NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

Mock Exam 2011/2012

MAS3219

Geometric Group Theory

Time allowed: 1 hour 30 minutes

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are THREE questions in Section A and TWO questions in Section B.

SECTION A

A1. Let f be the isometry of $\mathbb{R}^2$ given by

$$f(x,y) = (y, x + 4\sqrt{2}).$$

(a) Express f (with respect to the standard basis) in the form $(\mathbf{v}, M)$, where $\mathbf{v}$ is a vector and $M \in O_2(\mathbb{R})$.

(b) State whether f is direct or opposite. If it is direct, and a rotation, find its centre and its angle of rotation.

If it is opposite state whether it is a reflection or a glide reflection and find vectors $\mathbf{a}$ and $\mathbf{b}$ so that f can be expressed as $(2\mathbf{a}+\mathbf{b}, M)$, and has axis a line with slope orthogonal to $\mathbf{a}$ which passes through the point $\mathbf{a}$.

[20 marks]

A2. (a) Let G be a group generated by a subset X. Define the *Cayley graph* $\Gamma(G, X)$ of G with respect to X.

(b) Draw the Cayley graph $\Gamma(C_8, \{x\})$ of the cyclic group C_8 of order 8 with respect to a generating set $\{x\}$.

(c) Let D_∞ be the group given by the presentation $\langle x, y | y^2, (xy)^2 \rangle$. Every element of D_∞ can be written as $x^m y^\varepsilon$, where $m \in \mathbb{Z}$ and $\varepsilon = 0$ or 1. Also, $x^m y^\varepsilon = x^n y^\delta$, with $\varepsilon, \delta = 0$ or 1, in D_∞ if and only if $m = n$ and $\varepsilon = \delta$. Use this information to sketch part of the Cayley graph $\Gamma(D_\infty, \{x, y\})$ of D_∞, showing the elements x^m and $x^m y$ for $m = 0, \pm 1, \pm 2$ and all their incident edges.

[10 marks]

A3. (a) State the Nielsen-Schreier theorem for subgroups of a free group.

(b) Find a free basis for the subgroup H of the free group $F(a,b)$ generated by the set of elements $\{ab^2, a^2b, ba, aba\}$.

[**10 marks**]

SECTION B

B4. Let W be a wallpaper group, with lattice $L = \{\alpha\mathbf{a}+\beta\mathbf{b} : \alpha, \beta \in \mathbb{Z}\}$ and point group $O = \{1, \rho, \rho^2\}$, where $\rho = A_{2\pi/3}$.

(a) Show, by reference to standard presentations in the notes, that L has a presentation $\langle \tau_0, \tau_1 | [\tau_0, \tau_1] \rangle$ (where $[\tau_0, \tau_1] = \tau_0^{-1}\tau_1^{-1}\tau_0\tau_1$).

Show that O has a presentation $\left\langle \rho | \rho^3 \right\rangle$.

(**5 marks**)

(b) Let K and H be groups with presentations $\langle X|R\rangle$ and $\langle Y|S\rangle$, respectively, and let Φ be a homomorphism from H to $\mathrm{Aut}(K)$.

State a result which describes a presentation of $K \rtimes_\Phi H$ in terms of X, Y, R, S and certain other relators.

(**4 marks**)

(c) Assume that $W \cong L \rtimes_\Phi O$, where Φ is the map sending ρ to the automorphism Φ_ρ of L such that $\Phi_\rho(\tau_0) = \tau_0^{-1}\tau_1$ and $\Phi_\rho(\tau_1) = \tau_0^{-1}$. Use this information to find a presentation for W with generating set $X = \{\rho, \tau_0, \tau_1\}$: quoting results from the notes, or other parts of this question, at appropriate points.

(**10 marks**)

(d) Show that $\rho\tau_0 = \tau_0^{-1}\tau_1\rho$ and $\rho\tau_1 = \tau_0^{-1}\rho$.

(e) Show that every element of W may be written in the form $\tau_0^{\alpha_1}\tau_1^{\alpha_2}\rho^\beta$, where $\alpha_i \in \mathbb{Z}$ and $\beta = 0, 1$, or 2.

(**7 marks**)

(f) Does the word

$$u = \rho\tau_0\rho^2\tau_0^{-1}\tau_1^3$$

represent the identity element of W? Justify your answer.

(**4 marks**)

Question B4 continued on next page

[30 marks]

B5. Let G and H be finite groups of orders m and n, respectively, where $\gcd(m,n) = 1$. Consider the direct product $G \times H$ of G and H. This question goes through a proof that $\mathrm{Aut}(G\times H)$ is isomorphic to $\mathrm{Aut}(G)\times \mathrm{Aut}(H)$.

(a) In this part of the question (and others if necessary) you may use the following fact. If $g \in G$ and $h \in H$ such that $g \neq 1_G$ and $h \neq 1_H$ then $|(g, 1_H)| \neq |(g, h)| \neq |(1_G, h)|$ and $|(g, 1_H)| \neq |(1_H, h)|$.

Show that if $\alpha \in \mathrm{Aut}(G \times H)$ then $\alpha(G \times \{1_H\}) \subseteq G \times \{1_H\}$.

Explain why this means that $\alpha(G \times \{1_H\}) = G \times \{1_H\}$.

Use the symmetry of the situation to deduce that $\alpha(\{1_G\} \times H) = (\{1_G\} \times H$.

(5 marks)

(b) Let $\pi_G : G \times H \longrightarrow G$ be the map given by $\pi_G(g, h) = g$ and similarly let $\pi_H : G \times H \longrightarrow G$ be given by $\pi_H(g, h) = h$. You may use the fact that π_G and π_H are surjective homomorphisms, without justification.

Let $\iota_G : G \longrightarrow G \times H$ be the map given by $\iota_G(g) = (g, 1_H)$ and let $\iota_H : H \longrightarrow G \times H$ be the map given by $\iota_H(h) = (1_G, h)$, for all $g \in G$ and $h \in H$. You may use the fact that ι_G and ι_H are injective homomorphisms, without justification.

Let α be an automorphism of $\mathrm{Aut}(G\times H)$. Let α_G be the composite map $\pi_G \circ \alpha \circ \iota_G$. Show that α_G is an automorphism of G. [Hint. A composition of homomorphisms is a homomorphism.]

Similarly let $\alpha_H = \pi_H \circ \alpha \circ \iota_H$. Use the symmetry of the situation to explain briefly why α_H is also an automorphism, of H.

(5 marks)

(c) Show that $\alpha(g, h) = (\alpha_G(g), \alpha_H(h))$, for all $(g, h) \in G \times H$.

(3 marks)

(d) Define the subset A of $\mathrm{Aut}(G \times H)$ to be all those automorphisms α such that $\alpha(g, h) = (g', h)$, for some $g' \in G$. That is

$$A = \{\alpha \in \mathrm{Aut}(G \times H) \,:\, \alpha_H(h) = h, \forall h \in H\}.$$

Show that A is a subgroup of $\mathrm{Aut}(G \times H)$.

Let

$$B = \{\alpha \in \mathrm{Aut}(G \times H) \,:\, \alpha_G(g) = g, \forall g \in G\}.$$

Explain (very briefly) why B is also a subgroup of $\mathrm{Aut}(G \times H)$.

(5 marks)

(e) Show that if $\sigma \in \mathrm{Aut}(G)$ and $\tau \in \mathrm{Aut}(H)$ then $\gamma : G\times H \longrightarrow G\times H$

THE END