4.9 For each of the following curves find multiplicity and all tangents at $(0,0)$. (Use Corollary 4.21.)

(a) $y^2 - x^2 + x^4$.

(b) $xy(x^2 - y^2) - (x^2 + y^2) = 0$.

(c) $(x^4 + y^4)^2 - x^2y^2 = 0$.

(d) $(x^4 + y^4 - x^2 - y^2)^2 - 9x^2y^2 = 0$.

4.10 Let C_f be a curve of degree d and let (x_0, y_0) be a point of C_f. Define a polynomial g by

$$g(x,y) = f(x + x_0, y + y_0).$$

(a) Show that g has degree d and that $(0,0)$ is a point of C_g.

(b) Show that

$$f_{x^iy^j}(u + x_0, v + x_0) = g_{x^iy^j}(u,v),$$

for all $(u,v) \in \mathbb{A}_2(k)$.

(c) Show that (x_0, y_0) is a point of C_f of multiplicity r if and only if $(0,0)$ is is a point of C_g of multiplicity r.

(d) Show that (x_0, y_0) is a point of C_f of multiplicity r if and only if the lowest degree terms of g are of degree r.

4.11 Find the multiplicity of the given points of the following curves. (Use the previous question.)

(a) $(1,2)$ on the curve

$$x^2y^2 - 4x^2y + x^2 - 2xy^2 + 9xy - 4x + y^2 - 5y + 3 = 0;$$

(b) $(3,0)$ on the curve

$$yx^4 - 12yx^3 + 54x^2y - 108xy + 81y - 5x^3 + 45x^2 - 135x + 135 + 2xy^2 - 6y^2 + y^4 = 0.$$

5 Projective space and Projective curves

5.1 Let (a,b,c) be a non–zero vector in k^3. Define a map $\theta : k^3 \longrightarrow k$ by $\theta(x,y,z) = ax + by + cz$.

(a) Show that θ is a linear transformation from the vector space k^3 to the vector space k.

(b) Write down the formula relating the dimensions of the domain, image and kernel of a linear transformation of vector spaces. Use this to show that $\dim_k(\ker(\theta)) = 2$.

(c) Show that the projective line l with equation $ax + by + xz = 0$ is the set of points
$$\{(u : v : w) \in \mathbb{P}_2(k) : (u, v, w) \in \ker(\theta)\}.$$

(d) Given a 2–dimensional vector subspace of V of k^3 show that there is a unique projective line l such that $(a : b : c) \in l$ if and only if $(a, b, c) \in V$.

(e) Conclude from the above that there is a one to one correspondence between projective lines and 2–dimensional subspaces of k^3.

(f) Use the previous part of the question to show that two distinct points of $\mathbb{P}_2(k)$ lie on a *unique* projective line.

5.2 (a) Find the equation and parametric form of the lines through

i. $(1 : 1 : 2)$ and $(3 : 2 : 1)$;

ii. $(1 : 7 : -1)$ and $(3 : -1 : 5)$.

(b) Dehomogenize the equations found in the previous part of the question (with respect to $z = 1$) to find equations of corresponding affine lines.

(c) Find the cartesian coordinates of the points of the affine plane corresponding to the homogeneous coordinates of the 4 points of part 5.2a. Check that the affine lines you found in the previous part of the question pass through these affine points.

5.3 Find the intersection $l_1 \cap l_2$ of each of the following pairs of lines in $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{C})$.

(a) $l_1 = \{(x : y : z) \in \mathbb{P}_2 | x = 0\}$, $l_2 = \{(x : y : z) \in \mathbb{P}_2 | z = 0\}$,

(b) $l_1 = \{(x : y : z) \in \mathbb{P}_2 | x + y + z = 0\}$, $l_2 = \{(x : y : z) \in \mathbb{P}_2 | x + y + 3z = 0\}$,

(c) $l_1 = \{(x : y : z) \in \mathbb{P}_2 | 6x + 4y + 2z = 0\}$, $l_2 = \{(x : y : z) \in \mathbb{P}_2 | 3x + 2y + z = 0\}$.

(d) $l_1 = \{(x : y : z) \in \mathbb{P}_2 | x + 7y - z = 0\}$, $l_2 = \{(x : y : z) \in \mathbb{P}_2 | 3x - y + 5z = 0\}$.

If you find that $l_1 \cap l_2$ has more than one point explain why.

5.4 (a) Let l_1 and l_2 be distinct projective lines with equations $ax + by + cz = 0$ and $ux+vy+wz = 0$, respectively. Show that l_1 and l_2 intersect at $(\alpha : \beta : \gamma)$ where
$$\alpha = \begin{vmatrix} b & c \\ v & w \end{vmatrix}, \beta = - \begin{vmatrix} a & c \\ u & w \end{vmatrix} \text{ and } \gamma = \begin{vmatrix} a & b \\ u & v \end{vmatrix}.$$

Verify this formula for each of part of Question 5.3

(b) If V and W are subspaces of a vector space U then we define $V+W = \{a\mathbf{v}+b\mathbf{w} : a, b \in k, \mathbf{v} \in V, \mathbf{w} \in W\}$. The following formula then follows for the dimensions of $V + W$ and $V \cap W$:
$$\dim_k(V + W) = \dim_k(V) + \dim_k(W) - \dim_k(V \cap W).$$

(See notes for MAS261.) Show that if V and W are distinct 2–dimensional subspaces of k^3 then $\dim_k(V \cap W) = 1$. Use this to show that distinct lines l_1 and l_2 in $\mathbb{P}_2(k)$ intersect in exactly one point.

5.5 Find the homogenization of each of the following polynomials. In each case find the points of intersection of the corresponding projective curve and the line $z = 0$. Use maple to plot each of the real affine curves using the given parametrisation. (More than one plot may be needed for a particular curve, in order to avoid points where the denominator of the parametrisation vanishes.) Print out the plots and the commands used to generate them and describe briefly how the drawings relate to the intersections with $z = 0$ that you found.

(a) $x^4 + y^4 - x^2y$ (setting $y = tx$ gives a parametrisation

$$x = \frac{t}{1+t^4},\ y = \frac{t^2}{1+t^4},$$

for this curve).

(b) $x^4 - y^4 - x^2y$ (setting $y = tx$ gives a parametrisation

$$x = \frac{t}{1-t^4},\ y = \frac{t^2}{1-t^4},$$

for this curve).

(c) $x^3 + y^3 + x^2 - 2y^2$ (setting $y = tx$ gives a parametrisation

$$x = \frac{2t^2-1}{1+t^3},\ y = \frac{t(2t^2-1)}{1+t^3},$$

for this curve).

(d) $x^4 + y^4 + x^3 + y^3$ (setting $y = tx$ gives a parametrisation

$$x = \frac{-(1+t^3)}{1+t^4},\ y = \frac{-t(1+t^3)}{1+t^4},$$

for this curve).

5.6 Prove that a finite union of projective curves is a projective curve.

5.7 (a) Given a linear transformation $\theta : k^3 \longrightarrow k^3$ show that there is a well–defined map $\phi : \mathbb{P}_2(k) \longrightarrow \mathbb{P}_2(k)$ given by $\phi(a : b : c) = (u : v : w)$, where $(u, v, w) = \theta(a, b, c)$. (You have to show that if $(a : b : c) = (a' : b' : c')$ then $\phi(a : b : c) = \phi(a' : b' : c')$.) Show further that if θ is invertible then so is ϕ. A map such as ϕ, which is induced by an invertible linear transformation, is called a **projective transformation**.

(b) Given that we have fixed a basis of k^3, an invertible linear transformation of k^3 to k^3 corresponds to an invertible 3×3 matrix with entries in k. Let $\mathbf{A} = (a_{ij})$ be a matrix corresponding to the invertible linear transformation θ: that is

$$\mathbf{A}\mathbf{x} = \theta(\mathbf{x}),$$

where elements of k^3 are written in column form and we use the canonical basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ of k^3, ($\mathbf{e}_i$ has 1 in the ith position and 0's elsewhere). Let $A^{-1} = (b_{ij})$ and let F be a homogeneous polynomial of degree $d > 0$. Show that the polynomial G given by

$$G(x, y, z) = F(b_{11}x + b_{12}y + b_{13}z, b_{21}x + b_{22}y + b_{23}z, b_{31}x + b_{32}y + b_{33}z)$$

is homogeneous of degree d (in particular show that G is non–zero.)

(c) Let ϕ be a projective transformation and let C be a projective curve of degree d. Show that $\phi(C)$ is a projective curve of degree d. Show further that C is irreducible if and only if $\phi(C)$ is irreducible. The curves C and D are said to be **projectively equivalent** if there is a projective transformation ϕ such that $\phi(C) = D$.

5.8 **Classification of Real Projective Conics**: in this question real projective conics are classified, up to projective equivalence. A matrix (a_{ij}) is **symmetric** if $a_{ij} = a_{ji}$, whenever $i \neq j$. A matrix N is **orthogonal** if $N^T N = I$. We shall say that a projective transformation is **orthogonal** if the linear transformation to which it corresponds is given by an orthogonal matrix (with respect to the canonical basis). Recall, from second year Linear Algebra, that if A is a symmetric matrix, with real entries, then there exists an orthogonal matrix N such that $N^{-1}AN = D$, where D is a diagonal matrix ($D = (d_{ij})$ with $d_{ij} = 0$ whenever $i \neq j$).

A curve of degree 2 is called a **conic**. Let C be a projective conic in $\mathbb{P}_2(\mathbb{R})$. Then C has equation

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0, \tag{5.1}$$

where a, b, c, d, e and f are real constants and not all zero.

(a) Show that there exists a symmetric matrix A such that equation (5.1) can be written as

$$(x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

(b) Deduce from 5.8a and the fact recalled from Linear Algebra above, that there exists an orthogonal projective transformation ϕ, such that the equation of $\phi(C)$ is

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0, \tag{5.2}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$ and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

(c) Using 5.8b (and a further projective transformation if necessary) show that the conic C with equation (5.2) is one of the following:

i. a circle (that is, has equation $x^2 + y^2 - z^2 = 0$);
ii. the empty set;
iii. a pair of distinct lines;
iv. a single point;
v. a single line (repeated).

(d) Which of the above correspond to the affine ellipse, parabola and hyperbola.

(e) How many complex projective conics do you think there are?

5.9 Find the multiplicity of each singular point of the following curves. (Use your answers to Question 4.1, noting that in this exercise you found all the required singularities except for those that may occur on the line $z = 0$.)

(a) $xz^2 - y^3 + xy^2 = 0$;
(b) $(x + y + z)^3 - 27xyz = 0$;
(c) $x^2y^2 + 36xz^3 + 24yz^3 + 108z^4 = 0$.
(d) Find equations for all the tangents to these projective curves at all their points of intersection with the line $z = 0$. Hence find equations for all asymptotes to the the complex affine curves of Question 4.1.

5.10 Find the condition on k for the cubic

$$x^3 + y^3 + z^3 = kxyz$$

to be non-singular.

5.11 Find equations for all the tangents to the curves below at points where they meet the line $z = 0$ (see Question 5.5). Hence find all asymptotes to the affine curves of the first 3 parts of Question 5.5.

Plot each curve and its real asymptotes on the same drawing, using maple (reuse the plots from Question 5.5 if you have them). Use "thickness=5" for the curves and "thickness=3" for the asymptotes (or some similar ratio). Note that a line with parametric form $(3s + 2, 5s - 4)$ can be plotted using

```
>plot([3*s+2,5*s+2,s=-10..10],x=-5..5,y=-5..5,axes=none,thickness=3);
```

for example. To display more than one plot at a time do something like the following

```
>l1:=plot([3*s+2,5*s+2,s=-10..10],x=-5..5,y=-5..5,axes=none,thickness=3):
>l2:=plot([s,-s-1,s=-10..10],x=-5..5,y=-5..5,axes=none,thickness=3):
>C:=implicitplot(x^2+y^2-3,x=-3..3,y=-3..3,numpoints=5000):
>display([l1,l2,C]);
```

Describe briefly how the drawings you've plotted relate to the asymptotes that you found.

(a) $x^4 + y^4 - x^2yz = 0$;
(b) $x^4 - y^4 - x^2yz = 0$;
(c) $x^3 + y^3 + x^2z - 2y^2z = 0$.

5.12 Let C be an irreducible projective curve of degree 4 in $\mathbb{P}_2(\mathbb{C})$ and let p be a triple point of C. Show that p is the only singular point of C and that every line through p meets C in exactly one further point.

5.13 Suppose that $F(x, y, z)$ is a homogeneous polynomial with dehomgenisation f.

(a) Prove that if all partial derivatives of F order r vanish at (α, β, γ), then so do its partial derivatives of order less than r.
(b) Prove that all partial dervivatives of f of order less than or equal to r vanish at (α, β) if and only if all partial derivatives of F of order equal to r vanish at $(\alpha, \beta, 1)$.
(c) Complete the proof of Theorem 5.21.

5.14 Show that if $p = (u : v : w)$ is a non–singular point of the projective curve C_F then the tangent to C_F at p is the line with equation

$$xF_x(u, v, w) + yF_y(u, v, w) + zF_x(u, v, w) = 0.$$

5.15 Use the maple package "algcurves" to find all singularities of the curve with polynomial $f = xy(x^2 - y^2) - (x^2 + y^2)$. You need to load the package with the command

```
>with(algcurves);
```

The output is described on the help pages. (We have not covered delta invariants, so ignore them.) Note that maple first homogenises the curve and then attempts to find all singularities of the projective homogenisation. Find multiplicity and tangents at each singular point. Find any asymptotes to the curve. Now use the "implicitplot" command to sketch this curve in $\mathbb{A}_2(\mathbb{R})$. Set the x and y ranges from -5 to 5 and "numpoints" = 5000. (Don't forget to load the plots package and remember that you can make the output look better by using something like

```
>setoptions(thickness=5,xtickmarks=3,ytickmarks=3,
axesfont=[TIMES,ROMAN,32],labelfont=[TIMES,ITALIC,32]);
```

to start with. Plot the curves and asymptotes on one diagram (one diagram for each curve). Make the curves thicker than the asymptotes using "thickness". Use "implicitplot" to plot the curves and "plot" for the asymptotes (see question 5.11). Print out your maple commands and the plot of the curve and mark any asymptotes on the printout.

5.16 Use the maple package "algcurves" to find all singularities of the following curves. (See the previous question.)

(a) $(x^4 + y^4)^2 - x^2y^2 = 0$.

(b) $(x^4 + y^4 - x^2 - y^2)^2 - 9x^2y^2 = 0$.

Now use "implicitplot" to sketch the real affine curves with these equations. Set the x and y ranges from -3 to 3 and "numpoints" = 5000 for the first curve and 10000 for the second. Implicitplot is not really up to this, as it cannot understand the curve near the origin. Use the solution to question 4.9 to make a better sketch by hand: assume that at the origin the curve looks like the union of its tangents at the origin.

6 Bézout's Theorem

6.1 **Pascal's mystic hexagon:** Take 6 points $P_1, \cdots, P_6$, on a curve C. Let $L_1, \cdots, L_6$ be the lines $L_1 = P_1P_2$, $L_2 = P_2P_3, \cdots, L_5 = P_5P_6$ and $L_6 = P_6P_1$. We call $L_1 \cup L_2 \cup \cdots \cup L_6$ a *hexagon inscribed in* C. The side L_1 is said to be *opposite* L_4, L_2 opposite L_5 and L_3 opposite L_6. Prove that "The pairs of opposite sides of a hexagon inscribed in an irreducible projective conic meet in three collinear points" as follows. Let the conic be C and the inscribed hexagon be $L_1 \cup L_2 \cup \cdots \cup L_6$ as above.

(a) Consider the curves $A = L_1 \cup L_3 \cup L_5$ and $B = L_2 \cup L_4 \cup L_6$. Show that A and B have degree 3.

(b) Show that A and B meet at the points $P_1, \cdots, P_6$ and the three points, Q_1, Q_2 and Q_3, of intersection of opposite sides of the hexagon.

(c) Use the result of Question 6.3 to show that Q_1, Q_2 and Q_3 are collinear, as required.

(d) Draw a sketch to illustrate this theorem in the case where C lies in the real plane.

6.2 Two pairs of opposite sides of a hexagon inscribed in a circle are parallel. Prove that the third pair is parallel. [**Hint:** Pascal's mystic ...]

6.3 Let k be an algebraically closed field and let C and D be projective curves of degree n in $\mathbb{P}_2(k)$ which intersect in exactly n^2 points. Assume that precisely mn of these points lie on an irreducible curve E of degree m, with $m < n$. Let f, g and h be the polynomials defining C, D and E respectively.

(a) Let $(a : b : c)$ be a point of E which is not in $C \cap D$. Let $\lambda = g(a, b, c)$ and $\mu = -f(a, b, c)$. Show that the polynomial $s = \lambda f + \mu g$ has degree n and that $C_s \cap E$ contains at least $nm + 1$ points (where C_s is the curve with equation $s = 0$).

(b) Show that E and C_s have a common component. Conclude that E is a component of C_s.

(c) Show that $s = ht$ for some polynomial t of degree at most $n - m$.

(d) Prove that the $n(n - m)$ points of $C \cap D$ which do not lie on E lie on a curve of degree at most $n - m$.

7 Inflexions

7.1 Find the inflexions of the curves with equations

(a) $x^3 - y^2 z - 2z^3 = 0$;

(b) $x^3 + y^3 + xz^2 = 0$;

(c) $zy^2 - x^3 + 2z^3 = 0$.

7.2 It can be shown that if C_f is an irreducible curve in $\mathbb{P}_2(k)$ such that $H_f(u : v : w) = 0$ at each point $(u : v : w)$ of C_f then C_f must be a line. This fact may be useful in answering the first half of this question. Let C be a non–singular curve of degree d in $\mathbb{P}_2(k)$, where k is algebraically closed.

(a) Show that if $d \geq 2$ then C has at most $3d(d - 2)$ inflexions.

(b) Show that if $d \geq 3$ then C has at least on inflexion.

7.3 Show that the complex curve C_F, where $F = x^3 + y^3 - xy(x + y + z)$, has one singular point and 3 inflexions. Show that the 3 inflexions all lie on one line.

7.4 Show that the complex curve C_F, where $F = z(x^2 + y^2) - (3y^2 - x^2)$, has a unique singularity at $(0 : 0 : 1)$. Show that the Hessian vanishes if and only if $z(x^2 + y^2) + 3(3y^2 - x^2) = 0$. Deduce that C_F has exactly 3 inflexions all of which lie on a line.

7.5 Find all 12 inflexions of the curve with polynomial $F = x^4 + y^4 + z^4$ in $\mathbb{P}_2(\mathbb{C})$.

8 Cubics and the group law

8.1 Show that the curve C of question 7.1a is non–singular. Let O be the point $(0 : 1 : 0)$ of this curve and consider the group law with O as identity. Find the points

(a) $(0 : \sqrt{2} : i) + (2 : \sqrt{6} : 1)$;

(b) $(0 : \sqrt{2} : i) + (0 : \sqrt{2} : -i)$;

(c) inverse of $(2 : \sqrt{6} : 1)$.

8.2 Let $\mathcal{C}$ be an irreducible cubic and let O be an inflexion of $\mathcal{C}$. The Group Law is defined on $\mathcal{C}$, taking O to be the identity element.

(a) Show that if A is a point of $\mathcal{C}$ then $-A$ is the third point of intersection of the line OA with $\mathcal{C}$.

(b) Show that if A, B, C are points of $\mathcal{C}$ then

$$A + B + C = O$$

if and only if A, B and C are collinear.

(c) In the terminology of group theory an element A of a group has *order* n if $nA = 0$ and $A, 2A, \ldots, (n-1)A$ are all non–identity elements. Show that if $P \neq O$ is a point of $\mathcal{C}$ then P has order 2 if and only if the tangent to $\mathcal{C}$ at P passes through O. Show that $P \neq O$ is of order 3 if and only if P is an inflexion.

8.3 Let

$$f = xy(x + y) + z^3$$

and let $\mathcal{C}$ be the complex curve with equation $f = 0$.

(a) Show that $\mathcal{C}$ is non–singular.

(b) Find all points of inflexion of $\mathcal{C}$.

(c) The Group Law is defined on $\mathcal{C}$, taking the identity element O to be the inflexion $(1 : -1 : 0)$. Let $\omega = (-1 + i\sqrt{-3})/2$, $A = (0 : 1 : 0)$ and $B = (\omega : 1 : 1)$. Find the points $A + B$, $-A$, $-B$, $A - B$, $B - A$ and $-A - B$ of $\mathcal{C}$ and verify that, together with O, these are precisely the inflexions of $\mathcal{C}$.

8.4 Let

$$f = x^3 + y^3 - z^3$$

and let $\mathcal{C}$ be the complex curve with equation $f = 0$.

(a) Show that $\mathcal{C}$ is non–singular and find all its points of inflexion.

(b) The Group Law is defined on $\mathcal{C}$, taking the identity element O to be the inflexion $(1 : 0 : 1)$. Let $A = (1 : -1 : 0)$ and $B = (0 : 1 : \omega^2)$, where $\omega = e^{i2\pi/3}$. Find the points $A + B$, $-A$, $-B$, $A - B$, $B - A$ and $-A - B$ of $\mathcal{C}$ and verify that, together with O, these are precisely the inflexions of $\mathcal{C}$.