^i\zeta_3, \tag{2.28}$$

highly suggestive of a saw-tooth structure with three teeth. Note that both of these cumulant forms exhibit the characteristic of $\kappa_{i+1}/\kappa_i \to 3$ as $i \to \infty$, in contrast to the value 1 for the Poisson process generated by the single-immigration-death process. So in the context of light intensity the triple immigration-death process corresponds to non-classical, and not classical, light (see the comment following (1.1)).

To obtain the associated marginal population and counting probabilities we need to extract the coefficients of z^n and s^c in expressions (2.24) and (2.26), respectively. Inserting the Hermite expansion (2.6) for the exponential of a quadratic function into the exponential cubic representation (2.24), and paralleling the derivation for the paired-immigration process, yields the marginal population size probabilities

$$p_n(t) = \mathrm{e}^{\xi_0} \sum_{r=0}^{[n/3]} \frac{\xi_3^r(-\xi_2)^{(n-r)/2}}{r!(n-r)!} H_{n-r}(\omega_1). \tag{2.29}$$

Note that it is the presence of $[x]$ (i.e. the integer part of x) in the upper limit of the summation that induces the triple effect. Specifically, on splitting $p_n(t)$ into $p_{3n}(t)$, $p_{3n+1}(t)$ and $p_{3n+2}(t)$, we have for $i = 0, 1, 2$,

$$p_{3n+i}(t) = \mathrm{e}^{\xi_0} \sum_{r=0}^{n} \frac{\xi_3^r(-\xi_2)^{3(n-r+i/3)/2}}{r![3(n-r+i/3)]!} H_{3(n-r+i/3)}(\omega_1). \tag{2.30}$$

The strong degree of similarity between these three expressions means that on denoting

$$G_m^{(i)}(x_1, x_2, x_3, z) \equiv \sum_{r=0}^{m} \frac{x_3^r(-x_2)^{3(m-r+i/3)/2}}{r![3(m-r+i/3)]!} H_{3(m-r+i/3)/2}(x_1) z^{3r+i}, \tag{2.31}$$

they can be written in the much simpler, unified, form

$$p_{3n+i}(t) = \mathrm{e}^{\xi_0} G_n^{(i)}(\omega_1, \xi_2, \xi_3, 1) \qquad \text{where} \qquad \omega_1 = \frac{\sqrt{\alpha_3}}{\sqrt{-2\mu}} \frac{(1-\mathrm{e}^{-\mu t})^2}{\sqrt{1+2\,\mathrm{e}^{-\mu t}}}. \tag{2.32}$$

As with the paired-immigration probabilities (2.8), these triple-immigration probabilities exhibit a structure that corresponds directly to the number of immigrants that simultaneously enter the process. Letting $t \to \infty$ yields (relatively) simple expressions for the equilibrium probabilities, namely

$$p_{3n+i}(\infty) = \exp\left(\frac{-11\alpha_3}{6\mu}\right) G_n^{(i)}\left(\sqrt{\frac{\alpha_3}{-2\mu}}, \frac{\alpha_3}{2\mu}, \frac{\alpha_3}{3\mu}\right) \tag{2.33}$$

for $i = 0, 1, 2$ and $n = 0, 1, 2, \ldots$. A parallel argument for the counting probabilities, this time based on the p.g.f. (2.26), produces similar triple-effect characteristics.

To calculate the factorial population moments $M_r(t)$ we first substitute $z = 1 + z'$ into $R(z, 1; t)$ to form

$$\begin{aligned} R(1+z', 1; t) &= \exp\{\xi_0 + \xi_1(1+z') + \xi_2(1+z')^2 + \xi_3(1+z')^3\} \\ &= \exp\{(\xi_1 + 2\xi_2 + 3\xi_3)z' + (\xi_2 + 3\xi_3)(z')^2 + \xi_3(z')^3\}, \end{aligned} \tag{2.34}$$

where $\xi_0, \ldots, \xi_3$ are defined by (2.25). Then on extracting the coefficient of $(z')^n$ we obtain for $i = 0, 1, 2$

$$M_{3r+i}(t) = \sum_{i=0}^{r} \frac{\xi_3^i(-\xi_2 - 3\xi_3)^{3(r-i+i/3)/2}}{i![3(r-i+i/3)]!} H_{3(r-i+i/3)}(\omega_2), \tag{2.35}$$

where $\omega_2 = (\xi_1 + 2\xi_2 + 3\xi_3)/(2\sqrt{-\xi_2 - 3\xi_3})$. As with the population probabilities (2.32), use of the representation (2.31) yields the much simpler, unified, form,

$$M_{3r+i}(t) = G_r^{(i)}(\omega_2, \xi_2 + 3\xi_3, \xi_1 + 2\xi_2 + 3\xi_3, 1). \tag{2.36}$$

Note that this result is a direct parallel of expression (2.32) for the $p_{3n+i}(t)$, and provides a clear exposition of the triple effect in exactly the same way. The marginal counting factorial moments $\left\{M_{3r+i}^c(t)\right\}$ can be calculated in similar manner, based on the counting p.g.f. (2.26) for $R(1, s; t)$ with s replaced by $1 + s'$.

Figure 3(*a*) shows the equilibrium population probabilities (2.33) for $M_1(\infty) = 1, \ldots, 6$. When the mean is close to unity, the gradient changes clearly illustrate a tripled effect, which diminishes as $M_1(\infty)$ increases; this effect parallels that for the paired process, where an odd–even effect is witnessed only when the mean is close to 1 (Jakeman *et al* 1995). Figure 3(*b*) shows the associated time-dependent p.d.f. at $t = 0.1$, 1 and 10 for $n_0 = 1$; taking $\alpha_3 = 2$ and $\mu = 1$ gives an equilibrium mean of 6. Though the gradient changes exhibit a strong triple effect at $t = 0.1$, this has already substantially diminished by $t = 1$, and has (visually) disappeared by $t \simeq 10$. This swift decline is supported by figure 4(*a*), which shows the ratio $p_i(t)/p_{i+1}(t)$ for $i = 0, 1, \ldots, 20$. This ratio not only demonstrates that the triple effect disappears as time increases, but also that the effect itself holds throughout the range of i-values. Note that a comparison of figures 3(*b*) and 4(*a*) shows that this probability ratio reveals the presence of a triple effect much more strongly than a straight probability plot. Further comparison with figure 4(*b*) shows that the power of the factorial population moment ratio, $M_i(t)/M_{i+1}(t)$, to highlight the triple effect lies inbetween these two. Here, as with the probability ratio shown in figure 4(*a*), the tripled effect is (visually) present only when t is small. A parallel analysis for the counting probability ratio $p_i^c(t)\big/p_i^{c+1}(t)$ (figure 5(*a*)), and factorial counting moment ratio $M_i^c(t)\big/M_{i+1}^c(t)$ (figure 5(*b*)), produces similar results, i.e. a strong tripled effect is present only for small time values. This close degree of similarity once again highlights the power of the counting process to capture a considerable amount of the information generated by the underlying population process.

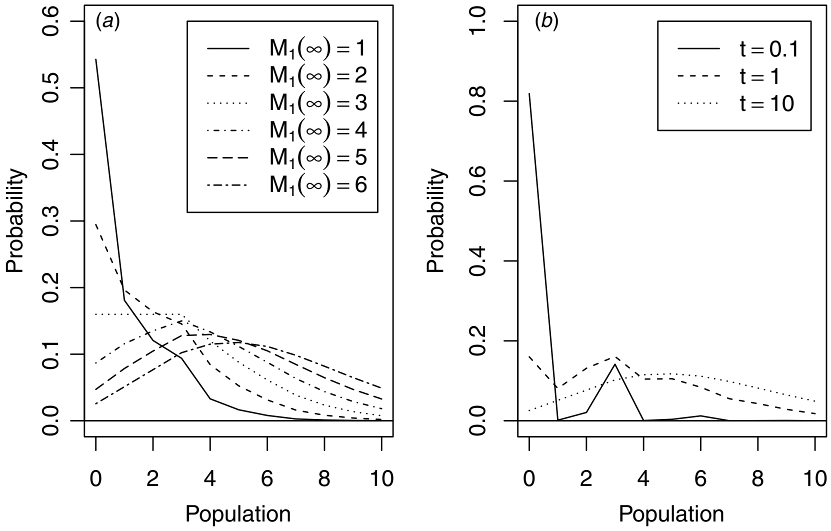

Figure 3. Triple immigration-death p.d.f.s for $\mu = 1$ and $\alpha_3 = 2$ with $n_0 = 0$: (*a*) $p_n(\infty)$ for equilibrium means $M_1(\infty) = 1, \ldots, 6$; (*b*) $p_n(t)$ for $t = 0.1, 1, 10$.

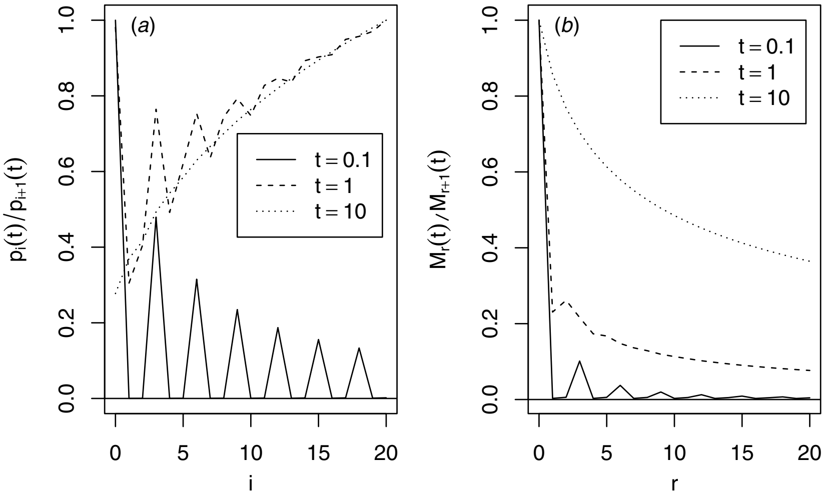

Figure 4. Ratios corresponding to figure 3(*b*) for (*a*) population probabilities, $p_i(t)/p_{i+1}(t)$, and (*b*) factorial moments, $M_i(t)/M_{i+1}(t)$, normalized to ensure a maximum value of 1.

2.2.1. The effect of a large death rate. We have already seen that when paired immigrants enter the population, a strong odd–even effect is present when the death rate becomes very large relative to the other parameters. To show that this phenomenon carries over to the triple immigration process, we note that when $\delta = \mu + \eta$ is large in comparison to α_3, the associated counting p.d.f. is a direct parallel of the equilibrium population result (2.33), namely

$$p^c_{3c+i}(t) \simeq \exp\left(\frac{\alpha_3(\mu^3 - \delta^3)t}{\delta^3}\right) G^i_c\left(-\sqrt{-\frac{3\alpha_3 t\mu^3}{4\delta^3}}, \frac{3\alpha_3\eta^2\mu t}{\delta^3}, \frac{\alpha_3\eta^3 t}{\delta^3}, 1\right), \tag{2.37}$$

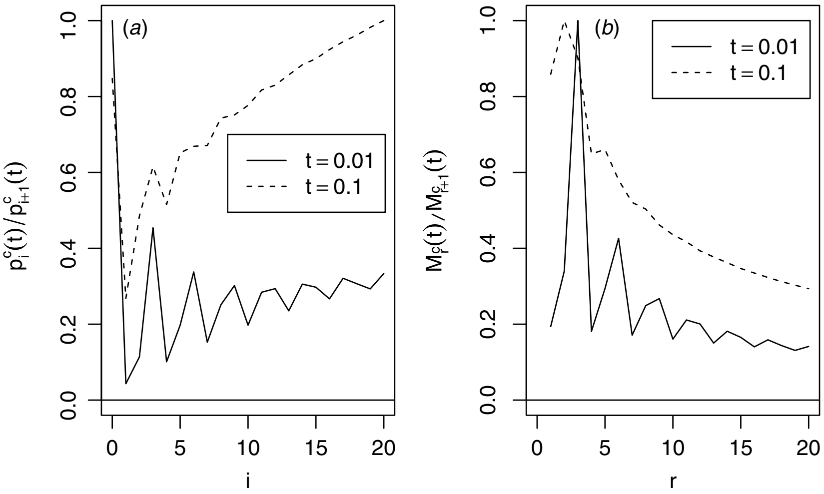

Figure 5. (*a*) The normalized counting ratios (*a*) $p_i^c(t)/p_{i+1}^c(t)$ and (*b*) $M_i^c(t)/M_{i+1}^c(t)$ corresponding to figure 4 for $\mu = 0, \eta = 1, n_0 = 0, \alpha_3 = 2$ and $t = 0.01, 0.1$.

for $i = 0$, 1 and 2. For the special case $\mu = 0$ (i.e. counting but no death), this reduces to the Poisson distribution with rate $\alpha_3 t$ defined purely on $c = 0, 3, 6, \ldots$, namely $p_{3c}^c(t) \simeq (\alpha_3 t)^c\,e^{-\alpha_3 t}/c!$ Since the death rate is large in comparison to the immigration rate, immigrants have a very short life-span and consequently $p_{3c+1}^c(t) \simeq p_{3c+2}^c(t) \simeq 0$. This structure results in 'probability ridges' appearing in the p.d.f. at $c = 0, 3, 6 \ldots$.

2.2.2. The autocorrelation function. The derivation of the population and counting autocorrelation functions follows the same route as that developed for the paired-immigration process, with the former leading to

$$\rho(s) = \frac{[3\alpha_3(2 + e^{-\mu t}) + n_0\mu\,e^{-\mu t}](1 - e^{-\mu t})\,e^{-\mu s}}{\mu\sqrt{\kappa_2(t)\kappa_2(t+s)}}, \tag{2.38}$$

where the variance $\kappa_2(t) = (3\alpha_3/\mu)(1 - e^{-\mu t})(2 + e^{-\mu t}) + n_0\,e^{-\mu t}(1 - e^{-\mu t})$. So α_3 only features in the opening, highly transient, stage of the process, with $\rho(s)$ converging swiftly towards the equilibrium solution $\rho_{eq}(s) = e^{-\mu s}$ as $t \to \infty$. Thus whilst the probability and moment structures of the triple-immigration death-process are influenced extensively by the rate of immigrant entry into the population, the population autocorrelation function is not. Indeed, in equilibrium this autocorrelation function is identical to that of the pairs and singleton processes. The corresponding counting autocorrelation function is given by

$$\rho^c(c_1, c_2) = \frac{e^{-\delta t_2}\eta^2[6\alpha_3\,e^{\delta t_1}(\cosh(\delta T) - 1) - e^{-\delta t_1}(1 - e^{-\delta T})^2(3\alpha_3 + \delta n_0)]}{\delta^3\sqrt{\mathrm{Var}[c_1;\,(t_1, t_1 + T)]\,\mathrm{Var}[c_2;\,(t_2, t_2 + T)]}}, \tag{2.39}$$

where for $i = 1, 2$,

$$\begin{aligned}\mathrm{Var}[c_i;\,(t_i, t_i + T)] &= 3\alpha_3\eta(3\eta + \mu)\delta^{-2}T - 6\alpha_3\eta^2\delta^{-3}(1 - e^{-\delta T})\\ &\quad - \eta^2(3\alpha_3 + \delta n_0)\delta^{-3}\,e^{-2\delta t_i}(1 - e^{-\delta T})^2 + \eta(\delta n_0 - 3\alpha_3)\delta^{-2}\,e^{-\delta t_i}(1 - e^{-\delta T})\end{aligned} \tag{2.40}$$

and $\rho^c(c_1, c_2) \equiv \rho(c_1; (t_1, t_1 + T), c_2; (t_2, t_2 + T))$ is evaluated as per expression (2.18). To examine the equilibrium correlation structure we set $t_2 = t_1 + \tau + T$, for $\tau \geqslant 0$, and allow $t_1 \to \infty$, whence (2.39) yields

$$\rho^c(c_1, c_2) = \frac{\eta\, \mathrm{e}^{-\delta\tau}(1 - \mathrm{e}^{-\delta T})^2}{T(2\delta^2 + \eta^2 - \mu^2) - 2\eta(1 - \mathrm{e}^{-\delta T})}. \tag{2.41}$$

Thus as the distance, τ, separating the two non-overlapping time intervals $(t_1, t_1 + T)$ and $(t_2, t_2 + T)$ increases, the correlation $\rho^c(c_1, c_2)$ reduces as $\mathrm{e}^{-\delta\tau}$, as also happens with the pairs autocorrelation function (2.21). However, whilst both correlation functions have an $\mathrm{e}^{-\delta\tau}$ term, the correlation in the triples process is always greater for $T > 0$, since $2\delta^2 + \eta^2 - \mu^2 < \delta^2 + \eta^2 + \eta\mu$.

3. Birth–death process

Although the p.g.f. (2.4) of Gillespie and Renshaw (2005) provides the general 'solution' to the (additive) batch immigration-death-counting process, we have shown that detailed analysis of specific cases (here double and triple immigration) yields far more transparent solutions that illuminate the saw-tooth behaviour of the underlying probability and moment structures. Development of specific results for quadruple, etc., immigration could well be rewarding, but let us now explore the effect of introducing birth, in order to determine whether the counting process can still provide a high level of information content on the development of an unobservable (multiplicative) population process.

The simplest and most transparent way of modelling this construct is to employ the simple birth–death (BD) process (Cox and Miller 1965, Bartlett 1966). For this is one of the few multiplicative continuous-time processes for which the probability structure can be expressed in closed form, and such mathematical tractability suggests that the parallel counting probabilities will not be too opaque to prevent the extraction of process structure. Moreover, on including immigration, the resulting birth-immigration-death (BDI) process not only provides an immediate extension to our earlier immigration-death counting model, but it is also directly associated with a kind of behaviour commonly encountered in optics (Jakeman *et al* 2003). For when the birth and immigration rates are equal, thermal or Bose–Einstein statistics are predicted for the single-fold population distribution, which is a simple geometric progression. This feature is found to characterize the photon statistics of a laser below threshold (Shimoda *et al* 1957) and also Gaussian speckle noise generated when laser light is scattered by particles or rough surfaces (Berlolotti 1974). In the case of the laser model, spontaneous and stimulated emission within a population of photons in a cavity is analogous to immigration and birth, respectively, and absorption is the analogue of death. Now a key point of Jakeman *et al* (2003) is that, in equilibrium, the MID process results in a negative binomial probability structure that closely resembles that of the BDI process. So their main objective is to devise and optimize a strategy for distinguishing between these two scenarios when measurement is restricted to the external monitoring scheme. Here, we shall concentrate on developing the full time-dependent probability structure for the simpler BD process; that for the BDI model involves straightforward, albeit more tedious, algebraic development, and in this sense adds little to our overall understanding.

For a population of size n at time t, in the small time interval $(t, t + h)$ there is a probability $\lambda nh + \mathrm{o}(h)$ that a particular individual gives birth, a probability $\mu nh + \mathrm{o}(h)$ that it dies unobserved, and a probability $\eta nh + \mathrm{o}(h)$ that it dies and is counted. So the forward Kolmogorov equation for this population counting process is

$$\mathrm{d}p_{nc}/\mathrm{d}t = \lambda(n-1)p_{n-1,c} + \mu(n+1)p_{n+1,c} + \eta(n+1)p_{n+1,c-1} - n(\lambda + \mu + \eta)p_{nc}, \tag{3.1}$$

where $t \geqslant 0$ and $p_{n,-c}(t) = 0$ for $c = 1, 2, \ldots$. This can be solved in terms of the joint generating function (2.2) to give

$$R(z, s; t) = \left(\frac{\theta_2 z\,\mathrm{e}^{\lambda(\theta_1-\theta_2)t} - \theta_1\theta_2\,\mathrm{e}^{\lambda(\theta_1-\theta_2)t} - \theta_1 z + \theta_1\theta_2}{z\,\mathrm{e}^{\lambda(\theta_1-\theta_2)t} - \theta_1\,\mathrm{e}^{\lambda(\theta_1-\theta_2)t} - z + \theta_2}\right)^{n_0}, \tag{3.2}$$

where

$$\theta_1, \theta_2 = \left[\lambda + \eta + \mu \mp \sqrt{(\lambda + \eta + \mu)^2 - 4\lambda(\mu + \eta s)}\right]\big/(2\lambda). \tag{3.3}$$

Note that on setting $s = 1$ in (3.2) we recover the well-known generating function for the simple birth–death population process.

3.1. Counting mean and variance

On forming $[\partial R(1, s; t)/\partial s]_{s=1}$ it immediately follows that the counting mean

$$\kappa_1^c(t) = \begin{cases} \eta n_0(\mathrm{e}^{(\lambda-\mu-\eta)t} - 1)/(\lambda - \mu - \eta) & \text{for} \quad \lambda \neq \mu + \eta \\ \eta n_0 t & \text{for} \quad \lambda = \mu + \eta. \end{cases} \tag{3.4}$$

Whence allowing $t \to \infty$ yields

$$\lim_{t\to\infty} \kappa_1^c(t) = \begin{cases} n_0\eta/(\mu + \eta - \lambda) & \text{for} \quad \lambda < \mu + \eta \\ \infty & \text{for} \quad \lambda \geqslant \mu + \eta. \end{cases} \tag{3.5}$$

So when the population explodes, i.e. $\lambda \geqslant \mu + \eta$, the counts explode. Conversely, when the population is certain to become extinct, i.e. $\lambda < \mu + \eta$, the counting mean approaches the limit $n_0\eta/(\mu + \eta - \lambda)$. Similarly, forming $[\partial^2 R(1, s; t)/\partial s^2]_{s=1}$ leads to the counting variance

$$\kappa_2^c(t) = n_0\eta t[3(1 - t\eta) + 2t^2\lambda\eta]/3$$

for $\lambda = \eta + \mu$, and

$$\begin{aligned} \kappa_2^c(t) = n_0\eta\{&\mathrm{e}^{(\lambda-\eta-\mu)t}[(\lambda - \mu)^2 - \eta^2 - 4\lambda\eta t(\lambda - \eta - \mu)] \\ &+ \mathrm{e}^{2(\lambda-\eta-\mu)t}\eta(\lambda + \eta + \mu) - (\lambda - \mu)^2 - \eta(\lambda + \mu)\}/(\lambda - \eta - \mu)^3 \end{aligned} \tag{3.6}$$

for $\lambda \neq \eta + \mu$. Whence for large $t = T$, we see that

$$\begin{aligned} &\kappa_2^c(T) \simeq n_0\eta^2\,\mathrm{e}^{2(\lambda-\eta-\mu)T}(\lambda + \eta + \mu)/(\lambda - \eta - \mu)^3 && \text{for} \quad \lambda > \eta + \mu \\ &\kappa_2^c(T) \simeq 2n_0T^3\lambda\eta^2/3 && \text{for} \quad \lambda = \eta + \mu \\ &\kappa_2^c(T) = n_0\eta[(\lambda - \mu)^2 + \eta(\lambda + \mu)]/(-\lambda + \eta + \mu)^3 && \text{for} \quad \lambda < \eta + \mu. \end{aligned} \tag{3.7}$$

Thus unlike the population variance which tends to zero as $t \to \infty$ for $\lambda < \mu + \eta$, the counting variance reaches a finite limit. For ultimate extinction is certain to occur, at which point there are no longer any individuals available for counting.

3.2. Probabilities

To derive the (marginal) counting probability $p_c^c(t)$ we need to extract the coefficient of s^c from the generating function $Q(1, s; t)$, and to effect this we first need to construct a general framework. On noting that the modified Bessel function of the first kind is defined (A&S section 9.6.10) as

$$I_x(\xi) = \left(\frac{\xi}{2}\right)^x \sum_{k=0}^{\infty} \frac{(\xi/2)^{2k}}{k!\Gamma(x + k + 1)},$$

we can write down two particularly useful forms, namely

$$\begin{aligned} I_{x-1/2}(\xi) &= \frac{2^{x+1/2}}{\xi^{x+1/2}}\frac{x!}{\sqrt{\pi}}\sum_{k=x}^{\infty}\binom{k}{x}\frac{\xi^{2k}}{(2k)!} \\ I_{1/2-x}(\xi) &= \frac{2^{x+1/2}}{\xi^{x+1/2}}\frac{x!}{\sqrt{\pi}}\sum_{k=0}^{\infty}\binom{k+1/2}{x}\frac{\xi^{2k+1}}{(2k+1)!}. \end{aligned} \tag{3.8}$$

It then follows that

$$\begin{aligned} \mathrm{e}^{\xi_2\sqrt{1+s/\xi_1}} &= \sum_{n=0}^{\infty}\frac{\xi_2^n(1+s/\xi_1)^{n/2}}{n!} = \sum_{n=0}^{\infty}\frac{\xi_2^n}{n!}\sum_{i=0}^{\infty}\binom{n/2}{i}\frac{s^i}{\xi_1^i} \\ &= \sum_{i=0}^{\infty}\frac{s^i}{\xi_1^i}\left(\sum_{n=i}^{\infty}\binom{n}{i}\frac{\xi_2^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\binom{n+1/2}{i}\frac{\xi_2^{2n+1}}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty}\zeta_n(\xi_2)s^n \end{aligned} \tag{3.9}$$

where

$$\zeta_n(\xi_2) = \frac{2^{-1/2-n}\sqrt{\pi}\xi_2^{1/2+n}[I_{n-1/2}(\xi_2)+I_{1/2-n}(\xi_2)]}{\xi_1^n n!}. \tag{3.10}$$

Moreover, a similar expansion leads to

$$[(1+\mathrm{e}^{\xi_2\sqrt{1+s/\xi_1}})(1+\xi_3\sqrt{1+s/\xi_1})-2]^{-1} = \sum_{c=0}^{\infty}\varphi_c s^c, \tag{3.11}$$

where

$$\varphi_c = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\left\{\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n}\binom{n}{j}\xi_3^j\sum_{k=0}^{c}\binom{j/2}{c-k}\frac{\zeta_k(i\xi_2)}{\xi_1^{c-k}}\right\}. \tag{3.12}$$

Whence on observing that the denominator of the p.g.f. $Q(1,s;t)$ takes the form

$$\hat{Q}(1,s;t) = \frac{\xi_4}{(1+\mathrm{e}^{\xi_2\sqrt{1+s/\xi_1}})(1+\xi_3\sqrt{1+s/\xi_1})-2}, \tag{3.13}$$

where $\xi_1 = [4\lambda\mu-(\lambda+\eta+\mu)^2]/[4\lambda\eta]$, $\xi_2 = -t\sqrt{(\lambda+\eta+\mu)^2-4\lambda\mu}$, $\xi_3 = -\xi_2\xi_4/(2\lambda t)$ and $\xi_4 = 2\lambda/(\lambda-\mu-\eta)$, it follows that the general coefficient of s^c in the denominator of $Q(1,s;t)$ is $\xi_4\varphi_c$ where φ_c is given by (3.12). Finally, on defining $\xi_5 = \sqrt{(\lambda+\eta+\mu)^2-4\lambda\mu}$, the numerator of $Q(1,s;t)$ can be expressed as

$$\begin{aligned} &\theta_2\,\mathrm{e}^{\lambda(\theta_1-\theta_2)t} - \theta_1\theta_2\,\mathrm{e}^{\lambda(\theta_1-\theta_2)t} - \theta_1 + \theta_1\theta_2 \equiv \sum_{c=0}^{\infty}\vartheta_c s^c \\ &\quad = \frac{1}{2\lambda}[\mathrm{e}^{\xi_2}(\lambda+\eta+\xi_5-\mu)+\xi_5+\mu-\lambda-\eta]s^0 \\ &\qquad + \frac{\eta}{\lambda\xi_5}(\mathrm{e}^{\xi_2}\lambda t(\lambda+\eta+\xi_5-\mu)-\lambda(1+\mathrm{e}^{\xi_2})+\xi_5(1-\mathrm{e}^{\xi_2}))s^1 \\ &\qquad + \sum_{n=2}^{\infty}\left\{\zeta_n(\xi_2)(\lambda+\mu+\eta)(1-2\mu)-\zeta_{n-1}(\xi_2)2\eta\right. \\ &\qquad \left. -\xi_5\binom{1/2}{n}\frac{1}{\xi_1^n}+\sum_{i=0}^{n}\zeta_i(\xi_2)\binom{1/2}{n-i}\frac{\xi_5}{\xi_1^{n-i}}\right\}\frac{s^n}{2\lambda}. \end{aligned} \tag{3.14}$$

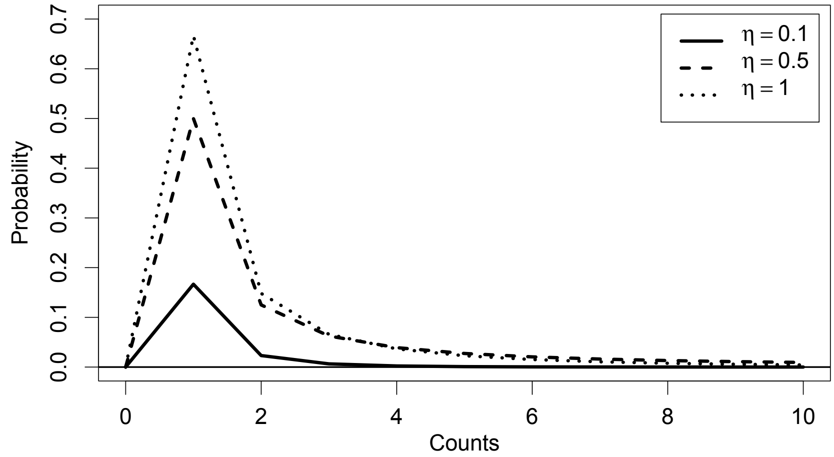

Figure 6. The counting p.d.f. corresponding to extinction for $\lambda = 0.5, \mu = 0, n_0 = 1$ and $\eta = 0.1, 0.5$ and 1.

On matching coefficients of s^i and s^{c+i} in the denominator and numerator, respectively, it then follows that the probability of observing c counts by time t is given by

$$p_c(t) = \xi_4 \sum_{i=0}^{c} \varphi_i \vartheta_{c-i}. \tag{3.15}$$

Unlike the earlier immigration-death scenario, not only is extinction now a real possibility, but it is certain to occur if $\lambda \leqslant \mu + \eta$. Now

$$\sum_{c=0}^{\infty} p_c^c(\infty) = \begin{cases} 1 & \text{for} \quad \lambda \leqslant \mu + \eta \\ (\eta + \mu)/\lambda & \text{for} \quad \lambda > \mu + \eta. \end{cases} \tag{3.16}$$

Since if extinction does occur then the counts remain fixed thereafter, otherwise they increase indefinitely. So the probabilities obtained by letting $t \to \infty$ in $Q(1, s; t)$ (for $n_0 = 1$), namely

$$\begin{aligned} p_0^c(\infty) &= \frac{\lambda + \eta - \mu - \sqrt{(\lambda + \eta + \mu)^2 - 4\lambda\mu}}{\lambda - \eta - \mu - \sqrt{(\lambda + \eta + \mu)^2 - 4\lambda\mu}} \\ p_c^c(\infty) &= \left(\frac{-\sqrt{(\lambda + \eta + \mu)^2 - 4\lambda\mu}}{2\lambda}\right) \binom{1/2}{c} \left(\frac{-4\lambda\eta}{(\lambda + \eta + \mu)^2 - 4\lambda\mu}\right)^c \end{aligned} \tag{3.17}$$

for $c = 1, 2, \ldots$ and $\lambda \neq \mu + \eta$, relate only to situations involving extinction.

When $\mu = 0$ the probabilities (3.17) reduce to $p_0^c(\infty) = 0$ with

$$p_c^c(\infty) = -\frac{\lambda + \eta}{2\lambda} \binom{1/2}{c} \left(-\frac{4\lambda\eta}{(\lambda + \eta)^2}\right)^c \qquad (c > 0). \tag{3.18}$$

Figure 6 shows this counting distribution for $n_0 = 1, \lambda = 0.5, \mu = 0$ and $\eta = 0.1, 0.5, 1.0$, and we see that although ultimate extinction is only certain in the first two cases, the three distributions are similar, sharing a large peak at $c = 1$, corresponding to the first event being a death.

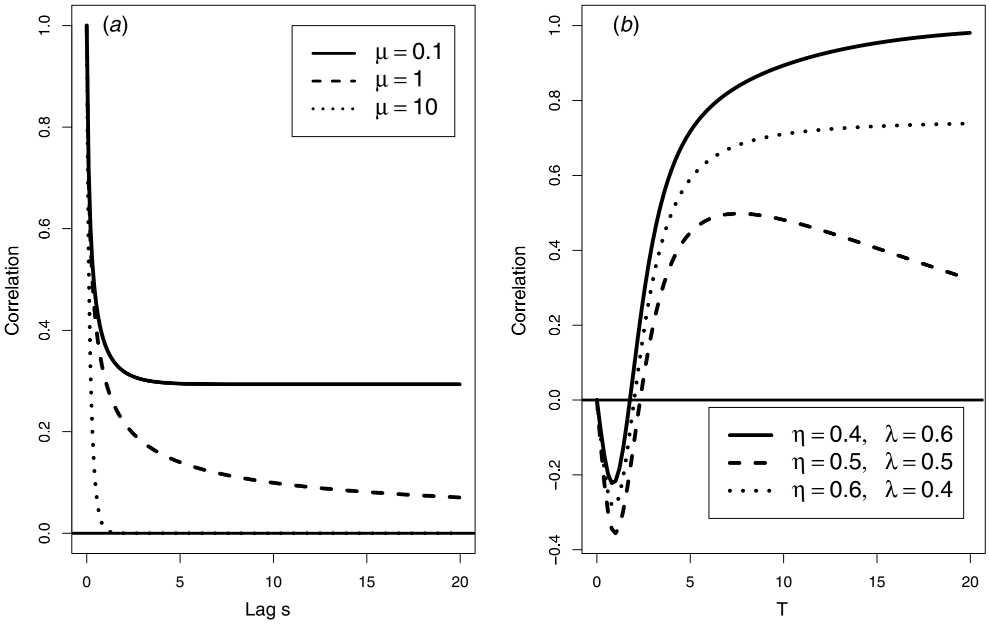

Figure 7. (*a*) Simple birth–death autocorrelation function at $t = 0.1$ for $\lambda = 1$ and $\mu = 0.1, 1$ and 10. (*b*) Simple birth–death counting autocorrelation function with $t_1 = 0, t_2 = T, \mu = 0$ and $n_0 = 1$.

3.3. The autocorrelation function

The population autocorrelation function takes a considerably simpler form than that for the multiple immigration-death process, namely

$$\rho(s) = \frac{e^{(\lambda-\mu)s}(e^{(\lambda-\mu)t} - 1)}{\sqrt{e^{(\lambda-\mu)s}(e^{(\lambda-\mu)t} - 1)(e^{(\lambda-\mu)(t+s)} - 1)}}. \tag{3.19}$$

Figure 7 shows $\rho(s)$ at $t = 0.1$, i.e. in the initial highly transient stage of the process, for $\lambda = 1$ and $\mu = 0.1$, 1 and 10. When $\lambda \leqslant \mu$ we see that $\rho(s) \to 0$, since for large s extinction is certain which contrasts with the initial positive population size. Whilst for $\mu < \lambda$ it follows that $\rho(s)$ reaches a positive limit, with the counting rate for $n > 0$ increasing exponentially fast. Indeed, on letting $s \to \infty$ in expression (3.19) we see that

$$\lim_{s\to\infty} \rho(s) = \begin{cases} \sqrt{1 - e^{(\mu-\lambda)t}} & \text{for} \quad \lambda > \mu \\ 0 & \text{for} \quad \lambda \leqslant \mu. \end{cases} \tag{3.20}$$

It is clear from (3.20) that when $\lambda = \mu$, $\rho(s) \to 0$ as $s \to \infty$. However, for $t \to \infty$ expression (3.19) yields

$$\lim_{t\to\infty} \rho(s) = \begin{cases} 1 & \text{for} \quad \lambda > \mu \\ \sqrt{e^{(\lambda-\mu)s}} & \text{for} \quad \lambda \leqslant \mu. \end{cases} \tag{3.21}$$

So $\lim_{t\to\infty} \rho(s) = 1$ when $\lambda \geqslant \mu$. Evaluating the corresponding counting autocorrelation function yields, for $\lambda \neq \mu + \eta$ and $\zeta_1 = \lambda - \mu - \eta$,

$$\rho^c[c_1; (t_1, t_1 + T), c_2; (t_2, t_2 + T)] = \frac{n_0\eta^2(1 - e^{\zeta_1 T})\, e^{\zeta_1 t_2}[2\lambda T\zeta_1 + (\lambda + \eta + \mu)\, e^{\zeta_1 t_1}(1 - e^{\zeta_1 T})]}{(\lambda - \mu - \eta)^3\sqrt{\mathrm{Var}[(t_1, t_1 + T)]\,\mathrm{Var}[(t_2, t_2 + T)]}} \tag{3.22}$$

where for $i = 1, 2$,

$$\mathrm{Var}[(t_i, t_i + T)] = \frac{n_0\eta\, \mathrm{e}^{\zeta_1 t_i}[\eta\, \mathrm{e}^{\zeta_1 t_i}(\lambda + \eta + \mu)(1 - \mathrm{e}^{\zeta_1 T})^2 - \mathrm{e}^{\zeta_1 T} 4\eta\lambda\zeta_1 T]}{(\lambda - \eta - \mu)^3} - \frac{n_0\eta\, \mathrm{e}^{\zeta_1 t_i}(\zeta_1^2 + 4\eta\lambda)(1 - \mathrm{e}^{\zeta_1 T})}{(\lambda - \eta - \mu)^3}. \tag{3.23}$$

Figure 7(b) shows (3.22) for $t_1 = 0, t_2 = T, n_0 = 1$ and $\mu = 0$. The most surprising feature of this a.c.f. is the presence of negative correlation for $0 \leqslant T \leqslant 2$, which contrasts with the population a.c.f. where correlation is always non-negative. The explanation for this is that if a count is observed in the time period $(0, T)$ for small T, then this is likely to have resulted in extinction. Hence, no counts will be observed in the subsequent time interval $(t_2, t_2 + T)$. Note that, as before, the correlation is one or zero for λ larger or smaller than μ.

On setting $t_2 = t_1 + T + \tau$ in (3.22), for $\tau > 0$, and letting $T \to \infty$, it follows that $\rho[c_1; (t_1, t_1 + T), c_2; (t_2, t_2 + T)] \to 1$ for $\lambda > \mu + \eta$, $3/4$ for $\lambda = \mu + \eta$ and 0 for $\lambda < \mu + \eta$. So we obtain the intuitively reasonable result that when λ is greater or less than μ, the counting correlation is either one or zero. A surprising feature, however, occurs when $\lambda = \mu$, for this results in a correlation of $3/4$. This anomaly occurs because although extinction is certain, the event itself can take an infinitely long time to happen (see figure 7(b)).

4. Summary and discussion

The construction of models is the most basic activity in science, lying at the heart of understanding, prediction and practical applications, and uncovering the dominant mechanisms governing the stochastic evolution of dynamical systems in an essential component of this activity (Jakeman *et al* 2003). Now in many situations of practical interest, it will not be possible to make a direct measurement of the population itself, but only of some external manifestation of its evolution. Here we assume that it is only possible to monitor the rate of evolution of individuals from the population. So in quantum optics we might count photon emissions, in biological populations emigrants from the system, and in epidemiology people either dying, being isolated, or exhibiting specific symptoms.

The time-dependent paired-immigration-death process, which has the merit of being both easy to formulate and is analytically tractable, produces some neat, and highly illuminating features, especially the strikingly clear odd–even population effects in both the probability and moment structures. That these features are similarly present in the counting probabilities and moments is witness to the strength of the information in the counting process in allowing us to make inferences on the (unseen) population process. We see that these effects are, nevertheless, highly susceptible to the presence of single immigrants, only a small infusion of which are necessary to mask the odd–even phenomenon. A key question relates to the correlation between counts in non-overlapping time intervals, and whilst the algebraic development is non-trivial it is perfectly possible to develop relations that are sufficiently transparent to expose the underlying correlation features. Extending single- and paired-immigration to k-immigration produces intriguing mathematical results; here we just consider $k = 3$ in order to demonstrate that both the approach and the associated conclusions carry over neatly to the multiple-immigration scenario.

Even under the generalized k-immigration scheme, once individuals enter the population 'black-box' they cannot produce yet more individuals, so there is a direct balance between them and those who are subsequently either counted or die. However, once we introduce birth the situation changes dramatically. For individuals can be created within the black-box that are totally unseen from outside, and so the extent to which the population statistics can be gleaned from the counting statistics is far less intuitive. Indeed, being multiplicative by nature, this process has the potential for offering a much greater theoretical challenge. Although the counting probability structure is algebraically more awkward to derive, in essence it nevertheless just involves a basic series expansion, which is itself sufficiently tractable to allow basic inferences to be drawn without further calculation. However, unlike the immigration processes, we now have to distinguish between the cases $\lambda \leqslant \mu+\eta$, for which ultimate extinction is certain, and $\lambda > \mu + \eta$, in which case both extinction and unlimited exponential population/count growth are possible. Intriguingly, the correlation coefficient (3.22) exhibits an anomalous departure from the values 0 ($\lambda < \mu + \eta$) and 1 ($\lambda > \mu + \eta$) at $\lambda = \mu + \eta$, since in this case it equals 3/4.

The approach clearly offers considerable future potential for expanding into more complex situations. Not only is the analysis of multi-type processes, such as predator–prey or general epidemic models, a potentially huge field covering applied situations of immediate practical and intrinsic use, but the associated inferential problems of determining an unknown stochastic process from known counts offers rich theoretical rewards. Overall the problem of determining what information is contained within the counting statistics of a stochastic process is complex. Future research should concentrate on determining methods and techniques that would enable us to state explicitly the information that the observed process could tell us about the underlying latent system. Current methods of analysis are based almost exclusively on numerical techniques centred on Markov chain Monte Carlo procedures (see, for example, Boys *et al* 2008 and Golightly and Wilkinson 2008). So the development of parallel theoretical results would be both novel and timely.

Acknowledgments

We wish to thank Professor Eric Jakeman, Dr Keith Hopcraft and Jonathan Matthews for valuable discussion. The work was supported by an EPSRC Research Studentship. We also thank the anonymous referees for their insightful comments that helped to strengthen this paper.

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