MAS3219/8219 Geometric Group Theory

Assignment Exercises: Solutions

1 Exercises for Friday 17th February

1.1 Let K and H be groups. Show

(a) that $H \oplus K \cong K \oplus H$ and

(b) that $K \lhd K \oplus H$. Deduce, in no more than one line, that $H \lhd K \oplus H$. (More than one line = 0 marks.)

Solution

(a) Define a map $\theta : H \oplus K \longrightarrow K \oplus H$ by $\theta(h, k) = (k, h)$. If (h_1, k_1) and $(h_2, k_2) \in H \oplus K$ then

$$\begin{aligned}\theta((h_1, k_1)(h_2, k_2)) &= \theta(h_1h_2, k_1k_2)\\ &= (k_1k_2, h_1h_2)\\ &= (k_1, h_1)(k_2, h_2)\\ &= \theta(h_1, k_1)\theta(h_2, k_2).\end{aligned}$$

Therefore θ is a homomorphism.
If $(k, h) \in K \oplus H$ then $(k, h) = \theta(h, k)$, so θ is surjective.
If $\theta(h, k) = 1_{K\oplus H} = (1_K, 1_H)$, then $(k, h) = (1_K, 1_H)$ so $k = 1_K$ and $h = 1_H$. Therefore $(h, k) = (1_H, 1_K) = 1_{H\oplus K}$. Hence θ is injective.
As θ is an isomorphism the required result follows.

(4 marks)

(b) Really the question should ask if $K' = \{(k, 1) : k \in K\}$ is normal in $K \oplus H$; so this is the question answered here.
It is shown in the notes that K' is a subgroup of $K \oplus H$. (Also, from the notes, $K' \cong K$.) To see that K' is normal in $K \oplus H$, let $(k, 1) \in K'$ and $(a, b) \in K \oplus H$. Then

$$(a, b)^{-1}(k, 1)(a, b) = (a^{-1}, b^{-1})(ka, b) = (a^{-1}ka, b^{-1}b) = ((a^{-1}ka, 1).$$

Thus, for all $(a, b) \in K \oplus H$ and $(k, 1) \in K'$ we have $(a, b)^{-1}(k, 1)(a, b) \in K'$, so K' is normal in $K \oplus H$.
From the first part of the question $K \oplus H \cong H \oplus K$, and under the isomorphism defined there $H' = \{(1, h) : h \in H\}$ maps to the subgroup $H'' = \{(h, 1) : h \in H\}$ of $H \oplus K$.

From this part of the question $H'' \triangleleft H \oplus K$. Hence H' is normal in $K \oplus H$ (as isomorphism preserves normality).

(0 marks)

1.2 Let G be a group with normal subgroups A and B. Then it is a fact (which you should check if you're not sure of it) that $A \cap B$ is normal in G. Define

$$\psi : G/(A \cap B) \longrightarrow (G/A) \oplus (G/B), \text{ by } \psi(g(A \cap B)) = (gA, gB).$$

(a) Show that ψ is a well-defined map.

(b) Show that ψ is an injective homomorphism.

(c) Show that if G/A and G/B are Abelian then $G/(A \cap B)$ is Abelian: in no more than two lines. (More than two lines ...)

Solution

(a) We must check that if $g(A \cap B) = h(A \cap B)$, for some $g, h \in G$, then $\psi(g(A \cap B)) = \psi(h(A \cap B))$. Suppose then that $g(A \cap B) = h(A \cap B)$; so $h^{-1}g \in A \cap B$. In this case $h^{-1}g \in A$ and $h^{-1}g \in B$, so we have $gA = hA$ and $gB = hB$. Thus

$$\psi(g(A \cap B)) = (gA, gB) = (hA, hB) = \psi(h(A \cap B)).$$

Therefore ψ is well-defined.

(2 marks)

(b) First we show that ψ is a homomorphism. If $g(A \cap B)$ and $h(A \cap B)$ are elements of $G/(A \cap B)$ then

$$\begin{aligned}\psi([g(A \cap B)][h(A \cap B)]) &= \psi(gh(A \cap B) = (ghA, ghB)\\ &= (gAhA, gBhB)\\ &= (gA, gB)(hA, hB)\\ &= \psi(g(A \cap B))\psi(h(A \cap B)),\end{aligned}$$

as required.

To see that ψ is injective suppose that $g(A \cap B)$ maps to the identity; that is $\psi(g(A \cap B) = (1, 1)$. As the identity of G/A is A and the identity of G/B is B this means that $gA = A$ and $gB = B$; so $g \in A$ and $g \in B$. Thus $g(A \cap B) = A \cap B$, as required.

(4 marks)

(c) Now suppose that G/A and G/B are Abelian. If H and K are Abelian groups then it follows that $H \oplus K$ is Abelian; since $(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2) = (h_2h_1, k_2k_1) = (h_2, k_2)(h_1, k_1)$, for all $h_i \in H$ and $k_i \in K$. Hence $G/A \oplus G/B$ is Abelian. As $G/(A \cap B)$ maps injectively into $G/A \oplus G/B$ it must also be Abelian.

(2 marks)

1.3 Let G be a group with normal subgroups A and B, such that $G = AB$. Using the previous question, show that

$$G/(A \cap B) \cong G/A \oplus G/B.$$

Solution

Let ψ be the injective homomorphism of the previous question. If $(gA, hB) \in G/A \oplus G/B$ then there exist $a, c \in A$ and $b, d \in B$ such that $g = ab$ and $h = cd$. Thus

$$(gA, hB) = (abA, cdB) = (Aab, cB) = (Ab, cB) = (Acb, cbB) = (cbA, cbB).$$

Therefore $(gA, hB) = \psi(cb(A \cap B))$, and so ψ is surjective and an isomorphism.

(3 marks)

1.4 In this question you will prove Theorem 3.10 in the notes. To this end recall Theorem 3.9 and the discussion following it. That is, let B be a f.g. Abelian group generated by $\{z_1, \ldots, z_n\}$ and let A be a free Abelian group, freely generated by $\{x_1, \ldots, x_n\}$. The homomorphism ϕ is given by $\phi(r_1x_1 + \cdots + r_nx_n) = r_1z_1 + \cdots + r_nz_n$, as in Theorem 3.9, so

$$B \cong A/\mathrm{Ker}(\phi).$$

We must therefore find the structure of $A/\mathrm{Ker}(\phi)$. Free generators $\{y_1, \ldots, y_n\}$ are chosen for A such that

$$\mathrm{Ker}(\phi) = \langle r_1y_1, \ldots, r_my_m\rangle,$$

for some $m \leq n$, and integers $r_i > 0$ such that $r_i | r_{i+1}$. (The case $n = 1$ is considered in the notes.) The generators y_i fall into three types.

- y_i such that $i > m$. Define $c_1 = y_{m+1}, \ldots, c_{n-m} = y_n$.
- y_i such that $i \leq m$ and $r_i = 1$. Suppose that $r_1 = \ldots = r_l = 1$ and $r_{l+1} > 1$. Then define $e_1 = y_1, \ldots, e_l = y_l$.
- y_i such that $i \leq m$ and $r_i > 1$: that is $y_{l+1}, \ldots, y_m$. Define $d_1 = y_{l+1}, \ldots, d_{m-l} = y_m$.

Then

$$A = \langle c_1, \dots, c_{n-m}, d_1, \dots, d_{m-l}, e_1, \dots e_l \rangle,$$

and

$$\text{Ker}(\phi) = \langle s_1 d_1, \dots, s_{m-l} d_{m-l}, e_1, \dots, e_l \rangle,$$

where $s_i = r_{i+l}$. Hence $s_i | s_{i+1}$.

For $a \in A$ let $\bar{a} = a\text{Ker}(\phi)$, the image of a in $A/\text{Ker}(\phi)$.

(a) Show that $\bar{c}_i$ has infinite order in $A/\text{Ker}(\phi)$.

(b) Show that $\bar{d}_i$ has finite order s_i in $A/\text{Ker}(\phi)$.

(c) Show that $\bar{e}_i$ is the identity element of $A/\text{Ker}(\phi)$.

(d) Show that $A/\text{Ker}(\phi) = \langle \bar{c}_1, \dots, \bar{c}_{n-m}, \bar{d}_1, \dots, \bar{d}_{m-l} \rangle$.

(e) Show that if

$$\sum_{i=1}^{n-m} \alpha_i \bar{c}_i + \sum_{i=1}^{m-l} \beta_i \bar{d}_i = 0$$

then $\alpha_1 = \cdots = \alpha_{n-m} = 0$ and $\beta_1 = \cdots = \beta_{m-l} = 0$. **No - the latter condition should be: $s_i | \beta_i$, for $i = 1, \dots, m-l$.**

(f) Use the last two parts of the question to show that $A/\text{Ker}(\phi)$ is the direct sum of the cyclic subgroups generated by the $\bar{c}_i$ and $\bar{d}_i$.

(g) Bring all these facts together to complete a proof of Theorem 3.10.

Solution.

(a) We have $\text{Ker}(\phi) = \langle r_1 y_1, \dots, r_m y_m \rangle$. The free Abelian group A is freely generated by $y_1, \dots, y_n$; so for $j > m$, we have $dy_j \in \text{Ker}(\phi)$ if and only if $d = 0$. Thus $d\bar{y}_j \neq 0 \in A/\text{Ker}(\phi)$, unless $d = 0$. Hence $\bar{y}_j = \bar{c}_{j-m}$ has infinite order in $A/\text{Ker}(\phi)$.

(2 marks)

(b) For $l < j \leq m$ we have $r_j > 1$ so $dy_j \in \text{Ker}(\phi)$ if and only if $r_j | d$. Hence $d\bar{y}_j = 0 \in A/\text{Ker}(\phi)$ if and only if $r_j | d$. Therefore the order of $\bar{y}_j = \bar{d}_{j-l}$ is $r_j = s_{j-l}$.

(1 mark)

(c) For $1 \leq j \leq l$ we have $r_j = 1$ so $y_j \in \text{Ker}(\phi)$ and $\bar{y}_j = 0 \in A/\text{Ker}(\phi)$.

(1 mark)

(d) $A/\text{Ker}(\phi)$ is generated by $\{\bar{y}_1, \dots, \bar{y}_n\} = \{\bar{c}_1, \dots, \bar{c}_l, \bar{d}_1, \dots, \bar{d}_{m-l}, \bar{e}_1, \dots, \bar{e}_{n-m}\}$. As $\bar{e}_1 = \cdots = \bar{e}_{n-m} = 0$ it follows that $A/\text{Ker}(\phi)$ has the given generators.

(1 mark)

(e)
$$\sum_{i=1}^{n-m} \alpha_i \bar{c}_i + \sum_{i=1}^{m-l} \beta_i \bar{d}_i = 0 \quad \iff \quad \sum_{i=1}^{n-m} \alpha_i c_i + \sum_{i=1}^{m-l} \beta_i d_i \in \mathrm{Ker}(\phi). \tag{1.1}$$

As
$$\mathrm{Ker}(\phi) = \langle s_1 d_1, \ldots, s_{m-l} d_{m-l}, e_1, \ldots, e_m \rangle$$
the right hand expression of (1.1) belongs to $\mathrm{Ker}(\phi)$ if and only if $\alpha_1 = \cdots = \alpha_{n-m} = 0$ and $s_i | \beta_i$, for $i = 1, \ldots, m-l$, as claimed.

(0 marks)

(f) From part 4d,
$$A/\mathrm{Ker}(\phi) = \sum_{i=1}^{n-m} \langle \bar{c}_i \rangle + \sum_{i=1}^{m-l} \langle \bar{d}_i \rangle.$$
From part 4e, if the intersection of any two of the cyclic subgroups in this sum is $\{0\}$. Therefore
$$A/\mathrm{Ker}(\phi) = \bigoplus_{i=1}^{n-m} \langle \bar{c}_i \rangle \oplus \bigoplus_{i=1}^{m-l} \langle \bar{d}_i \rangle.$$

(0 marks)

(g) From the above $A/\mathrm{Ker}(\phi) = C \oplus D$, where
$$C = \bigoplus_{i=1}^{n-m} \langle \bar{c}_i \rangle \quad \text{and} \quad D = \bigoplus_{i=1}^{m-l} \langle \bar{d}_i \rangle.$$
As $B \cong A/\mathrm{Ker}(\phi)$, C is free Abelian and the groups $\langle \bar{d}_i \rangle$ have order s_i, where $s_i | s_{i+1}$, this proves Theorem 3.10.

(0 marks)

2 Exercises for Friday 2nd March

2.1 The group S_n of permutations of $\{1, \ldots, n\}$ has a subgroup A_n consisting of even permutations. (Recall from MAS3202, that every element of S_n is generated by the transpositions, and that a permutation is even if it is a product of an even number of transpositions, and odd otherwise: and this depends only on the element of S_n not on a particular expression as a product of transpositions.) In fact A_n is normal in S_n. This question shows that S_n is a semidirect product, using the normal subgroup A_n.

Let $n \geq 3$ and regard $S_2 = \{(), (12)\}$ as a subgroup of S_n.

(a) Show that if σ is an odd permutation then $\sigma(12)$ is an even permutation. Use this fact to show that $S_n = A_n S_2$.

(b) Show that $A_n \cap S_2 = \{()\}$.

(c) Combine all the above to show that $S_n = A_n \rtimes S_2$, quoting an appropriate theorem from the notes.

(d) Now consider the particular case of S_4.

Show that the permutation $\tau = (243)$ is **even**. (A method of writing any permutation as a product of transpositions was explained in MAS3202.)

Find $(12)\tau(12)$ as a product of disjoint cycles.

Let $\sigma = (13)(24)$. In the semidirect product $A_4 \rtimes S_2$ calculate

$$(\sigma, (12))(\tau, ()),$$

giving the answer in the form (ρ, α), where $\rho \in A_4$ and $\alpha \in S_2$, both expressed as disjoint products of cycles.

(7 marks)

Solution.

(a) σ is an odd permutation if and only if it can be written as a product of an odd number of transpositions, in which case $\sigma(12)$ is even, say $\sigma(12) = \alpha \in A_n$. Thus, if σ is odd then $\sigma = \alpha(12)^{-1} = \alpha(12) \in A_n S_2$. Every even permutation is in $A_n \subseteq A_n S_2$, so $S_n = A_n S_2$.

(2 marks)

(b) As (12) is odd, the only even element of S_2 is (), so $A_n \cap S_2 = \{()\}$.

(1 mark)

(c) As $A_n \lhd S_n$ and $S_n = A_n S_2$ and $A_n \cap S_2 = \{()\}$ it follows from Theorem 4.11 that $S_n = A_n \rtimes S_2$.

(1 mark)

(d) $\tau = (243) = (23)(24)$, so is in A_n.
The product $(12)\tau(12) = (12)(243)(12) = (143)$.
From Theorem 4.11 the product is

$$(\sigma[(12)\tau(12)], (12)) = (\sigma(143), (12)) = ((13)(24)(143), (12)) = ((124), (12).$$

(3 marks)

2.2 Show that the isometry $(\mathbf{v}, B)$ of $\mathbb{R}^2$, given by

$$g(x, y) = \left(\frac{3x}{5} + \frac{4y}{5} - 14, \frac{4x}{5} - \frac{3y}{5} + 3\right)$$

is an opposite isometry and a glide reflection and express it in the form $(2\mathbf{a}+\mathbf{b}, B)$ for appropriate $\mathbf{a}, \mathbf{b}$ and B. Find the equation of its axis of reflection and the distance of its translation parallel to this axis.

(9 marks)

Solution

$$g(0, 0) = (-14, 3)$$

so

$$\mathbf{v} = \begin{pmatrix} -14 \\ 3 \end{pmatrix}$$

and

$$g(x, y) - g(0, 0) = \left(\frac{3x}{5} + \frac{4y}{5}, \frac{4x}{5} - \frac{3y}{5}\right)$$

so

$$B\mathbf{x} = \begin{pmatrix} \frac{3x}{5} + \frac{4y}{5} \\ \frac{4x}{5} - \frac{3y}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}.$$

(2 marks)

Thus $g = (\mathbf{v}, B)$ and $\det(B) = \frac{1}{25}(-9 - 16) = -1$, so g is opposite.

$$B\mathbf{v} = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} -14 \\ 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -30 \\ -65 \end{pmatrix} = \begin{pmatrix} -6 \\ -13 \end{pmatrix} \neq -\mathbf{v},$$

so g is a glide reflection.

(2 marks)

$$\mathbf{a} = \frac{1}{4}\left(\mathbf{v} - B\mathbf{v}\right) = \frac{1}{4}\left(\begin{pmatrix} -14 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 13 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

and

$$\mathbf{b} = \mathbf{v} - 2\mathbf{a} = \begin{pmatrix} -10 \\ -5 \end{pmatrix}.$$

(2 marks)

[**Check**: B should satisfy $B\mathbf{a} = -\mathbf{a}$:

$$B\mathbf{a} = \frac{1}{5}\begin{pmatrix} 10 \\ -20 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = -\mathbf{a},$$

as it should.]

The axis of reflection of g is a line in direction of the vector $\mathbf{b}$ and passing through the point $\mathbf{a}$. The line through the origin in direction of $\mathbf{b}$ is the set of points

$$t\mathbf{b} = t\begin{pmatrix} -10 \\ -5 \end{pmatrix}, \text{ for } t \in \mathbb{R},$$

that is

$$t\begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Therefore the axis of reflection consists of points

$$t\mathbf{b} + \mathbf{a} = t\begin{pmatrix} -2 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2t - 2 \\ -t + 4 \end{pmatrix}, \quad t \in \mathbb{R}.$$

These are points $(x, y)^t$ with

$$\begin{aligned} x &= -2t - 2 \\ y &= -t + 4 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x + 2 &= -2t \\ y - 4 &= -t \end{aligned}$$

or

$$x + 2 = 2(y - 4).$$

Rearranging this we find that the axis of reflection is the line m with equation

$$y = \frac{x}{2} + 5.$$

[Alternatively apply the standard formula: the line parallel to **b** and passing through **a** has equation $-10(y-4) = -5(x+2)$. After rearranging this gives the same answer.]

The vector **b** has length $||\mathbf{b}|| = \sqrt{100 + 25} = 5\sqrt{5}$, so **b** is a translation parallel to m through a distance of $5\sqrt{5}$.

(3 marks)

2.3 This question completes the detail of one case of the classification of wallpaper groups: which turns out to be $*2222$.

Let W be a wallpaper group and let L and O be its lattice and point group, respectively. Assume that L is rectangular so $L = \{\alpha\mathbf{a} + \beta\mathbf{b} \,|\, \alpha, \beta \in \mathbb{Z}\}$, for fixed vectors $\mathbf{a} = (x, 0)^t$ and $\mathbf{b} = (0, y)^t$, where we may assume $0 < x < y$ in $\mathbb{R}$ and that $O = \{I, -I, B_0, B_\pi\}$, where I is the identity matrix (so $-I$ is the matrix of rotation through π) and B_0 and B_π are the matrices of reflection in the x and y axes, respectively.

Assume further that W contains reflections g and h in lines parallel to the x and y axes, respectively.

As g is reflection in a line parallel to the x-axis it is of the form $(\mathbf{v}, B_0)$, for some vector $\mathbf{v}$ such that $B_0\mathbf{v} = -\mathbf{v}$. Assume that $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b}$, for some $\alpha, \beta \in \mathbb{R}$. Then

$$B_0\mathbf{v} = \alpha B_0\mathbf{a} + \beta B_0\mathbf{b} = \alpha\mathbf{a} - \beta\mathbf{b} = -\mathbf{v}$$

if and only if

$$\alpha\mathbf{a} - \beta\mathbf{b} = -\alpha\mathbf{a} - \beta\mathbf{b}$$

if and only if $\alpha = 0$. Thus $(\mathbf{v}, B_0)$ is a reflection if and only if $\mathbf{v} = \beta\mathbf{b}$, for some $\beta \in \mathbb{R}$. Writing $b = 2\beta$, we conclude that

$$g = (2b\mathbf{b}, B_0) \text{ for some } b \in \mathbb{R}.$$

(a) Use a similar argument to show that $h = (2a\mathbf{a}, B_\pi)$, for some $a \in \mathbb{R}$.

Now, choose coordinate axes so that the x-axis is the axis of reflection of g and the y-axis is the axis of reflection of h. Then W contains the reflections $\sigma = g = (\mathbf{0}, B_0)$ and $\tau = h = (\mathbf{0}, B_\pi)$.

Consider the coset $(L \times \{I\})\sigma$ of $L \times \{I\}$ in W. We have

$$\begin{aligned}(L \times \{I\})\sigma &= \{(\alpha\mathbf{a} + \beta\mathbf{b}, I_2)(\mathbf{0}, B_0) : \alpha, \beta \in \mathbb{Z}\}\\ &= \{(\alpha\mathbf{a} + \beta\mathbf{b}, B_0) : \alpha, \beta \in \mathbb{Z}\}.\end{aligned}$$

Thus

$$(\beta\mathbf{b}, B_0) \in (L \times \{I\})\sigma \subseteq W,$$

for all $\beta \in \mathbb{Z}$, and $(\beta\mathbf{b}, B_0)$ is a reflection (as $B_0\beta\mathbf{b} = -\beta\mathbf{b}$) with axis parallel to the x-axis and passing through the point $\frac{\beta}{2}\mathbf{b}$.

That is, W contains a reflection with axis parallel to the x-axis and passing through the point $(\beta/2)\mathbf{b}$, for all $\beta \in \mathbb{Z}$.

$(L \times \{I\})\sigma$ also contains elements $(\alpha\mathbf{a} + \beta\mathbf{b}, B_0)$ with $\alpha \neq 0$, and these are glide reflections with axis parallel to the x-axis, passing through the point $\frac{\beta}{2}\mathbf{b}$, and with translation through a distance of α in the direction of the x-axis.

(b) Similarly, by considering $(L \times \{I\})\tau$, show that W contains a reflection with axis parallel to the y-axis and passing through the point $(\alpha/2)\mathbf{a}$, for all $\alpha \in \mathbb{Z}$.

(c) As before, give a geometric description of all symmetries in $(L \times \{I\})\tau$ other than those described in the previous part of the question.

(d) By considering the composition of σ and τ show that W contains the rotation $\rho = (\mathbf{0}, -I)$.

(e) By considering $(L \times \{I\})\rho$, show that W contains a rotation through π about $(\alpha\mathbf{a} + \beta\mathbf{b})/2$, for all $\alpha, \beta \in \mathbb{Z}$.

Using Lagrange's theorem $[W : (L \times \{I\})] = |O|$. Given this fact we may write W as a union of cosets of $L \times \{I\}$: that is

$$W = (L \times \{I\}) \cup (L \times \{I\})\sigma \cup (L \times \{I\})\tau \cup (L \times \{I\})\rho.$$

Therefore we have now found all elements of W.

(f) Draw a diagram showing some points of L, vectors $\mathbf{0}, \mathbf{a}$ and $\mathbf{b}$ and marking one instance of each distinct type of centre of rotation and axis of reflection. (It should follow from the diagram that W is a wallpaper group of type $*2222$.)

(g) Sketch a wallpaper pattern with symmetry group W.

(9 marks)

Solution

(a) h is reflection in a line parallel to the y-axis so it is of the form $(\mathbf{u}, B_\pi)$, for some vector $\mathbf{u}$ such that $B_\pi\mathbf{u} = -\mathbf{u}$. If $\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b}$ then

$$B_\pi\mathbf{u} = \alpha B_\pi\mathbf{a} + \beta B_\pi\mathbf{b} = -\alpha\mathbf{a} + \beta\mathbf{b} = -\mathbf{u}$$

if and only if

$$-\alpha\mathbf{a} + \beta\mathbf{b} = -\alpha\mathbf{a} - \beta\mathbf{b}$$

if and only if $\beta = 0$. Writing $a = 2\alpha$, we conclude that

$$h = (2a\mathbf{a}, B_\pi) \text{ for some } a \in \mathbb{R}.$$

(1 mark)

(b) Consider the coset $(L \times \{I\})\tau$ of $L \times \{I\}$ in W. We have

$$(L \times \{I\})\tau = \{(\alpha\mathbf{a} + \beta\mathbf{b}, B_0) : \alpha, \beta \in \mathbb{Z}\}.$$

Thus, for all $\beta \in \mathbb{Z}$, W contains the reflection $(\alpha\mathbf{a}, B_\pi)$ with axis parallel to the y-axis and passing through the point $\frac{\alpha}{2}\mathbf{a}$.

(1 mark)

(c) $(L \times \{I\})\tau$ also contains elements $(\alpha\mathbf{a} + \beta\mathbf{b}, B_\pi)$ with $\beta \neq 0$, and these are glide reflections with axis parallel to the y-axis, passing through the point $\frac{\alpha}{2}\mathbf{a}$, and with translation through a distance of β in the direction of the y-axis.

(1 mark)

(d)

$$\sigma\tau = (\mathbf{0}, B_0)(\mathbf{0}, B_\pi) = (\mathbf{0}, A_\pi) = (\mathbf{0}, -I) = \rho.$$

(1 mark)

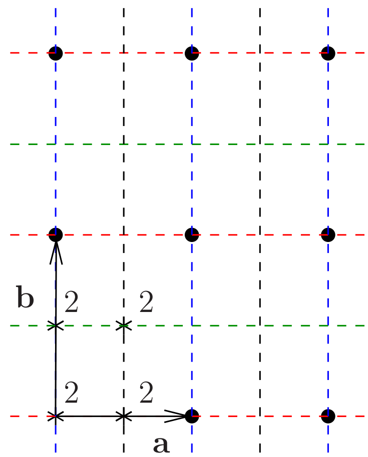
(e) $(L \times \{I\})\rho = \{(\alpha\mathbf{a} + \beta\mathbf{b}, -I) : \alpha, \beta \in \mathbb{Z}\}$. The isometry $(\alpha\mathbf{a} + \beta\mathbf{b}, -I)$ is a rotation through π about

$$\begin{aligned}\mathbf{c} &= (I - (-I))^{-1}(\alpha\mathbf{a} + \beta\mathbf{b}) \\ &= \frac{1}{2}I(\alpha\mathbf{a} + \beta\mathbf{b}) \\ &= \frac{\alpha}{2}\mathbf{a} + \frac{\beta}{2}\mathbf{b}.\end{aligned}$$

Thus W contains rotations through π about $(\alpha\mathbf{a} + \beta\mathbf{b})/2$, for all $\alpha, \beta \in \mathbb{Z}$.

(2 marks)

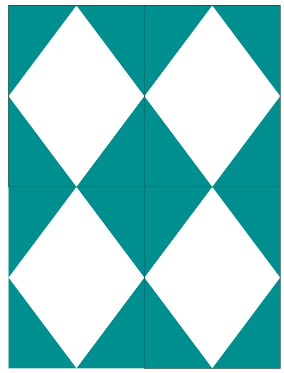
(f)



- There are two orbits of horizontal axes of reflection: red and green.
- There are two orbits of vertical axes of reflection: blue and black.
- There are four orbits of centres of rotation, all on axes of reflection. One representative of each orbit is marked with an "*" and labelled 2.

(2 marks)

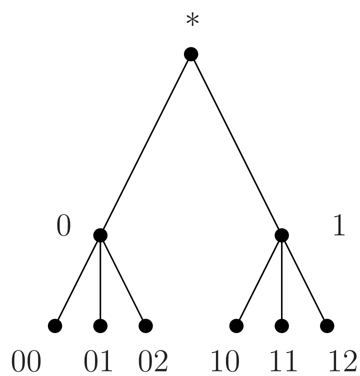
(g)



(1 mark)

3 Exercises for Friday 16th March

3.1 Let Γ be the graph shown below.



This question will determine the structure of $\mathrm{Sym}(\Gamma)$. We shall see this is generated by isomorphisms which swap 0 and 1, and essentially do nothing else: as well as those which fix both 0 and 1 and permute the leaves below them. First it's necessary to set up some notation.

Let Id denote the identity map of Γ and let σ denote the map such that

$$\sigma(0) = 1, \quad \sigma(1) = 0, \quad \sigma(0j) = 1j, \quad \sigma(0j) = 1j.$$

Let S_3 denote the group of bijections of $\{0, 1, 2\}$ (i.e. the symmetric group of degree 3). For $\rho \in S_3$, let ρ^L be the isomorphism of Γ such that

$$\rho^L(0) = 0, \quad \rho^L(1) = 1, \quad \rho^L(0j) = 0\rho(j), \quad \rho^L(1j) = 1j.$$

That is, ρ^L fixes all vertices except those below 0, which are permuted by ρ acting on their right hand digit. Similarly, for $\rho \in S_3$, let ρ^R be the isomorphism of Γ such that

$$\rho^R(0) = 0, \quad \rho^R(1) = 1, \quad \rho^R(0j) = 0j, \quad \rho^R(1j) = 1\rho(j).$$

Let us assume without further comment that σ, ρ^L and ρ^R are isomorphisms of Γ. To avoid interminable, tedious calculation let's also assume that we have checked that both

$$L = \{\rho^L : \rho \in S_3\} \text{ and } R = \{\rho^R : \rho \in S_3\}$$

are subgroups of $\mathrm{Sym}(\Gamma)$.

(Also, isomorphisms of this graph are completely determined by their effect on vertices, so to check that two of them are the same it's sufficient to check their effects on vertices.)

(a) Show that if $\rho_1, \rho_2 \in S_3$ then

$$\rho_1^L \rho_2^R = \rho_2^R \rho_1^L.$$

[Consider the effect of the lhs on vertices, 0, 1, $0j$ and $1j$. Then do the same for the rhs.]

(3 marks)

(b) Show that $L \cap R = \{\mathrm{Id}\}$.

(2 marks)

(c) By quoting an appropriate definition from the notes show that $LR = L \times R$, the internal direct product of L and R. (This is the same as the direct sum $L \oplus R$, but the product notation and terminology seems more fitting here.)

(1 mark)

(d) Show that the subgroup $T = \langle \sigma \rangle$ of $\mathrm{Sym}(\Gamma)$ is cyclic of order 2.

(2 marks)

(e) Show that $L \times R$ is normal in the subgroup of $\mathrm{Sym}(\Gamma)$ generated by $L \times R$ and T. [It's sufficient to show that if $x \in T$ and $y \in L \times R$ then $x^{-1}yx \in L \times R$. Consider the effect of $x^{-1}yx$ on the various different types of vertex, as in part (a).]

(5 marks)

(f) Show that $(L \times R) \cap T = \{\mathrm{Id}\}$.

(2 marks)

(g) By quoting an appropriate theorem from the notes show that $(L \times R)T$ is the internal semi-direct product $(L \times R) \rtimes T$ of $(L \times R)$ and T.

(1 mark)

(h) Explain why every isomorphism of Γ is an element of $(L \times R) \rtimes T$ and hence that $\mathrm{Sym}(\Gamma) = (L \times R) \rtimes T$.

(3 marks)

(Now it's not too much effort to show that $\mathrm{Sym}(\Gamma) \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2$, but this is not part of the question.)

3.2 Draw Cayley graphs for the following groups G with generating sets S.

(a) $G = D_4$, $S = \{\sigma, \tau\}$ as on pages 9–11 of the notes.

(2 marks)

(b) $G = D_\infty$, $S = \{\sigma, \tau\}$. D_∞ is the group with elements $\{\sigma^n, \sigma^n\tau : n \in \mathbb{Z}\}$, where $\tau^2 = 1$ and $\sigma^{-1}\tau = \tau\sigma$. Draw a diagram showing the vertices σ^n and $\sigma^n\tau$, for $-3 \leq n \leq 3$, and all incident edges.

(2 marks)

(c) $G = \mathbb{Z} \times \mathbb{Z}$, with binary operation addition, $S = \{x = (1,1), y = (1,-1), z = (1,2)\}$. Show vertices (m,n) for $-1 \leq m, n \leq 2$.

(2 marks)

4 Exercises for Friday 27th April

4.1 In this question the proof of one half of Theorem 8.10 is constructed. Let X_1 and X_2 be sets such that $|X_1| = |X_2|$. The aim is to show that $F(X_1) \cong F(X_2)$. By definition $|X_1| = |X_2|$ means that there exists a bijection $\alpha : X_1 \longrightarrow X_2$ (we're allowing infinite sets of course).

(a) Use Theorem 8.8 and the Universal Mapping Theorem 8.9 to show that there exists a unique homomorphism $\phi_1 : F(X_1) \longrightarrow F(X_2)$, such that $\phi_1(x_1) = \alpha(x_1)$, for all $x_1 \in X_1$.

(b) Argue (more briefly this time) that similarly there exists a unique homomorphism $\phi_2 : F(X_2) \longrightarrow F(X_1)$, such that $\phi_2(x_2) = \alpha^{-1}(x_2)$, for all $x_2 \in X_2$.

(c) Show that $\phi_2 \circ \phi_1(x_1) = x_1$, for all $x_1 \in X_1$.

(d) Note that the identity ι_1 from $F(X_1)$ to $F(X_1)$ has the property that $\iota_1(x_1) = x_1$, for all $x_1 \in X_1$. Apply Theorem 8.9, and this fact, to show that $\phi_2 \circ \phi_1 = \iota_1$.
Show, similarly, that $\phi_1 \circ \phi_2 = \iota_2$, where ι_2 is the identity map of $F(X_2)$ to itself. (If you like, use the symmetry of the situation rather than repeating the whole argument again.)

(e) From this deduce that ϕ_1 and ϕ_2 are isomorphisms, and so $F(X_1) \cong F(X_2)$.

4.2 Apply the Stallings folding algorithm to find a free basis for each of the following subgroups of $F(x, y)$.

(a) $H_1 = \langle xy^2, yx^2, (xy)^2 \rangle$.

(b) $H_2 = \langle xy^2, x^2y, (xy)^2 \rangle$.

(c) $H_3 = \langle xy^2, y^{-2}x^{-1}, xy^2xy^2 \rangle$.

4.3 (a) Show that the subgroup of $F(x, y)$ generated by x, $y^{-1}xy$ and $y^{-2}xy^2$ has rank 3.

(b) Generalise this to show that the free group $F(x, y)$ contains a free group of rank n, for all $n \geq 3$ (and hence for all $n \geq 1$). [Use an informal argument based on Stallings foldings.]

(c) Assuming that Stallings' theorem holds for infinitely generated subgroups H briefly outline an argument to show that $F(x, y)$ contains a free group of infinite rank.

5 Exercises for Friday 11th May

5.1 **Baumslag-Solitar groups.** The family of groups with presentations

$$BS(m,n) = \langle a, b \,|\, a^{-1}b^m a = b^n \rangle,$$

for positive integers m, n are called *Baumslag-Solitar groups* (as they were first investigated by Graham Higman). Here we consider $BS(1,2)$, the group with presentation

$$\langle a, b \,|\, a^{-1}ba = b^2 \rangle.$$

First we shall show that every element of $BS(1,2)$ can be written as an element of the set

$$L = \{a^n : n \in \mathbb{Z}\} \cup \{a^{-k}b^{2m+1}a^{k+n} : k, m, n \in \mathbb{Z}\}.$$

This can be done by induction on the length of words in the generators. First it helps to establish the following fact for elements of $BS(1,2)$.

(a) Show, by induction on r, that for all $r \geq 1$,

$$a^{-r}ba^r = b^{2^r}.$$

The unique word of length 0 is the empty word, and this represents the identity element which can be written as a^0, so is in L. Assume that all words in $F(a,b)$, of length at most n, are equal in $BS(1,2)$ to elements of L. To complete the inductive step we must show that if w is a word of length $n+1$ then it is equal to an element of L. Suppose then that $w = xv$, where $x \in \{a^{\pm 1}, b^{\pm 1}\}$ and v is represented by an element of L. We check each possible form for v, for each possible value of x. The possible forms of v are $v = a^n$ or $v = a^{-k}b^{2m+1}a^{k+n}$.

(b) Suppose that $v = a^n$. Check that $a^{\pm 1}v$ and $b^{\pm 1}v$ are elements of L. (No more than two lines required.)

(c) Suppose that $v = a^{-k}b^{2m+1}a^{k+n}$ and $x = a^{\pm 1}$. Show (one line only) that xv is in L.

(d) Suppose that $v = a^{-k}b^{2m+1}a^{k+n}$ and $x = b^{\pm 1}$. Show that xv can be represented by an element of L. [Hint. Replace $a^{-k}b^{2m+1}a^{k+n}$ by a power of b, using the first part of the question. It may help to remember that for all group elements g, h it's true that $g^{-1}h^k g = (g^{-1}hg)^k$.]

This completes the inductive step, so now we have shown that every element of $BS(1,2)$ can be written as an element of L. Next we shall show that $BS(1,2)$ acts on the real line. Consider the two maps α and β of $\mathbb{R}$ to itself given by

$$\alpha(x) = x/2$$

and

$$\beta(x) = x + 1.$$

Together α and β generate a subgroup A of the group of invertible maps from $\mathbb{R}$ to $\mathbb{R}$, under composition of maps: that is $A = \langle \alpha, \beta \rangle$. Define a map $f : \{a, b\} \longrightarrow A$ by $f(a) = \alpha$ and $f(b) = \beta$.

(e) Use von Dyck's Theorem to show that there is a homomorphism θ from $BS(1, 2)$ to A such that $\theta(a) = \alpha$ and $\theta(b) = \beta$.

This means that we have an action of $BS(1, 2)$ on $\mathbb{R}$. Next we shall show that in fact $BS(1, 2) \cong A$.

(f) Is θ surjective and if so why?

(g) If $g \in \ker(\theta)$ then, in particular $\theta(g)$ maps 0 to 0 and 1 to 1. Moreover g is represented by an element of L. Show that the image of 1 under $\theta(a^n)$ is not 1, unless $n = 0$. Show that the image of 0 under $\theta(a^{-k}b^{2m+1}a^{k+n})$ is not 0, for any k, m, n. Conclude that θ is injective; and so $BS(1, 2) \cong A$.

(h) Find the image of 1 under $\theta(a^{-k}b^{2m+1}a^{k+n})$. Hence show that no two elements of L represent the same group element. That is, θ maps L bijectively to A.

(i) Give yourself a pat on the back: you've done some serious geometric group theory.

(j) Match the names "Gilbert Baumslag", "Graham Higman" and "Donald Solitar" to the pictures below.