Grad, Div and Curl in Cylindrical and Spherical Coordinates

In applications, we often use coordinates other than Cartesian coordinates. It is important to remember that expressions for the differential operators of vector calculus are DIFFERENT in different coordinates. Here we give explicit formulae for cylindrical and spherical coordinates.

Cylindrical Polar Coordinates (ρ, ϕ, z)

$$x = \rho\cos\phi\,, \qquad y = \rho\sin\phi\,, \qquad z = z\,, \qquad \mathbf{r} = (x, y, z) = (\rho, 0, z)\,,$$

$$\nabla f = \hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \widehat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}\,,$$

$$\nabla\cdot\mathbf{u} = \frac{1}{\rho}\frac{\partial(\rho u_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z}\,,$$

$$\nabla\times\mathbf{u} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\widehat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ u_\rho & \rho u_\phi & u_z \end{vmatrix}$$

$$= \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right) + \widehat{\boldsymbol{\phi}}\left(\frac{\partial u_\rho}{\partial z} - \frac{\partial u_z}{\partial \rho}\right) + \hat{\mathbf{z}}\frac{1}{\rho}\left[\frac{\partial(\rho u_\phi)}{\partial \rho} - \frac{\partial u_\rho}{\partial \phi}\right].$$

NB! The cylindrical radius $\rho = \sqrt{x^2+y^2}$ is often denoted s or r; in the latter case, it can be confused with the length of the position vector, $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. Therefore, the notation r for the cylindrical radius, if used, is ALWAYS explained explicitly.

Spherical Polar Coordinates (r, θ, ϕ)

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta\ , \qquad \mathbf{r} = (x, y, z) = (r, 0, 0)\ ,$$

$$\nabla f = \hat{\mathbf{r}}\,\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\,\frac{\partial f}{\partial \theta} + \widehat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \phi}\ ,$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\,\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\,\frac{\partial u_\phi}{\partial\phi}\ ,$$

$$\nabla\times\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \widehat{\boldsymbol{\phi}}\,r\sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ u_r & r u_\theta & r u_\phi\sin\theta \end{vmatrix}$$

$$= \hat{\mathbf{r}}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r u_\phi)\right] + \widehat{\boldsymbol{\phi}}\,\frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$