MAS3210 Geometries and Designs

Semester 1, 2009/2010

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We are all familiar with shapes, patterns and pictures from our earliest consciousness. So, too, geometry, concerning particular spatial configurations which have pictorial elegance and beauty or practical significance, has its origins at the beginning of mathematical explorations. The serious study of geometry involves the systematic development of its basic concepts and subsequent formation of appropriate algebraic and analytic methods. This can occur in settings other than the ordinary Euclidean world; for example, in the projective plane which has a line at infinity or in planes where the co-ordinates belong to a finite field. Figures in the latter involve a finite number of points, lines and conics etc., which have combinational properties. These generalise to the ideas of designs, which figure in applications like the design of experiments. There is a wealth of possible material. The module will seek to convey an appreciation of some of this.

Books

1. Combinatorial Mathematics, H.J. Ryser (Math. Assoc. of America).
2. A Course in Combinatorics, J.H. Van Lint and R.M. Wilson (C.U.P.).
3. **Library §512.5, §511.6**

Notes

The printed notes consist of lecture notes, intended to supplement the notes you make during the lectures, exercises and a mock exam with solutions. Material given on slides in the lectures is covered in the printed notes, what is written on the blackboard during lectures may not be. There should be enough space in the printed notes for you to write down the notes you take in lectures. The notes, exercises and other course information can be found on the web at

`www.mas.ncl.ac.uk/~najd2/teaching/mas3210/`

from where they can be viewed or printed out.

AJ Duncan August 2009

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AJD September 15, 2009

1 Combinatorial Designs

Broadly speaking, a (combinatorial) design is a set together with a number of special subsets satisfying two conditions. The subject has its roots in Statistics, in the design and analysis of experiments.

(1) **The Prussian Officers Problem (Euler, 1779)** In the Prussian army there are 6 different officer ranks. One officer of each rank is selected from each of 6 regiments, making 36 officers in all. Can you arrange the officers in a 6×6 square so that each row and each column contains just one officer of each rank and just one officer from each regiment?

Answer Euler conjectured that it could not be done, G. Tarry confirmed this in 1900.

(2) **Kirkman's Schoolgirls Problem (Kirkman, 1850)**

Answer

These are **combinatorial problems**, concerned with a finite set of n objects which can be 'arranged' in a number (N say) of ways. Two sorts of questions arise.

(A) Among the N arrangements, is there one which satisfies certain prescribed conditions?

(B) If the answer to (A) is YES, how many of the N arrangements satisfy the conditions?

Of course it is in principle possible to resolve such questions by **exhaustion**, i.e., by examining each of the N arrangements in turn. In practice, however, such an approach is ruled out because N is impossibly large, even when n is quite small. This phenomenon is called the **combinatorial explosion**. For example, the number of ways of arranging a class of 30 students in order of merit is $30!$, roughly 2.65×10^{32}.

This is where mathematics comes in. It can be used to work out the implications of the 'prescribed conditions', and hence reduce the number of arrangements that we need to consider (in effect, to replace N by a smaller number). Having cut down the number of possibilities we may then use exhaustion to administer the coup de grace (as Tarry did with Euler's problem). But exhaustion is usually regarded as a last resort.

1.1 A Simpler Problem

Problem A swimming club wants to send a squad of 10 to a swimming gala consisting of $b \leq 20$ races. From this squad it has to enter a team of size k in each race, with different teams for different races. Also any two squad members are to swim together exactly twice. Is it possible to do this, and if so what are the possible values for b and k?

Answer

1.2 2-Designs

Let us first fix some notation.

Notation 1.1. Suppose that $v, k \in \mathbb{N}$ with $v \geq k$. By a v-*set* we mean a set with v elements. Given a v-set X, a k-*subset* is a subset of X with k elements.

Remark 1.2.

Definition 1.3. A $2-(v,k,\lambda)$ design consists of

Remarks 1.4. (a) We refer to the design $(X,\mathcal{B})$. The numbers v,k,λ are *parameters* of the design.

(b)

(c)

(d)

(e)

(f) We shall always assume that $2 \leq k \leq v$. If a $2-(v,k,\lambda)$ design exists, then $\lambda \geq 1$ (for if we take a block B and take two distinct points $P, Q \in B$, then the 2-subset $\{P, Q\}$ lies in at least one block). [If we allowed $k = 1$, then we should always have a $2-(v,k,\lambda)$ design with $\lambda = 0$.]

(g) The fundamental question is: For given, v, k and λ, does a $2-(v,k,\lambda)$ design exist? The answer is: sometimes but not always.

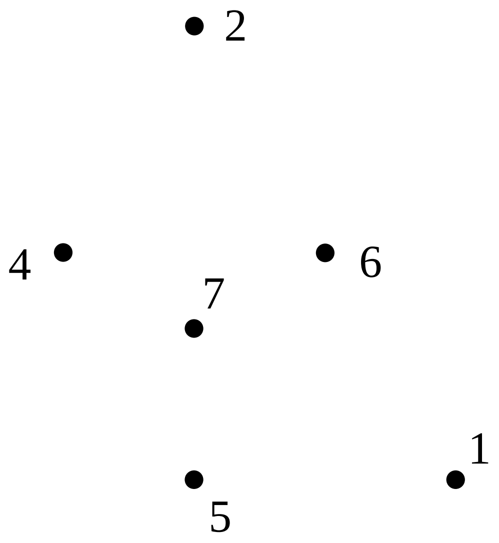
(h)

(i)

Notation 1.5. We use b to denote the number of blocks: $b = |B|$. (We do not build b into the definition: it turns out that it is determined by v, k and λ.)

Example 1.6. Here is an example of a $2-(7,3,1)$ design (the **Fano Plane**).

The following diagram has seven points and seven 'lines'.



Here $X = \{1,2,3,4,5,6,7\}$ so $|X| = 7$. Each block is a set of three points on a line, where we count the circle $4, 5, 6$ as a 'line'. Thus

$$\mathcal{B} = \{\{1,6,2\},\{2,4,3\},\{3,5,1\},\{1,7,4\},\{2,7,5\},\{3,7,6\},\{4,5,6\}\}.$$

We can use symmetry to help us check efficiently that any two points lie in exactly one block. The rotational symmetry of the triangle means that an argument applied to the point 1 applies equally well to the points 2 and 3, and an argument applied to the point 4 applies equally well to the points 5 and 6.

Observe that

- Precisely 3 lines pass through each point.
- 2 distinct lines have just one point in common.

1.3 Trivial Designs

Theorem 1.7. *Let X be any v-set with $v \geq 2$, let $2 \leq k \leq v$ and let $\mathcal{B}$ be the set of all k-subsets of X. Then $(X, \mathcal{B})$ is a $2-(v,k,\lambda)$ design with $\lambda = \binom{v-2}{k-2}$.*

Proof.

□

Definition 1.8. A $2-(v,k,\lambda)$ design is **trivial** if $\mathcal{B}$ is the set of all k-subsets of X.

Theorem 1.9. *If $k = 2$ then the only $2-(v,k,\lambda)$ design is the trivial one and $\lambda = 1$.*

Proof.

□

Theorem 1.10. *If* $k = v$ *then the only* $2-(v,k,\lambda)$ *design is the trivial one and* $\lambda = 1$.

Proof.

□

Corollary 1.11. *In a non-trivial* $2-(v,k,\lambda)$ *design* $2 < k < v$.

Proof. Follows immediately from the preceding theorems. □

1.4 Arithmetic of 2-designs

Theorem 1.12. *In a* $2{-}(v,k,\lambda)$ *design, the number of blocks is given by* $b = \dfrac{\lambda v(v-1)}{k(k-1)}$.

Proof.

$\square$

Remark 1.13. *This explains why* b *was not specified in the definition of a* $2-(v,k,\lambda)$ *design: it is completely determined by* v,k *and* λ.

Theorem 1.14. *In a* $2-(v,k,\lambda)$ *design, each point lies in the same number of blocks, namely,*

$$r=\lambda\frac{(v-1)}{(k-1)}.$$

Proof. Let P be any point of X and regard P as fixed. Let r be the number of blocks containing P. We count the number of pairs (Q,B), where $Q\in X\setminus\{P\}, B\in\mathcal{B}$ and $P,Q\in B$. We do this in two different ways and then compare.

(1) Choose B first: there are r blocks containing P so we can do this in r ways. For each choice of B there are $k-1$ choices for Q, making $r(k-1)$ pairs (Q,B).

(2) Choose Q first: there are $v-1$ points of X different from P so we can do this in $v-1$ ways. For each choice of Q there are λ choices for B containing P,Q, making $\lambda(v-1)$ pairs (Q,B).

The number of pairs does not depend on the way we count. Therefore $r(k-1)=\lambda(v-1)$, i.e., $r=\lambda\dfrac{(v-1)}{(k-1)}$. This expression does not depend on the choice of P, so r is the same for every point of X. $\square$

Corollary 1.15. *In a* $2-(v,k,\lambda)$ *design with* $v>k$*, it must be the case that* $r>\lambda$.

Proof. This follows immediately from Theorem 1.14, given that $\dfrac{(v-1)}{(k-1)}>1$. $\square$

Theorem 1.16. *In a* $2-(v,k,\lambda)$ *design,* $bk=vr$.

Proof. This follows immediately from Theorems 1.12 and 1.14. $\square$

1.5 Applications

Example 1.17. Show that there is no $2-(13,5,2)$ design.

Solution

Example 1.18. Show that there is no $2-(12,4,2)$ design.

Solution

Example 1.19. Show that there are at most two values of k for which a non-trivial $2-(31,k,1)$ design exists.

Solution

Example 1.20. Show that if a $2-(v,3,1)$ design exists then v must be of the form $6n+1$ or $6n+3$.

Solution

Example 1.21. Suppose that a schoolteacher is able to give each child in her class 3 differently coloured crayons, drawn from a box containing crayons of 6 different colours. No 2 children receives the same 3 colours. In fact, each pair of colours are given two exactly 2 children. How many children are in the class? How many crayons of each colour are used?

Solution

Exercise 1.22. Show that if $k = v - 1$, then the only $2 - (v, k, \lambda)$ design is the trivial one. [Hint: use the expression for r to get a lower bound for λ and use this to show that $b \geq v$; explain why in the case of a $2 - (v, v-1, \lambda)$ design it must happen that $b \leq v$.]

1.6 Steiner Systems

Definition 1.23. A $2-(v, k, 1)$ design is known as a **Steiner System of order** v. In particular if $k = 3$, then it is called a Steiner Triple System (STS).

Remarks 1.24. (a)

(b)

(c)

(d)

Definition 1.25 (Binary Projective Spaces)**.** Given any $n \geq 2$, let V_{n+1} be the set of all vectors of length $n+1$ with each entry either 0 or 1. Then we can add

vectors in V_{n+1} by performing operations $\pmod 2$. The zero vector is just the vector $(0, 0, \dots, 0)$. If u, v are distinct non-zero vectors, consider the set of vectors $\{u, v, u+v\}$: we observe that $u+(u+v) = 2u+v = v$, $v+(u+v) = 2v+u = u$, so the sum of any two elements is the third; moreover $u, v, u+v$ are distinct with $u+v$ non-zero (for if $u+v = 0$, then $u = v$; if $u = u+v$, then $v = 0$; if $v = u+v$, then $u = 0$).
Let X be the set of all non-zero vectors in V_{n+1} and let $\mathcal{B}$ be the set of all 3-subsets of X of the form $u, v, u+v$ where $u \neq v$. We call $(X, \mathcal{B})$ the **binary projective space of dimension** n.

Theorem 1.26. *The binary projective space of dimension n is an STS of order $2^{n+1} - 1$.*

Proof.

$\square$

Remark 1.27. *As a consequence of Theorem 1.26, we know that there are infinitely many STSs (at least one of order* $2^{n+1}-1$ *for each* $n \geq 2$*) and therefore infinitely many designs.*

Examples 1.28. (a) When $n = 2$, we have an STS of order 7, i.e., a $2-(7,3,1)$ design. X consists of the vectors:

$$(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1),(1,1,1).$$

The blocks then are:

$$\{(1,0,0),(0,1,0),(1,1,0)\},\{(1,0,0),(0,0,1),(1,0,1)\},$$
$$\{(0,1,0),(0,0,1),(0,1,1)\},\{(1,0,0),(0,1,1),(1,1,1)\},$$
$$\{(0,1,0),(1,0,1),(1,1,1)\},\{(0,0,1),(1,1,0),(1,1,1)\},$$
$$\{(1,1,0),(0,1,1),(1,0,1)\}.$$

We can label the Fano Plane as follows:

Thus the Fano Plane is simply the smallest binary projective space (and it is a plane because it has dimension 2).

(b) When $n = 3$ we get an STS of order 15. This is a candidate for a solution to the Kirkman Schoolgirl Problem.

(c) When $n = 4$ we get an STS of order 31, i.e., a $2-(31,3,1)$ design (see Example 1.19).

Theorem 1.29. *If there exist Steiner Triple Systems of orders v_1 and v_2, then there exists a Steiner Triple System of order v_1v_2.*

Proof.

$\square$

Example 1.30. Construct an STS of order 9.

Solution. We observe that $9 = 3 \times 3$ and that there is an STS of order 3, namely the trivial $2-(3,3,1)$ design. We may take $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ as the same design, with points labelled $1, 2, 3$ and each with one block, namely $\{1, 2, 3\}$. The set Z is given by

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}.$$

We construct blocks as indicated above:

(a) We have only one block in $\mathcal{B}$, namely $\{1, 2, 3\}$. For $j = 1, 2, 3$ in turn, we get 3-subsets: $\{(1,1),(2,1),(3,1)\}$, $\{(1,2),(2,2),(3,2)\}$, $\{(1,3),(2,3),(3,3)\}$.

(b) We have only one block in $\mathcal{C}$, namely $\{1, 2, 3\}$. For $i = 1, 2, 3$ in turn, we get 3-subsets: $\{(1,1),(1,2),(1,3)\}$, $\{(2,1),(2,2),(2,32)\}$, $\{(3,1),(3,2),(3,3)\}$.

(c) We have only one block, $\{1, 2, 3\}$, in $\mathcal{B}$ and one block, $\{1, 2, 3\}$,in $\mathcal{C}$. There are six subsets of Z (one for each ordering of $1, 2, 3$): $\{(1,1),(2,2),(3,3)\}$, $\{(1,1),(2,3),(3,2)\}$, $\{(1,2),(2,1),(3,3)\}$, $\{(1,3),(2,2),(3,1)\}$, $\{(1,2),(2,3),(3,1)\}$, $\{(1,3),(2,1),(3,2)\}$.

We can see these points in a diagram:

Examples 1.31. For each of the following values of v, determine whether or not there is an STS of order v:

(a) $v = 127$ **Solution.**

(b) $v = 21$ **Solution.**

(c) $v = 35$ **Solution.**

(d) $v = 105$ **Solution.**

(e) $v = 343$ **Solution.**

1.7 Projective Planes

Definition 1.32. A (finite) **projective plane of order** n is a $2-(n^2+n+1, n+1, 1)$ design.

Remark 1.33. *We shall see more of projective planes in a geometric context later in the course. For now it suffices to note:*

- *There is a projective plane of order* q *for every prime power* q*. In fact, quite often, there are two or more different known examples for many prime powers* q*.*
- *What we shall see later are particular examples of projective planes of order* p*, where* p *is prime.*
- *There are no known projective planes having an order that is not a prime power, but it has not yet been proved that no such planes exist.*
- *The Fano Plane is a projective plane of order* 2*.*
- *A projective plane of order* 5 *is a* $2-(31, 6, 1)$ *design (see Example 1.19).*

1.8 Fisher's Inequality

Theorem 1.34. *Given a* $2-(v, k, \lambda)$ *design with* $v > k$*, the number of blocks* b *is* $\geq v$*.*

Proof. Later. ☐

Examples 1.35. (a) Recall the swimming club problem in Section 1.2.

Problem A swimming club sends a squad of 10 to a swimming gala consisting of $b \leq 20$ races. From this squad it has to enter a team of size k in each race, with different teams for different races. Also any two squad members are to swim together exactly twice. Find b and k so that this is possible.

(b) Show that there is no $2-(136,51,10)$ design.

Solution.

1.9 Complementary designs

Theorem 1.36. *Let* $(X, \mathcal{B})$ *be a* $2-(v,k,\lambda)$ *design and let* $\bar{\mathcal{B}}$ *consist of the* complements *in* X *of the blocks in* $\mathcal{B}$*, i.e.,*

$$\bar{\mathcal{B}} = \{\bar{B} = X \setminus B : \; B \in \mathcal{B}\}.$$

(a) Show that if P *and* Q *are two particular points of* X*, then the number of blocks containing one or both of* P *and* Q *is* $2r-\lambda$*. Show that the number of blocks containing neither* P *nor* Q *is* $b-2r+\lambda$*.*

(b) Show that $(X, \bar{\mathcal{B}})$ *is a* $2-(v, v-k, b-2r+\lambda)$ *design.*

Proof.

$\square$

Definition 1.37. The design $(X, \bar{\mathcal{B}})$ is called the *complementary design* to $(X, \mathcal{B})$.

Remarks 1.38. (a) If $B \in \mathcal{B}$, then $\bar{\bar{B}} = B$. Thus the complementary design to $(X, \bar{\mathcal{B}})$ has X as its set of points and $\mathcal{B}$ as its set of blocks, i.e., is $(X, \mathcal{B})$.

(b) If there exists a $2-(v, k, \lambda)$ design, then there exists a $2-(v, v-k, b-2r+\lambda)$ design. Conversely, suppose we want to know if there exists a $2-(v, k, \lambda)$

design and we calculate the numbers $k' = v - k$ and $\lambda' = b - 2r + \lambda$. If there is a $2 - (v, k', \lambda')$ design (with b' blocks and each point in r' blocks), then its complement will be a $2 - (v, v - k', b' - 2r' + \lambda')$ design. We can calculate (EXERCISE!):

$$\lambda' = \frac{\lambda(v-k)(v-k-1)}{k(k-1)},\ b' = b,\ r' = \frac{\lambda(v-k)(v-1)}{k(k-1)}$$

$$v - k' = k,\ b' - 2r' + \lambda' = \lambda.$$

Thus if a $2 - (v, k', \lambda')$ design exists, so does a $2 - (v, k, \lambda)$ design.

Example 1.39. Determine whether or not there exists a $2 - (7, 4, 2)$ design.

Solution.

1.10 Symmetric Designs

Definition 1.40. A $2 - (v, k, \lambda)$ design is called **symmetric** if $b = v$. [We should really call this a ‘square’ design, but the term ‘symmetric’ is too well established to change.]

Remark 1.41. *If $v = k$, then there is only one block (i.e., $b = 1$). Therefore, in assuming $v \geq 2$ for all designs, we have $v > k$ for all symmetric designs.*

Theorem 1.42. *In a symmetric design*

(a) $k = r$;

(b) $\lambda = \dfrac{k(k-1)}{v-1}$ *and* $v = \dfrac{k(k-1)}{\lambda} + 1$, *so* $\lambda \big| k(k-1)$.

Proof.

□

Example 1.43. Show that a symmetric $2-(v,8,1)$ design is a projective plane.

Solution.

Theorem 1.44. *If there exists a symmetric* $2-(v,k,\lambda)$ *design with* v *even, then* $k-\lambda$ *must be a perfect square.*

Proof. Later. ☐

Example 1.45. Show that there is no $2-(22,7,2)$ design.

Solution.

Theorem 1.46. *If there exists a symmetric* $2-(v,k,\lambda)$ *design with* v *odd, then the equation*

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

has a solution in integers x, y, z not all 0.

Proof. Not given. ☐

Theorem 1.47. *A (finite) projective plane is a symmetric design with an odd number of points.*

Proof. Recall that a projective plane is a $2-(n^2+n+1, n+1, 1)$ design. The number of blocks is given by

$$b = \lambda \frac{v(v-1)}{k(k-1)} = 1 \times \frac{(n^2+n+1)(n^2+n)}{(n+1)n} = n^2+n+1$$

so the design is symmetric. Note that $n^2+n = n(n+1)$ is always even, so $v = n^2+n+1$ is always odd. ☐

Example 1.48. There is no projective plane of order 14. [Recall that a projective plane of order 14 would be a (symmetric) $2-(211, 15, 1)$ design.]

Solution.

1.11 Matrix Multiplication, J-Matrices and Determinants

We begin this section with a reminder about matrix multiplication. As we shall discover, it is often useful to have general arguments regarding matrix multiplication, and for that it is useful to have a general description.

Definition 1.49. Suppose that $A = [a_{ij}]$ is an $m \times n$ matrix and that $B = [b_{ij}]$ is an $n \times p$ matrix. Then the product $C = AB$ is an $m \times p$ matrix, $C = [c_{ij}]$, with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

All this means is that c_{ij} is the result of multiplying the i'th row of A by the j'th column of B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

so

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example 1.50. Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Find $A^T A$. [Recall that A^T (the transpose of A) is obtained from A by switching rows and columns (1st row becomes 1st column and vice-versa).]

Solution.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

We shall have occasion to study matrices for which the entries in each row add up to the same sum. We can use J-matrices to express this property as a matrix equation.

Definition 1.51. For each $n, p \geq 1$, we let J_{np} denote the $n \times p$ matrix all of whose entries are equal to 1.

Example 1.52.

$$J_{23} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Theorem 1.53. *Let A be an $m \times n$ matrix. Then the following statements are equivalent.*

(1) The entries in each row of A add up to r.

(2) $AJ_{np} = rJ_{mp}$ for some $p \geq 1$.

(3) $AJ_{np} = rJ_{mp}$ for every $p \geq 1$.

Proof. Let the row-sums of A be denoted by $r_1, r_2, \ldots, r_m$. [That is, if $A = (a_{ij})$, then $r_i = a_{i1} + a_{i2} + \cdots + a_{in}$, the sum of the entries in row i of A.] Then

$$AJ_{np} = \begin{bmatrix} r_1 & r_1 & \ldots & r_1 \\ r_2 & r_2 & \ldots & r_2 \\ \ldots & \ldots & \ldots & \ldots \\ r_m & r_m & \ldots & r_m \end{bmatrix}_{m \times p}.$$

Also

$$rJ_{mp} = \begin{bmatrix} r & r & \ldots & r \\ r & r & \ldots & r \\ \ldots & \ldots & \ldots & \ldots \\ r & r & \ldots & r \end{bmatrix}_{m \times p}.$$

Each of the statements is equivalent to saying that $r = r_1 = r_2 = \cdots = r_m$.

☐

Theorem 1.53 has a counterpart for columns:

Theorem 1.54. *Let A be an $m \times n$ matrix. Then the following statements are equivalent.*

(1) The entries in each column of A add up to c;

(2) $J_{pm}A = cJ_{pn}$ for some *$p \geq 1$;*

(3) $J_{pm}A = cJ_{pn}$ for every *$p \geq 1$.*

Proof. Recall that for any two matrices X and Y, we can write $(XY)^T$ (the transpose of XY) as Y^TX^T. Observe that $J_{pm}^T = J_{mp}$. Then

- (1) is equivalent to: the entries in each row of A^T add up to c.
- (2) is equivalent to: $A^TJ_{mp} = cJ_{np}$ for some $p \geq 1$.
- (2) is equivalent to: $A^TJ_{mp} = cJ_{np}$ for every $p \geq 1$.

It is now clear the Theorem is an immediate consequence of Theorem 1.53. ☐

We now recall some properties of determinants and ways of calculating them.

Fact 1.55. *(a) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then* $\det A = ad - bc$.

(b) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then $\det A = a(ei - hf) - b(di - gf) + c(dh - ge)$.

(c) We can build up to larger matrices by 'expanding' along the first row. Suppose we have a 4×4 matrix A, with first row $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$ and suppose that we denote by A_{1j} the 3×3 matrix obtained by deleting the first row and j'th column. Then

$$\det A = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} - a_{14}A_{14}.$$

(d) In fact we can expand by any row as long as we use an appropriate pattern of signs. The pattern $+ - +-$ used above comes from the first row of a sign matrix $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$. The same applies to any column. In each case

the 3×3 determinants arise by deleting the row and column through the appropriate entry. Thus

$$\begin{aligned}\det A &= -a_{21}A_{21} + a_{22}A_{22} - a_{23}A_{23} + a_{24}A_{24}\\ &= -a_{14}A_{14} + a_{24}A_{24} - a_{34}A_{34} + a_{44}A_{44}.\end{aligned}$$

(e) *The $n \times n$ sign matrix is* $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$.

(f) *The determinant of a diagonal matrix (i.e., one in which all the non-(leading) diagonal entries are 0) is the product of the diagonal entries.*

(g) *The determinant of an upper or lower triangular matrix (i.e., one in which all the entries below or above the leading diagonal are 0) is the product of the diagonal entries.*

(h) *Any matrix with a row of 0s or with a column of 0s has determinant 0 (because expansion along that row or column must give 0.*

(i) *The following operations do not change the value of the determinant:*

  (1) *To any row, add or subtract a multiple of another row.*

  (2) *To any column, add or subtract a multiple of another column.*

(j) *Using these row and column operations, we can 'reduce' a matrix to upper or lower triangular form.*

Examples 1.56. (a) Find the determinant of $A = \begin{bmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{bmatrix}$ by expanding along an appropriate row or column.

Solution. Expand along the third column. We get $0(dh - ge) - 0(ah - gb) + 0(ae - db) = 0$.

(b) Find the determinant of the following matrix, using row and column reduction to triangular form:

$$M = \begin{bmatrix} 6 & 3 & 3 & 3 & 3 \\ 3 & 6 & 3 & 3 & 3 \\ 3 & 3 & 6 & 3 & 3 \\ 3 & 3 & 3 & 6 & 3 \\ 3 & 3 & 3 & 3 & 6 \end{bmatrix}.$$

Solution. Subtract Column 1 from each of Columns $2, 3, 4, 5$ in turn (four operations):

$$\det M = \begin{vmatrix} 6 & 3 & 3 & 3 & 3 \\ 3 & 6 & 3 & 3 & 3 \\ 3 & 3 & 6 & 3 & 3 \\ 3 & 3 & 3 & 6 & 3 \\ 3 & 3 & 3 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6 & -3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 6 & 3 & 3 \\ 3 & 0 & 3 & 6 & 3 \\ 3 & 0 & 3 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6 & -3 & -3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 0 & 0 & 6 & 3 \\ 3 & 0 & 0 & 3 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & -3 & -3 & -3 & 3 \\ 3 & 3 & 0 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 0 & 3 & 3 \\ 3 & 0 & 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 6 & -3 & -3 & -3 & -3 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 3 \end{vmatrix}.$$

Now add each of Rows $2, 3, 4, 5$ to Row 1 in turn (four operations):

$$\det M = \begin{vmatrix} 9 & 0 & -3 & -3 & -3 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 12 & 0 & 0 & -3 & -3 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 15 & 0 & 0 & 0 & -3 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 18 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 3 \end{vmatrix}.$$

We arrive at the determinant of a lower triangular matrix, so

$$\det M = 18 \times 3^4 = 1458.$$

1.12 Incidence Matrices

We shall associate a matrix with each $2-(v,k,\lambda)$ block design. This will allow us to bring our knowledge of matrix theory to bear on our study of designs.

Definition 1.57. Let $(X, \mathcal{B})$ be a $2-(v,k,\lambda)$ design with $b = |\mathcal{B}|$ and with each point of X lying in r blocks. Label the points of X: $P_1, P_2, \ldots, P_v$ and the blocks of $\mathcal{B}$: $B_1, B_2, \ldots, B_b$ in some order. The **incidence matrix** of the design $2-(v,k,\lambda)$ is the matrix $A = [a_{ij}]_{b\times v}$ where $a_{ij} = 1$ if $P_j \in B_i$ and $a_{ij} = 0$ if $P_j \notin B_i$.

Thus row i tells you which points are in B_i (it has k $1s$ and $(v-k)$ 0s). Similarly, column j tells you which blocks P_j belongs to (it has r $1s$ and $(b-r)$ $0s$).

Remark 1.58.

Example 1.59. Write down an incidence matrix for the $2-(7,3,1)$ design in Example 1.6.

Theorem 1.60. *A $b \times v$ $\{0,1\}$ matrix is the incidence matrix of a $2-(v,k,\lambda)$ design if and only if*

(a) each row has k $1s$;

(b) all rows are different;

(c) given any two columns, there are exactly λ rows with a 1 in both columns.

Proof.

□

Remark 1.61. *By Theorem 1.14, each column of an incidence matrix has* r $1s$*, with all other entries* 0*, where* $r = \dfrac{bk}{v}$.

Example 1.62. Show that there is a $2-(6,3,2)$ design and that it is unique up to the labelling of points and blocks.

Solution

Also remember that we can permute the columns (renumber the points) and permute the columns (renumber the blocks) without altering the design. We shall talk in terms of choosing labels for points and columns. This is something we can do when we have a collection of points or blocks that are indistinguishable up to the stage of choosing labels, but not when they are distinguishable. We write A for the incidence matrix.

(A) Choose a block and label it B_1, and label the three points in it P_1, P_2, P_3. Then $a_{11} = a_{12} = a_{13} = 1$ and $a_{14} = a_{15} = a_{16} = 0$. [At this stage, B_1 is distinguishable from the other blocks (rows) but $B_2, \ldots, B_{10}$ are indistinguishable. The points P_1, P_2, P_3 are indistinguishable from each other, as are P_4, P_5, P_6, but each of P_1, P_2, P_3 is distinguishable from each of P_4, P_5, P_6.]

(B) Consider the point P_1. Now P_1 lies in 5 blocks: B_1 and a further 4 blocks that we label B_2, B_3, B_4, B_5. Thus $a_{i1} = 1$ for $2 \leq i \leq 5$ and $a_{i1} = 0$ for $6 \leq i \leq 10$. [At this stage, P_1 is distinguishable from P_2, P_3, and B_1 is distinguishable from B_2, B_3, B_4, B_5, but P_2, P_3 are indistinguishable, as are B_2, B_3, B_4, B_5.]

(C) Consider the point P_2. It lies together with P_1 in 2 blocks: B_1 and one other amongst B_2, B_3, B_4, B_5. We choose to label as B_2 the one containing P_2. This means that now P_2 does not lie in B_3, B_4, B_5. Thus $a_{i2} = 1$ for $i = 1, 2$ and $a_{i2} = 0$ for $i = 3, 4, 5$. [At this stage B_3, B_4, B_5 are still indistinguishable.]

(D) Consider further the point P_2. It lies in 5 blocks: B_1, B_2 and a further 3 blocks that cannot be B_3, B_4, B_5 and that we therefore label B_6, B_7, B_8. Thus $a_{i2} = 1$ for $i = 6, 7, 8$ and $a_{i2} = 0$ for $i = 9, 10$.

(E) The block B_2 must contain 1 more point, but this point cannot be P_3 because $B_2 \neq B_1$. Therefore $a_{23} = 0$. We label as P_4 the other point in B_2. Thus $a_{24} = 1$ and $a_{25} = a_{26} = 0$.

(F) Consider the point P_3. It lies together with P_1 in 2 blocks: B_1 and one other amongst B_3, B_4, B_5. We choose to label as B_3 the one containing P_3. This means that now P_3 does not lie in B_4, B_5. Thus $a_{33} = 1$ and $a_{i3} = 0$ for $i = 4, 5$. [At this stage B_4, B_5 are still indistinguishable.]

(G) P_3 also lies together with P_2 in 2 blocks: B_1 and one other amongst B_6, B_7, B_8. We choose to label as B_6 the one containing P_3. This means that now P_3 does not lie in B_7, B_8. Thus $a_{36} = 1$ and $a_{i3} = 0$ for $i = 7, 8$. [At this stage B_7, B_8 are still indistinguishable.]

(H) Consider further the point P_3. It lies in 5 blocks: B_1, B_3, B_6 and a further 2 blocks that can only be B_9, B_{10}. Thus $a_{i3} = 1$ for $i = 9, 10$.

(I) Rows 4 and 5 cannot both start 1000 (otherwise the remaining 2 1s will be in cols 5 and 6, and we would have 2 identical rows). Therefore either $a_{44} = 1$ or $a_{54} = 1$, i.e., at least one of B_4, B_5 contains P_4. We choose to label by B_4 so that it contains P_4. Thus $a_{44} = 1$.

(J) Apply the same argument to rows 7 and 8. Label so that B_7 contains P_4, i.e., $a_{74} = 1$.

(K) Apply the same argument to rows 9 and 10. Label so that B_9 contains P_4, i.e., $a_{94} = 1$.

(L) We already have 2 common 1s in columns 1 and 4, i.e., P_1, P_4 already lie in two blocks (B_2, B_4) , so P_4 cannot lie in B_3 or B_5, i.e., $a_{34} = a_{54} = 0$.

(M) We already have 2 common 1s in columns 2 and 4, i.e., P_2, P_4 already lie in two blocks (B_2, B_7) , so P_4 cannot lie in B_6 or B_8, i.e., $a_{64} = a_{84} = 0$.

(N) Column 4 must have 5 1s, so $a_{10\ 4} = 1$.

(O) Each row must have 3 1s, so rows $5, 8$ must be completed with 1s. Thus $a_{55} = a_{56} = a_{85} = a_{86} = 1$.

(P) Block 3 has one more point, either P_5 or P_6. We choose to label P_5 as the point in B_3. Thus $a_{35} = 1, a_{36} = 0$.

(Q) Blocks 9 and 10 each have one more point, but it cannot be the same point, so one contains P_5 and the other P_6. We choose to label B_9 as the block containing P_5. Thus $a_{95} = a_{10\ 6} = 1, a_{96} = a_{10\ 5} = 0$.

(R) Points P_2, P_6 lie in 2 blocks: B_5 is one, B_4 is the only possibility for the other. This means that B_4 now has 3 points, so P_5 is not in B_4. Thus $a_{45} = 0, a_{46} = 1$.

(S) Points P_4, P_5 lie in 2 blocks: B_9 is one, B_7 is the only possibility for the other. This means that B_7 now has 3 points, so P_6 is not in B_7. Thus $a_{75} = 1, a_{76} = 0$.

(T) Each column must have 5 1s, so we must have $a_{65} = 0, a_{66} = 1$, i.e., B_6 contains P_6 but not P_5.

If we followed these instructions correctly we should have the incidence matrix:

$$\left[\begin{array}{c|c|c|c|c|c}
1_A & 1_A & 1_A & 0_A & 0_A & 0_A \\ \hline
1_B & 1_C & 0_E & 1_E & 0_E & 0_E \\ \hline
1_B & 0_C & 1_F & 0_L & 1_P & 0_P \\ \hline
1_B & 0_C & 0_F & 1_I & 0_R & 1_R \\ \hline
1_B & 0_C & 0_F & 0_L & 1_O & 1_O \\ \hline
0_B & 1_D & 1_G & 0_M & 0_T & 1_T \\ \hline
0_B & 1_D & 0_G & 1_J & 1_S & 0_S \\ \hline
0_B & 1_D & 0_G & 0_M & 1_O & 1_O \\ \hline
0_B & 0_D & 1_H & 1_K & 1_Q & 0_Q \\ \hline
0_B & 0_D & 1_H & 1_N & 0_Q & 1_Q
\end{array}\right]$$

We can now verify by inspection that we have a $2 - (6, 3, 2)$ design: check that

The design is unique in that the only choices of label that we have made have been amongst points or blocks that were, at that time, indistinguishable.

Theorem 1.63. *Let* $A = [a_{ij}]$ *be a* $\{0, 1\}$ *matrix of dimension* $b \times v$ *and let* $M = [m_{ij}]_{v \times v}$ *be the matrix* A^TA*. Then* m_{ij} *is the number of occasions when the* i*'th and* j*'th columns of* A *both have entry* 1*. In particular,* m_{ii} *is the number of* 1*s in the* i*'th column of* A*.*

Proof. By the definition of matrix multiplication, m_{ij} is the result of multiplying the i'th row of A^T by the j'th column of A:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{b1} \\ a_{12} & a_{22} & \dots & a_{b2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1v} & a_{2v} & \dots & a_{bv} \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1v} \\ a_{21} & a_{22} & \dots & a_{2v} \\ \vdots & \vdots & \dots & \vdots \\ a_{b1} & a_{b2} & \dots & a_{bv} \end{bmatrix}$$

so

$$m_{ij} = \begin{bmatrix} a_{1i} & a_{2i} & \dots & a_{bi} \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{bj} \end{bmatrix} = \sum_{k=1}^{b} a_{ki}a_{kj}.$$

□

Theorem 1.64. *Let* $A = [a_{ij}]$ *be a* $\{0,1\}$ *matrix of dimension* $b \times v$ *and let* $M = [m_{ij}]_{v\times v}$ *be the matrix* $A^T A$.

(a) If A *is the incidence matrix of a* $2-(v,k,\lambda)$ *design, then* $M = (r-\lambda)I_v + \lambda J_{vv}$.

(b) If A *has distinct rows, each with* $k < v$ *entries equal to 1, and if* $M = (r-\lambda)I_v + \lambda J_{vv}$*, then* A *is the incidence matrix of a* $2-(v,k,\lambda)$ *design, where* $k = \dfrac{vr}{b}$.

Proof. (a)

(b)

□

Theorem 1.65. *Let* A *be an incidence matrix for a* $2-(v,k,\lambda)$ *design and let* $M = A^TA$. *Then* $\det M = (r-\lambda)^{v-1}(r+(v-1)\lambda)$.

Proof. By Theorem 1.64(a),

$$M = \begin{bmatrix} r & \lambda & \lambda & \dots & \dots & \lambda & \lambda \\ \lambda & r & \lambda & \dots & \dots & \lambda & \lambda \\ \lambda & \lambda & r & \dots & \dots & \lambda & \lambda \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \lambda & \lambda & \lambda & \dots & \dots & \lambda & r \end{bmatrix}.$$

We can perform row and column operations on M that do not change the determinant. First we subtract Column 1 from each of the other columns to give

$$\begin{bmatrix} r & \lambda - r & \lambda - r & \dots & \dots & \lambda - r & \lambda - r \\ \lambda & r - \lambda & 0 & \dots & \dots & 0 & 0 \\ \lambda & 0 & r - \lambda & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \lambda & 0 & 0 & \dots & \dots & 0 & r - \lambda \end{bmatrix}.$$

Second we add Row 2 to Row 1, then add each subsequent row in turn, to give

$$B = \begin{bmatrix} r + (v-1)\lambda & 0 & 0 & \dots & \dots & 0 & 0 \\ \lambda & r - \lambda & 0 & \dots & \dots & 0 & 0 \\ \lambda & 0 & r - \lambda & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \lambda & 0 & 0 & \dots & \dots & 0 & r - \lambda \end{bmatrix}.$$

Now B is a lower triangular matrix, so its determinant is given by the product of the diagonal entries. Thus

$$\det M = \det B = (r - \lambda)^{v-1}(r + (v-1)\lambda).$$

□

Proof of Theorem 1.34, Fisher's Inequality. We need to prove that if $v > k$, then $b \geq v$. Recall that, by Corollary 1.15, we have $r > \lambda$.

Suppose that $(X, \mathcal{B})$ is a $2 - (v, k, \lambda)$ design with $v > b$ and let A be an incidence matrix for the design. Let A^* be the $v \times v$ matrix formed from A by adding $v - b$ rows of zeros to the bottom of A. Thus

$$A^* = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1v} \\ a_{21} & a_{22} & \dots & a_{2v} \\ \vdots & \vdots & \dots & \vdots \\ a_{b1} & a_{b2} & \dots & a_{bv} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Observe that

$$(A^*)^T A^* = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{b1} & 0 & \dots & 0 \\ a_{12} & a_{22} & \dots & a_{b2} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{1v} & a_{2v} & \dots & a_{bv} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1v} \\ a_{21} & a_{22} & \dots & a_{2v} \\ \vdots & \vdots & \dots & \vdots \\ a_{b1} & a_{b2} & \dots & a_{bv} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

If we write $N = (n_{ij})$ for $(A^*)^T A^*$ and $M = (m_{ij})$ for $A^T A$, then N and M are both $v \times v$ matrices and

$$n_{ij} = \begin{bmatrix} a_{1i} & a_{2i} & \dots & a_{bi} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{bj} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{k=1}^{b} a_{ki} a_{kj} = m_{ij}.$$

Therefore $(A^*)^T A^* = A^T A$.

□

1.13 Symmetric Designs

We return to studying symmetric designs (where $b = v$). We use matrix tools to prove Theorem 1.44.

Proof of Theorem 1.44. Recall that we need to prove: In a symmetric $2-(v,k,\lambda)$ design with v even, $k-\lambda$ must be a perfect square.

□

Theorem 1.66. *If A is an incidence matrix for a symmetric $2-(v,k,\lambda)$ design, then A is invertible.*

Proof.

□

Theorem 1.67 (Ryser)**.** *In a symmetric $2-(v,k,\lambda)$ design, any two distinct blocks have precisely λ points in common.*

Proof. Let A be an incidence matrix for the design. Then, by Theorem 1.64, $A^TA = (r-\lambda)I_v + \lambda J_{vv}$. By Theorem 1.66, we know that A^{-1} exists. By Theorem's 1.53 and 1.54, we know that $AJ_{vv} = kJ_{vv}$ and $J_{vv}A = rJ_{vv} = kJ_{vv} = AJ_{vv}$. Therefore

$$(A^T)^TA^T = AA^T = A(A^TA)A^{-1} = A((r-\lambda)I_v + \lambda J_{vv})A^{-1}$$

$$= (r-\lambda)AI_vA^{-1} + \lambda AJ_{vv}A^{-1} = (r-\lambda)I_v + \lambda J_{vv}AA^{-1} = (r-\lambda)I_v + \lambda J_{vv}.$$

We now apply Theorem 1.63 to $(A^T)^TA^T$. In this context, $M = A^TA$ is also $(A^T)^TA^T$, and therefore m_{ij} is the number of occasions when the i'th and j'th columns of A^T both have entry 1. For $i \neq j$, it follows that $m_{ij} = \lambda$ is the number of occasions when the i'th and j'th rows of A both have entry 1, i.e., it is the number of points that lie in both of blocks i and j. □

Example 1.68. Show that there is a projective plane of order 3.

Solution Such a design would be a symmetric $2-(13,4,1)$ design. We look for a suitable incidence matrix. Assuming that a design exists, we have

- $v=b=13,\ k=r=4,\ \ \lambda=1.$

- Any two blocks have exactly one point in common.

Translating this information into matrix language, we see that we seek a 13×13 $\{0,1\}$ matrix such that:

(a) Each row has 4 $1s$ and 9 $0s$.

(b) All rows are different.

(c) Each column has 4 $1s$ and 9 $0s$.

(d) Any two columns have common 1s in exactly 2 rows.

(e) Any two rows have common 1s in exactly 2 columns.

Also remember that we can permute the columns (renumber the points) and permute the columns (renumber the blocks) without altering the design. We shall talk in terms of choosing labels for points and columns. This is something we can do when we have a collection of points or blocks that are indistinguishable up the stage of choosing labels, but not when they are distinguishable. We write A for the incidence matrix.

(1) Choose a block and label it B_1, and label the three points in it P_1, P_2, P_3, P_4. Then $a_{1j} = 1$ for $1 \leq j \leq 4$ and $a_{1j} = 1$ for $5 \leq j \leq 13$.

(2) Consider the point P_1. Now P_1 lies in 4 blocks: B_1 and a further 3 blocks that we label B_2, B_3, B_4. Thus $a_{i1} = 1$ for $2 \leq i \leq 4$ and $a_{i1} = 0$ for $5 \leq i \leq 13$.

(3) B_1 meets each of B_2, B_3, B_4 in exactly one point, namely P_1, so $a_{ij} = 0$ for $2 \leq i, j \leq 4$.

(4) B_2 contains 3 more points that we label P_5, P_6, P_7. Thus $a_{2j} = 1$ for $5 \leq j \leq 7$ and $a_{2j} = 0$ for $8 \leq j \leq 13$.

(5) B_2 meets each of B_3, B_4 in exactly one point, namely P_1, so $a_{ij} = 0$ for $i = 3, 4$, $5 \leq j \leq 7$.

(6) B_3 contains 3 more points that we label P_8, P_9, P_{10}. Thus $a_{3j} = 1$ for $8 \leq j \leq 10$ and $a_{3j} = 0$ for $11 \leq j \leq 13$.

(7) B_3 meets B_4 in exactly one point, namely P_1, so $a_{4j} = 0$ for $8 \leq j \leq 10$.

(8) B_4 contains 3 more points that can only be P_{11}, P_{12}, P_{13}. Thus $a_{4j} = 1$ for $11 \leq j \leq 13$.

(9) P_2 lies in 4 blocks: B_1 and a further 3 blocks that we label B_5, B_6, B_7. Thus $a_{i2} = 1$ for $i = 5, 6, 7$ and $a_{i2} = 0$ for $8 \leq i \leq 13$.

(10) B_1 meets each of B_5, B_6, B_7 in exactly one point, namely P_2, so $a_{ij} = 0$ for $5 \leq i \leq 7, j = 3, 4$.

(11) P_3 lies in 4 blocks: B_1 and a further 3 blocks that we label B_8, B_9, B_{10}. Thus $a_{i3} = 1$ for $8 \leq i \leq 13$ and $a_{i3} = 0$ for $11 \leq i \leq 13$.

(12) B_1 meets each of B_8, B_9, B_{10} in exactly one point, namely P_3, so $a_{i4} = 0$ for $8 \leq i \leq 10$.

(13) P_4 lies in 4 blocks: B_1 and a further 3 blocks that can only be B_{11}, B_{12}, B_{13}. Thus $a_{i4} = 1$ for $11 \leq i \leq 13$.

(14) P_2 and P_5 lie together in 1 block. This must be one of B_5, B_6, B_7. These blocks are currently indistinguishable. We choose to label as B_5 the one that contains P_5. Thus $a_{55} = 1$ and $a_i5 = 0$ for $i = 5, 6$.

(15) P_3 and P_5 lie together in 1 block. This must be one of B_8, B_9, B_{10}. These blocks are currently indistinguishable. We choose to label as B_8 the one that contains P_5. Thus $a_{58} = 1$ and $a_{i5} = 0$ for $i = 9, 10$.

(16) P_4 and P_5 lie together in 1 block. This must be one of B_{11}, B_{12}, B_{13}. These blocks are currently indistinguishable. We choose to label as B_{11} the one that contains P_5. Thus $a_{5\ 11} = 1$ and $a_{i5} = 0$ for $i = 12, 13$.

(17) B_5 meets B_2 in exactly one point, namely P_5, so $a_{5j} = 0$ for $j = 6, 7$.

(18) P_2 and P_6 lie together in 1 block. This must be one of B_6, B_7. These blocks are currently indistinguishable. We choose to label as B_6 the one that contains P_5. Thus $a_{66} = 1$ and $a_{76} = 0$.

(19) B_6 meets B_2 in exactly one point, namely P_6, so $a_{67} = 0$.

(20) P_2 and P_7 lie together in 1 block. This can only be B_7. Thus $a_{77} = 1$.

(21) B_8 meets B_2 in exactly one point, namely P_5, so $a_{8j} = 0$ for $j = 6, 7$.

(22) P_3 and P_6 lie together in 1 block. This must be one of B_9, B_{10}. These blocks are currently indistinguishable. We choose to label as B_9 the one that contains P_6. Thus $a_{96} = 1$ and $a_{10\ 6} = 0$.

(23) B_9 meets B_2 in exactly one point, namely P_6, so $a_{97} = 0$.

(24) P_3 and P_7 lie together in 1 block. This can only be B_{10}. Thus $a_{10\ 7} = 1$.

(25) B_{11} meets B_2 in exactly one point, namely P_5, so $a_{11\ j} = 0$ for $j = 6, 7$.

(26) P_4 and P_6 lie together in 1 block. This must be one of B_{12}, B_{13}. These blocks are currently indistinguishable. We choose to label as B_{12} the one that contains P_6. Thus $a_{12\ 6} = 1$ and $a_{13\ 6} = 0$.

(27) B_{12} meets B_2 in exactly one point, namely P_6, so $a_{12\ 7} = 0$.

(28) P_4 and P_7 lie together in 1 block. This can only be B_{13}. Thus $a_{13\ 7} = 1$.

(29) B_3 and B_5 meet in one point. This must be one of P_8, P_9, P_{10}. These points are currently indistinguishable. We choose to label as P_8 the one that lies in B_5. Thus $a_{85} = 1$ and $a_{5j} = 0$ for $j = 9, 10$.

(30) B_5 meets each of B_6 and B_7 in one point, namely P_2, so P_8 is not in B_6, B_7. Thus $a_{i8} = 0$ for $i = 6, 7$.

(31) B_3 and B_6 meet in one point. This must be one of P_9, P_{10}. These points are currently indistinguishable. We choose to label as P_9 the one that lies in B_6. Thus $a_{69} = 1$ and $a_{6\ 10} = 0$.

(32) B_6 meets B_7 in one point, namely P_2, so P_9 is not in B_7. Thus $a_{79} = 0$.

(33) B_3 and B_7 meet in a point and this can only be P_{10}. Thus $a_{7\ 10} = 1$.

(34) B_4 and B_5 meet in one point. This must be one of P_{11}, P_{12}, P_{13}. These points are currently indistinguishable. We choose to label as P_{11} the one that lies in B_5. Thus $a_{5\ 11} = 1$ and $a_{5j} = 0$ for $j = 12, 13$.

(35) B_5 meets each of B_6 and B_7 in one point, namely P_2, so P_{11} is not in B_6, B_7. Thus $a_{i\ 11} = 0$ for $i = 6, 7$.

(36) B_4 and B_6 meet in one point. This must be one of P_{12}, P_{13}. These points are currently indistinguishable. We choose to label as P_{12} the one that lies in B_6. Thus $a_{6\ 12} = 1$ and $a_{6\ 13} = 0$.

(37) B_6 meets B_7 in one point, namely P_2, so P_{12} is not in B_7. Thus $a_{7\ 12} = 0$.

(38) B_4 and B_7 meet in a point and this can only be P_{13}. Thus $a_{7\ 13} = 1$. [Alternatively, B_7 has one more point.]

(39) B_5 and B_8 meet in one point, namely P_5, so P_8 and P_{11} do not lie in B_8.

(40) B_6 and B_9 meet in one point, namely P_6, so P_9 and P_{12} do not lie in B_9.

(41) B_7 and B_{10} meet in one point, namely P_7, so P_{10} and P_{13} do not lie in B_{10}.

(42) B_3 and B_8 meet in one point, either P_9 or P_{10}. We choose P_9. At this stage, P_9 and P_{10} *are* distinguishable, so in making this choice we hope to be able to complete the matrix and show the existence of a design, but the choice is not just one of labelling so the resulting design is not necessarily unique.

(43) B_8 and B_{10} meet in one point, namely P_3, so P_9 does not lie in B_{10}.

(44) B_3 and B_{10} meet in one point, this can only be P_8.

(45) B_9 and B_{10} meet in one point, namely P_3, so P_8 does not lie in B_9.

(46) B_3 and B_9 meet in one point, this can only be P_{10}.

(47) B_6 and B_8 meet in one point, namely P_9, so P_{12} does not lie in B_8.

(48) B_7 and B_9 meet in one point, namely P_{10}, so P_{13} does not lie in B_9.

(49) B_5 and B_{10} meet in one point, namely P_8, so P_{11} does not lie in B_{10}.

(50) B_8, B_9, B_{10} each need one more point, but in each case only one is possible: B_8 contains P_{13}, B_9 contains P_{11} and B_{10} contains P_{12}.

(51) B_5 and B_{11} meet in one point, namely P_5, so P_8 does not lie in B_{11}.

(52) B_8 and B_{11} meet in one point, namely P_5, so P_9 does not lie in B_{11}.

(53) B_3 and B_{11} meet in one point, this can only be P_{10}.

(54) B_9 and B_{11} meet in one point, namely P_{10}, so P_{11} does not lie in B_{11}.

(55) B_8 and B_{11} meet in one point, namely P_5, so P_{13} does not lie in B_{11}.

(56) B_4 and B_{11} meet in one point, this can only be P_{12}.

(57) B_{11} meets each of B_{12}, B_{13} in P_4, so P_{10} and P_{12} do not lie in B_{12}, B_{13}.

(58) B_{10} and B_{13} meet in one point, namely P_7, so P_8 does not lie in B_{13}.

(59) B_3 and B_{13} meet in one point, this can only be P_9.

(60) P_8 must lie in one more block, and this can only be B_{12}.

(61) P_9 already lies in 4 blocks, so P_9 does not lie in B_{12}.

(62) B_5 and B_{12} meet in one point, namely P_8, so P_{11} does not lie in B_{12}.

(63) P_{11} must lie in one more block, and this can only be B_{13}.

(64) B_{12} has one more point and this can only be P_{13}.

(65) P_{13} already lies in 4 blocks, so P_{13} does not lie in B_{13}.

This gives the incidence matrix

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right]$$

We have not shown that the design is unique.

Remarks 1.69. (a) Observe that the matrix we have obtained is square, but it is not symmetric (this is why the term 'symmetric' for such designs is misleading).

(b) In fact, there is a unique $2-(13,4,1)$ design. To see this we have to note that the only arbitrary choice of label was in line (42). If we had chosen the other way, we should have arrived at another matrix (actually the transpose of the given one). We can pass from one to the other by the following row and column operations (performed in the given order): $C_9 \leftrightarrow C_{10}$, $R_6 \leftrightarrow R_7$, $C_6 \leftrightarrow C_7$, $R_9 \leftrightarrow R_{10}$, $R_{12} \leftrightarrow R_{13}$, $C_{12} \leftrightarrow C_{13}$.

(c) When we made the arbitrary choice, conceivably we could have arrived at no allowable matrix. In that case we would have returned to line (42) and made the other choice.

(d) If we find that we cannot construct an allowable matrix, then we would conclude that no design existed with the given parameters.

2 Geometry Of The Projective Plane

In 1812 Napoleon invaded Russia. Unlike Hitler 129 years later, he succeeded in capturing Moscow. On the other hand, like Hitler, he found that the Russian winter was a more formidable enemy than the Russian army. His Russian adventure turned into a disaster. Among the French officers captured by the Russians was one J.V. Poncelet. To pass the time in his Russian jail he decided to write down all the geometry that he could remember. This led him to invent Projective Geometry ....

2.1 The Euclidean Plane $\mathbb{E}$

Poncelet's starting point was the familiar Euclidean plane. The Euclidean plane consists of points $P = (\alpha, \beta)$, where α and β are real numbers (these are cartesian coordinates).

Definition 2.1. A **line** in the Euclidean plane consists of all points (x, y) satisfying an equation of the form

$$ax + by + c = 0 \qquad (*)$$

where $(a, b) \neq (0, 0)$. (When we write $(a, b) \neq (0, 0)$ we mean that either $a \neq 0$ or $b \neq 0$ or both.)

The **gradient** of (*) is defined to be

$$-\frac{a}{b} \text{ if } b \neq 0$$

and to be

$$\infty \text{ if } b = 0.$$

Note:

- The symbol ∞ is just that: a symbol. It is not the result of a calculation $\Delta\frac{a}{0}$ and it is neither positive nor negative.
- If $b \neq 0$ then (*) becomes $\Delta y = -\frac{a}{b}x - \frac{c}{b}$; if $b = 0$ it becomes $\Delta x = -\frac{c}{a}$.

Definition 2.2. Distinct lines

$$\begin{aligned} ax + by + c &= 0 \\ Ax + By + C &= 0 \end{aligned}$$

are said to be **parallel** if $Ab = aB$.

Exercise 2.3. (a) Show that if lines $ax + by + c = 0$ and $Ax + By + C = 0$ with $Ab = aB$ have at least one point in common, then they are equal. [Thus parallel lines do not intersect.]

(b) Show that distinct non-parallel lines $ax + by + c = 0$ and $Ax + By + C = 0$ both pass through the point $\Delta\left(\frac{bC-cB}{Ba-bA}, \frac{Ac-aC}{Ba-bA}\right)$.

N.B.: This shows that lines in the Euclidean plane are parallel if and only if they do not intersect.

Theorem 2.4. *Given a pair of (distinct) points (p, q) and (u, v) in $\mathbb{E}$ there is a unique line containing them, and it has equation*

$$\begin{vmatrix} x & y & 1 \\ p & q & 1 \\ u & v & 1 \end{vmatrix} = 0.$$

Proof.

□

Example 2.5. Find the equation of the line through the points $(-5, 6)$ and $(6, -7)$.

Solution.

2.2 The Line At Infinity

$\mathbb{E}$ is awkward to handle because of special cases:

- 2 lines meet at a unique point, unless they are parallel;
- $ax + by + c = 0$ is not a line if $a = b = 0$;
- gradient has a special definition if $b = 0$.

Poncelet had the bright idea of adding "points at infinity" where parallel lines would meet. We shall see that this removes the special cases. This new projective geometry had been foreshadowed by Renaissance painters such as Leonardo da Vinci, as they strove to make paintings more realistic. If you stand in the middle of a straight railroad track stretching into the distance, you will see that the two rails appear to converge "at infinity". We shall begin by introducing homogeneous coordinates. We shall represent a point of $\mathbb{E}$ not by a pair of real numbers (x, y) but by a triple (x, y, z) where $z \neq 0$.

Definition 2.6. In $\mathbb{E}$, the triple (x, y, z), with $z \neq 0$, is a **homogeneous coordinate representation** of the point with cartesian coordinates

$$\left(\frac{x}{z}, \frac{y}{z}\right).$$

Note that every point (u, v) of $\mathbb{E}$ has homogeneous coordinates $(u, v, 1)$. Also, if $k \neq 0$ then the triples (x, y, z) and (kx, ky, kz) represent the same point.

Homogeneous Coordinates		**Cartesian Coordinates**
$(9, -33, 1)$	$\longleftrightarrow$	$(9, -33)$
$(0, 0, 1)$	$\longleftrightarrow$	$(0, 0)$
$(35, 15, 5)$	$\longleftrightarrow$	$(7, 3)$
$(7, 3, 1)$	$\longleftrightarrow$	$(7, 3)$
$(14, -21, 7)$	$\longleftrightarrow$	$(2, -3)$
$(10, -15, 5)$	$\longleftrightarrow$	$(2, -3)$
$(2, -3, 1)$	$\longleftrightarrow$	$(2, -3)$
$(0, 0, 5)$	$\longleftrightarrow$	$(0, 0)$
$(\lambda x, \lambda y, \lambda z)$	$\longleftrightarrow$	$\Delta\left(\frac{x}{z}, \frac{y}{z}\right)$ if $\lambda \neq 0$.

Thus (x, y, z) and (u, v, w) represent the same point of the Euclidean plane if and only if there exists a number λ such that $u = \lambda x$, $v = \lambda y$, $w = \lambda z$. We write simply $(x, y, z) = (u, v, w)$.

Example 2.7. We can introduce homogeneous coordinates in other planes, *e.g.*, in

$$\mathbb{C}^2 = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{C}\}.$$

Here the homogeneous coordinates take the form (ξ, η, ζ), where $\xi, \eta, \zeta \in \mathbb{C}$ and $\zeta \neq 0$, representing the point $\Delta\left(\frac{\xi}{\zeta}, \frac{\eta}{\zeta}\right) \in \mathbb{C}^2$.

We shall now extend $\mathbb{E}$ by allowing points with $z = 0$. In fact, we allow all points (x, y, z) except $(0, 0, 0)$. As before, we have $(x, y, z) = (\lambda x, \lambda y, \lambda z)$ if $\lambda \neq 0$.

Definition 2.8. Let $\mathbb{K}$ be any field. Then the **projective plane** $\mathbb{P}_2(\mathbb{K})$ consists of all triples $(x, y, z) \neq (0, 0, 0)$, with $x, y, z \in \mathbb{K}$, and where

$$(x, y, z) = (\lambda x, \lambda y, \lambda z)$$

for every $\lambda \in \mathbb{K}$ with $\lambda \neq 0$. We call (x, y, z) the **homogeneous coordinates** of a point in $\mathbb{P}_2(\mathbb{K})$.

2.3 Lines in $\mathbb{P}_2(\mathbb{K})$

Recall that the lines in $\mathbb{E}$ are of the form $ax + by + c = 0$ with $(a, b) \neq (0, 0)$, *i.e.*, if we let $x = X/Z, y = Y/Z$,

$$a\left(\frac{X}{Z}\right) + b\left(\frac{Y}{Z}\right) + c = 0$$

i.e.,

$$aX + bY + cZ = 0.$$

This prompts the following definition:

Definition 2.9. Let $(l, m, n) \neq (0, 0, 0)$. Then the points (x, y, z) of $\mathbb{P}_2(\mathbb{K})$ such that $lx + my + nz = 0$ form a **line** in $\mathbb{P}_2(\mathbb{K})$ with homogeneous coordinates

$$[l, m, n].$$

Note the use of round brackets for points and square brackets for lines.

Remarks 2.10. (a)

(b)

(c)

Summary

$\mathbb{E}$: **Points** All (x, y, z) with $z \neq 0$ (with $(x, y, z) = (\lambda x, \lambda y, \lambda z)$ whenever $\lambda \neq 0$).

Lines All $[l, m, n]$ with $(l, m) \neq (0, 0)$ (with $[l, m, n] = [\lambda l, \lambda m, \lambda n]$ whenever $\lambda \neq 0$).

$\mathbb{P}_2(\mathbb{K}) - \mathbb{E}$: **Points** All $(x, y, 0)$ with $(x, y) \neq (0, 0)$ (with $(x, y, 0) = (\lambda x, \lambda y, 0)$ whenever $\lambda \neq 0$).

Lines The line $[0,0,1]$, also known as $z=0$ (the same as $[0,0,n]$ for any $n \neq 0$).

$\mathbb{P}_2(\mathbb{K})$: **Points** All $(x,y,z) \neq (0,0,0)$ (with $(x,y,z) = (\lambda x, \lambda y, \lambda z)$ whenever $\lambda \neq 0$).

Lines All $[l,m,n] \neq [0,0,0]$ (with $[l,m,n] = [\lambda l, \lambda m, \lambda n]$ whenever $\lambda \neq 0$).

Definition 2.11. The new line $[0,0,n]$ is called l_∞, the **line at infinity**. The points that lie on it, i.e., $(x,y,0)$ with $(x,y) \neq (0,0)$, are called the **points at infinity**.

Theorem 2.12. *Any two distinct lines in a projective plane meet in exactly one point. (Thus there are no parallel lines in the projective plane.)*

Proof.

$\square$

Fact 2.13. *Suppose that A is a square matrix. If one row of A is a scalar multiple of another or if one column of A is a scalar multiple of another, then $\det A = 0$. This is because subtracting a scalar multiple of one of the rows from another (or a scalar multiple of one of the columns from another) doesn't change the determinant, but leads to a matrix with a row or column of 0s. Switching two rows or switching two columns has the consequence of multiplying the determinant by -1.*

Theorem 2.14. *Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be distinct points of $\mathbb{P}_2(\mathbb{K})$. Then there is one and only one line which passes through these points. Its equation is*

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

This is the line

$$\left[\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right].$$

Proof.

□

Definition 2.15. A set of two or more points are said to be **collinear** if they lie on a common line. A set of two or more lines are said to be **concurrent** if they pass through a common point. Any two distinct points lie on a line, so are collinear. Any two distinct lines meet in a point so are concurrent.

Theorem 2.16. *Suppose that the points* $P_1 = (x_1, y_1, z_1)$*,* $P_2 = (x_2, y_2, z_2)$ *and* $P_3 = (x_3, y_3, z_3)$ *of* $\mathbb{P}_2(\mathbb{K})$ *are not all the same. They are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Proof.

$\square$

2.4 Duality

Notice that the points and lines of $\mathbb{P}_2(\mathbb{K})$ resemble each other. Both are composed of triples with at least one nonzero component, and both are unchanged if we multiply each component by a nonzero number. In fact the lines $[l, m, n]$ of $\mathbb{P}_2(\mathbb{K})$ can be regarded as "points" in another $\mathbb{P}_2(\mathbb{K})$, called the **dual plane**. The "lines" in the dual plane are given by triples $(x, y, z) \neq (0, 0, 0)$ such that $xl + ym + zn = 0$, *i.e.*, by the points in the original plane. We can construct a dictionary to translate concepts from one plane to the other

Original Plane	**Dual Plane**
Points	Lines
Lines	Points
Two lines intersect in a point	Two points lie on a line
Two points lie on a line	Two lines intersect in a point
Points are collinear	Lines are concurrent
Lines are concurrent	Points are collinear

Principle 2.17 (Principle of Duality)**.**

Thus we have

Theorem 2.18. *Let* $[l_1, m_1, n_1]$ *and* $[l_2, m_2, n_2]$ *be distinct lines of* $\mathbb{P}_2$*. Then there is one and only one point of intersection of these two lines. It is given by the* *equation*

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

This is the point with coordinates $\Delta\left(\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}\right)$.

Proof. Apply the Principle of Duality to Theorem 2.14. ☐

Example 2.19. Find the line in $\mathbb{P}_2(\mathbb{R})$ through $(1, 1, 1)$ and $(1, 5, 6)$ and the point where this line meets $x + y + z = 0$.

Solution:

Remark 2.20. *The equation*

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

calls for comment. In what sense is it the equation of a point? Recall that the equation

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

specifies a line by telling us precisely which points (x, y, z) *lie on it. In exactly the same way, the above equation specifies a point by telling us precisely which lines* $[l, m, n]$ *pass through it.*

2.5 The Projective Planes $\mathbb{P}_2(\mathbb{K})$

Recall that if p is a prime then $\mathbb{Z}_p$ (also known as $GF(p)$) is the field consisting of the elements

$$0, 1, 2 \ldots, p-1$$

with addition and multiplication performed $\pmod p$. Thus for example, in $\mathbb{Z}_{11}$,

$$8 + 7 = 4, 8 \times 7 = 1.$$

We can therefore construct $\mathbb{P}_2(GF(p)) = \mathbb{P}_2(\mathbb{Z}_p)$, which we write simply as $\mathbb{P}_2(p)$.

Lemma 2.21. *(a)* $\mathbb{P}_2(p)$ *has* $p^2 + p + 1$ *points.*

(b) Each line contains $p + 1$ *points.*

Proof

Lemma 2.22.

(a) $\mathbb{P}_2(p)$ *has* $p^2 + p + 1$ *lines.*

(b) Each point lies on $p + 1$ *lines.*

Proof. Dualise Lemma 2.21. This completes the proof, but we prove (b) as an example of the application of the Principle of Duality.

Let (x, y, z) be a point. Suppose that $x \neq 0$ (the arguments are very similar if $y \neq 0$ or if $z \neq 0$). Then the lines $[l, m, n]$ through the point are precisely those satisfying

$$xl + ym + zn = 0.$$

There is a line through the point for each choice of m and n, with l given by $l = -(ym + zn)/x$, except for $m = n = 0$, and each point on the line can be so expressed (note that $[l, 0, 0]$ cannot pass through the point). There are p choices for m and p choices for n, so there are $p^2 - 1$ combinations of m and n (excluding $m = n = 0$). Therefore there are $p^2 - 1$ triples $[l, m, n]$ such that $xl + ym + zn = 0$. Each line is represented by $p - 1$ triples. Therefore the number of lines through the point is

$$\frac{p^2 - 1}{p - 1} = p + 1.$$

□

Theorem 2.23. *Assume that* $K = \mathbb{Z}_p$*.*

(a) If $P = (x_1, y_1, z_1)$ *and* $Q = (x_2, y_2, z_2)$ *are distinct points then the line* PQ *consists of all points of the form*

$$\lambda P + \mu Q = (\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2, \lambda z_1 + \mu z_2)$$

with $(\lambda, \mu) \neq (0, 0)$*.*

(b) The points other than Q *are precisely those of the form* $P + \lambda Q = (x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$*.*

Proof. Since P and Q are distinct, neither of (x_1, y_1, z_1), (x_2, y_2, z_2) is a scalar multiple of the other. Observe that

$$\begin{vmatrix} x_1 + \lambda x_2 & y_1 + \lambda y_2 & z_1 + \lambda z_2 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

(first subtracting λ Row 3 from Row 1, then Row 2 from Row 1).

$\square$

Remark 2.24. *The above theorem is actually true for any field, but the proof is more tricky.*

Example 2.25. In $\mathbb{P}_2(11)$, find the coordinates of the point of intersection of the line p passing through $(1, 1, 0)$ and $(2, 1, 2)$, and the line q passing through $(0, 1, 1)$ and $(2, 1, 5)$.

Solution

Example 2.26. Identify the lines of the Fano Plane having equations.

A: $z = 0$

B: $y + z = 0$

C: $y = 0$

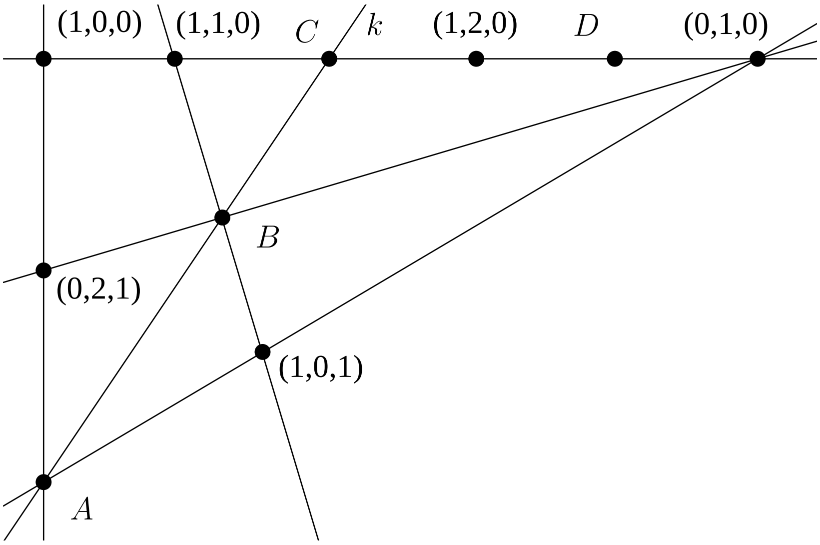
D: $x + z = 0$

E: $x = 0$

F: $x + y = 0$

G: $x + y + z = 0$.

Example 2.27. Identify the marked points and line in $\mathbb{P}_2(5)$ in the following diagram.



Solution.

Theorem 2.28. *The points of* $\mathbb{P}_2(p)$*, with lines as blocks, form a* $2-(p^2+p+1, p+1, 1)$ *design.*

Proof

Examples 2.29. (a) Take $p = 2$. Then we get a $2-(7,3,1)$ design.

(b) Take $p = 3$. Then we get a $2-(13,4,1)$ design.

(c) Take $p = 5$. Then we get a $2-(31,6,1)$ design.

(d) Take $p = 7$. Then we get a $2-(57,8,1)$ design.

Theorem 2.30. *If p is a prime then a $2-(p^2,p,1)$ design exists.*

Proof.

□

Example 2.31. It follows that there exist $2-(4,2,1)$, $2-(9,3,1)$, $2-(25,5,1)$ and $2-(49,7,1)$ designs.

Example 2.32. In a projective plane, a **triangle** is a set of three non-collinear points, and a **quadrangle** is a set of four points, no three of which are collinear. How many triangles does $\mathbb{P}_2(2)$ contain? How many quadrangles does it contain?

Solution

Example 2.33. How many points in total lie on the three sides of a triangle in $\mathbb{P}_2(p)$?

Solution

Definition 2.34. The set of lines passing through a point V is called the **pencil of lines** with vertex V.

Theorem 2.35. *(a) If* $[l_1, m_1, n_1]$ *and* $[l_2, m_2, n_2]$ *are distinct lines then the lines concurrent with them are precisely those of the form*

$$[\lambda l_1 + \mu l_2, \lambda m_1 + \mu m_2, \lambda n_1 + \mu n_2]$$

with $(\lambda, \mu) \neq (0, 0)$.

(b) The lines in this set other than $[l_2, m_2, n_2]$ *are precisely those of the form*

$$[l_1 + \lambda l_2, m_1 + \lambda m_2, n_1 + \lambda n_2]$$

Proof. Dualise Theorem 2.23. □

Example 2.36. In $\mathbb{P}_2(\mathbb{R})$, the lines $3x - 7y + 15z = 0$ and $45x + 51y - 37z = 0$ meet at a point A. Find the equation of the line joining A to the point $B = (1, 2, 3)$.

Solution

2.6 The Triangle of Reference and the Unit Point

Definition 2.37. In $\mathbb{P}_2(\mathbb{K})$, the triangle with vertices $X = (1,0,0)$, $Y = (0,1,0)$ and $Z = (0,0,1)$ is called the **triangle of reference**. Its sides are the lines $x = 0, y = 0, z = 0$.

The point $U = (1,1,1)$ is called the **unit point**. *Note that the unit point does NOT lie on any side of the Δ of reference.* A **quadrangle** is a set of four points, no three of which are collinear. Thus the vertices of the triangle of reference and the unit point form a quadrangle.

Theorem 2.38. *In any projective plane, given a quadrangle A, B, C, D, a coordinate system may be chosen so that $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$ and $D = (1,1,1)$.*

Proof. Not given. ☐

Example 2.39. Suppose we are given a triangle ΔXYZ, with points L on YZ, M on XZ and N on XY, such that the lines XL, YM and NZ intersect in the point U not on a side of ΔXYZ. Let the lines XY and ML intersect at P, the lines YZ and MN intersect at Q, and the lines ZX and LN intersect at R. Show that the points P, Q and R are collinear.

Solution We start by choosing coordinates so that ΔXYZ is the triangle of reference and U is the unit point. Thus

$$X = (1, 0, 0), Y = (0, 1, 0), Z = (0, 0, 1), U = (1, 1, 1).$$

XU:

YU: Either: by inspection both Y and U satisfy $x = z$ and so this is the line YU.
Or: YU has equation

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

i.e., $x - z = 0$, equivalently $x = z$.

ZU:

L:

M:

N:

LM:

MN: MN has equation

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

i.e., $-x+y+z=0$.

NL:

P:

Q: Q is the intersection point of MN and YZ, that is, of $-x+y+z=0$ and $x=0$. Thus $Q=(0,1,-1)$.

R: R is the intersection point of NL and ZX, that is, of $x-y+z=0$ and $y=0$. Thus $R=(-1,0,1)$.

Example 2.40. Let ΔABC be any triangle, and let D be any point which is not on a side of this triangle. Let the lines AD, BD, CD meet the lines BC, CA, AB in E, F, G, respectively. Let H be any point not on a side of ΔEFG. Let the lines EH, FH, GH meet the lines FG, GE, EF in the points I, J, K, respectively. Show that the lines AI, BJ, CK are concurrent.

Solution Choose coordinates so that ΔABC is the triangle of reference, and D is the unit point: $A=(1,0,0), B=(0,1,0), C=(0,0,1), D=(1,1,1)$.

E: The line AD has equation:

The line BC has equation:

Their point of intersection is $E =$

F: Similarly $F =$

G: Similarly $G = (1, 1, 0)$.

Let $H = (p, q, r)$. Since H does not lie on any side of ΔEFG

EF: By inspection, EF has equation

FG: By inspection, FG has equation

GE: By inspection, GE has equation

Thus the fact that H does not lie on any side of ΔEFG means that:

EH: The line EH has equation

FH: Calculated in the same way, FH has equation $(p-r)y+qz-qx=0$.

GH: Calculated in the same way, GH has equation $(q-p)z+rx-ry=0$.

I: The lines $EH : (q-r)x-py+pz=0$ and $FG : x-y-z=0$ meet at I, whose coordinates are calculated from

J: Calculated in the same way, $J=(q+p-r,2q,q+r-p)$.

K: Calculated in the same way, $K=(r+p-q,r+q-p,2r)$.

AI: The line AI has equation

BJ: Calculated in the same way, BJ has equation $(r-p-q)z+(r+q-p)x=0$.

CK: Calculated in the same way, CK has equation $(p-q-r)x+(p+r-q)y=0$.

[Recall that three lines $[l_i,m_i,n_i]$ ($i=1,2,3$) are concurrent if and only if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.]$$

In this case the determinant takes the form

$$\begin{vmatrix} 0 & q-r-p & q+p-r \\ r+q-p & 0 & r-p-q \\ p-q-r & p+r-q & 0 \end{vmatrix}.$$

Now calculate this determinant:

Note: we could have shown that the intersection of the first two lines lies on the third.

2.7 Desargues' Theorem

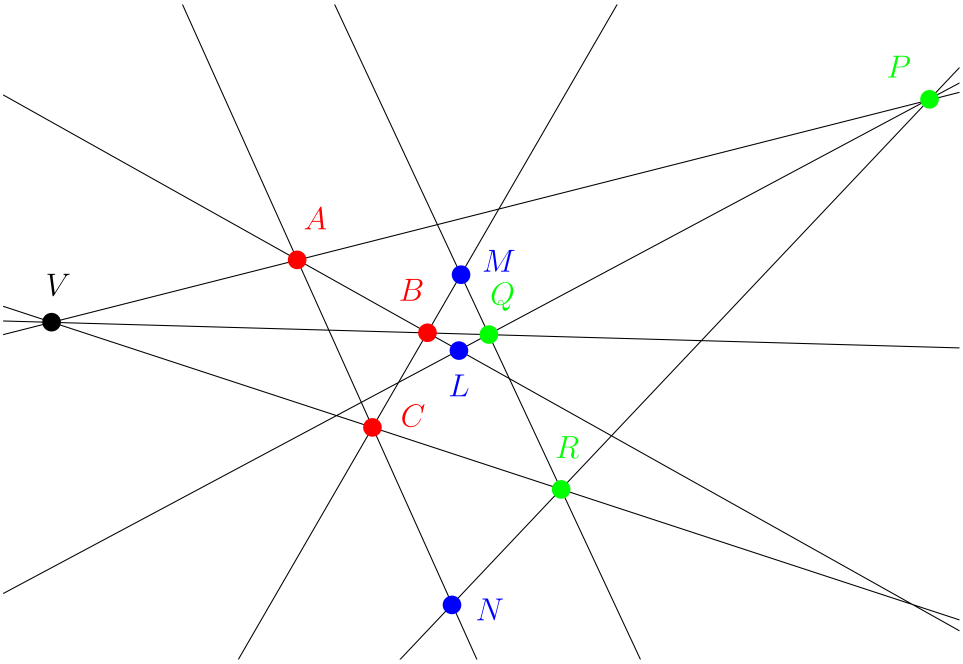
Definition 2.41. Two triangles ΔABC and ΔPQR are said to be **in perspective from the point** V if the lines AP, BQ and CR are concurrent at V.

In this definition it is assumed that the points A, B, C, D, E, F, V are distinct and that the lines AP, BQ, CR are distinct.

Theorem 2.42 (Desargues)**.** *If the triangles ΔABC and ΔPQR are in perspective from the point V then the points of intersection of corresponding sides:*

(i) AB, PQ, (ii) BC, QR; (iii) CA, RP

are collinear.



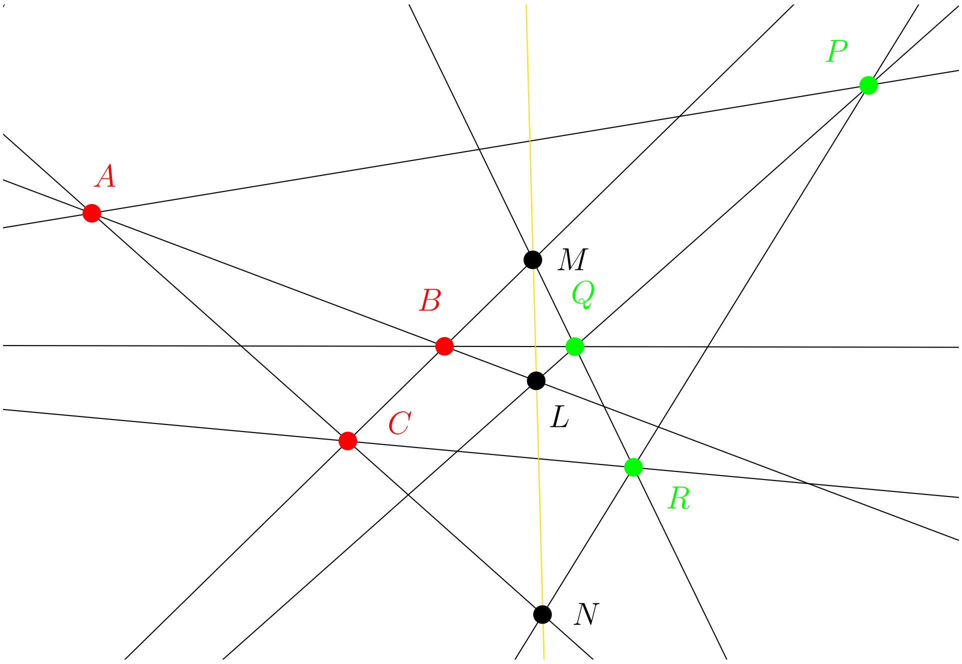
Proof.

□

Theorem 2.43 (Converse of Desargues)**.** *If the triangles ΔABC and ΔPQR are such that intersection of corresponding sides:*

(i) AB, PQ, *(ii)* BC, QR; *(iii)* CA, RP

are collinear, then the triangles are in perspective from a point.



Proof. We dualise Desargues' Theorem.

Desargues' Theorem says that, given two triangles ΔPQR and ΔABC, if the lines joining corresponding vertices, i.e.,

(i) AP (ii) BQ (iii) CR

are concurrent then the points of intersection of corresponding sides

AB, PQ and BC, QR and CA, RP

are collinear.

Since Desargues' Theorem is true, the same must be the case for its dual.

To begin with, let us consider the dual of a triangle. A triangle is defined as a set of three non-collinear points. Thus when we have a triangle XYZ, the lines XY, XZ, YZ are distinct non-concurrent lines. The dual of a triangle is a set of three non-concurrent lines. We call such an object a trilateral. When we have a trilateral abc we can construct the points of intersection $C = a \wedge b$, $A = b \wedge c$ and $B = a \wedge c$ and ABC is a triangle. [Notice that we write $a \wedge b$ for the point of intersection of the lines a and b.]

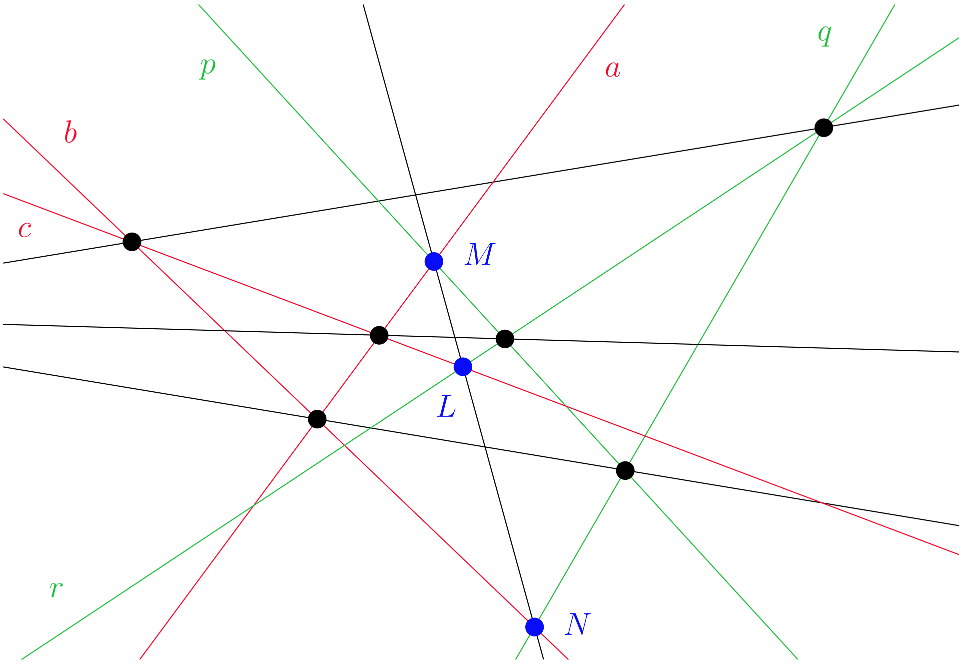
The dual of Desargues' Theorem says that, given two trilaterals pqr and abc, if the points of intersection of corresponding sides, i.e.,

(i) $a \wedge p$ (ii) $b \wedge q$ (iii) $c \wedge r$

are collinear then the lines joining of corresponding points

$$a \wedge b, p \wedge q \text{ and } b \wedge c, q \wedge r \text{ and } c \wedge a, r \wedge p$$

are concurrent.



□

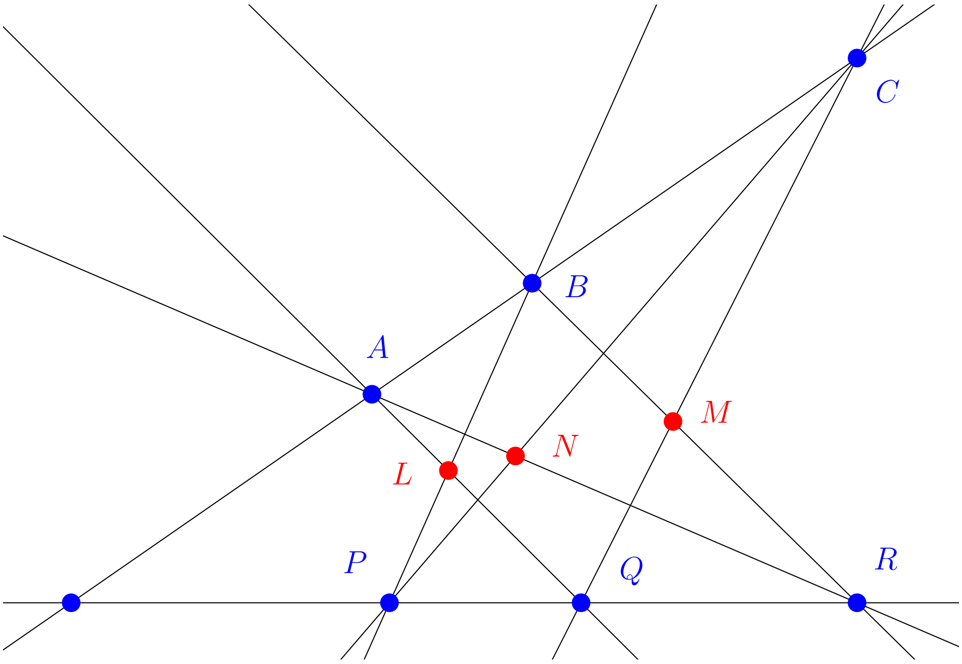
2.8 Pappus' Theorem

Theorem 2.44 (Pappus)**.** *Let the distinct points* A, B, C *lie on a line* l*, and let the distinct points* P, Q, R *lie on a line* m *(but not on* l*). Let*

(i) AQ*,* BP *meet at* L*, (ii)* BR*,* CQ *meet at* M*, (iii)* CP*,* AR *meet at* N*.*

Then the points L, M, N *are collinear.*

Proof 1.



□

Proof 2. Let the lines l and m meet at D. Choose coordinates so that ΔCDR is the triangle of reference and M is the unit point. Thus

$$C = (1, 0, 0), R = (0, 1, 0), D = (0, 0, 1), M = (1, 1, 1).$$

DC is the line $y = 0$ and MR is the line $x = z$. They meet at B, whose coordinates are therefore $(1, 0, 1)$.

DR is the line $x = 0$ and MC is the line $y = z$. They meet at Q, whose coordinates are therefore $(0, 1, 1)$.

A lies on DC so has coordinates of the form $(1, 0, a)$.

P lies on DR so has coordinates of the form $(0, 1, p)$.

CP has equation

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & p \end{vmatrix} = 0$$

i.e., $py - z = 0$.

AR has equation

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 1 & 0 & a \end{vmatrix} = 0$$

i.e., $-ax + z = 0$.

CP and AR meet at $N = (1/a, 1/p, 1) = (p, a, ap)$.

BP has equation

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 0 & 1 & p \end{vmatrix} = 0$$

i.e., $-x - py + z = 0$, or $z = x + py$.

AQ has equation

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ 1 & 0 & a \end{vmatrix} = 0$$

i.e., $-ax - y + z = 0$, or $z = ax + y$.

BP and AQ meet at L, where $x + py = ax + y$, or $(1 - a)x = (1 - p)y$. Set $x = 1 - p$. Then $y = 1 - a$ and $z = ax + y = a - ap + 1 - a = 1 - ap$. Therefore $L = (1 - p, 1 - a, 1 - ap)$.

Now L, M, N will be collinear if and only if

$$\begin{vmatrix} 1-p & 1-a & 1-ap \\ 1 & 1 & 1 \\ p & a & ap \end{vmatrix} = 0.$$

But if we first subtract Row 2 from Row 1 and then add Row 3 to Row 1, we get a top row consisting entirely of zeros. Therefore the determinant has value 0, and so L, M, N are collinear. $\square$

3 Conics

We shall assume that the underlying field is $\mathbb{C}$ or $\mathbb{Z}_p$ with $p \neq 2$. Conics over $\mathbb{R}$ or $\mathbb{Z}_2$ need special treatment: for example, they may have no points. Also, excluding $p = 2$ means that we can divide by 2 whenever we wish.

3.1 Introduction

Conic sections were first identified by the ancient Greeks, who made an extensive study of them. The Greeks defined them as the curves you get when you section (slice through) a double sided cone. We shall define them algebraically. A **conic** in $\mathbb{E}$ (over $\mathbb{R}$) is given by an equation of the form

$$ax^2 + by^2 + 2fxy + 2gy + 2hx + c = 0,$$

where at least one of $a, b, f \neq 0$.

Examples 3.1. (a) Circle:

(b) Parabola:

(c) Ellipse:

(d) Hyperbola:

(e) A Single Point:

(f) Two intersecting lines:

(g) Two identical lines:

(h) Empty set:

3.2 Conics in $\mathbb{P}_2(\mathbb{K})$

Definition 3.2. A **conic** in $\mathbb{P}_2(\mathbb{K})$ is given by an equation of the form

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2fxy + 2gyz + 2hzx = 0,$$

where at least one of $a, b, c, f, g, h \neq 0$. This means that the conic is the set of points (x, y, z) satisfying $\phi(x, y, z) = 0$.

Remark 3.3.

Let us emphasise:

- **We shall assume that $\mathbb{K} = \mathbb{C}$ or $\mathbb{Z}_p$ where $p > 2$. We shall assume that every conic has at least two points.**

3.3 Singular and Nonsingular Conics

Proposition 3.4. *Let M denote the matrix*

$$\begin{pmatrix} a & f & h \\ f & b & g \\ h & g & c \end{pmatrix}.$$

Then the equation of the conic can be recast in matrix form as $XMX^T = O$, where $X = (x, y, z)$.

Proof.

$\square$

We call M the matrix of the conic. Note that it is symmetric: $M^T = M$.

Remark 3.5.

Definition 3.6. A conic is **singular** if its equation factors into a product of linear expressions:

$$ax^2 + by^2 + cz^2 + 2fxy + 2gyz + 2hzx = (\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z),$$

so the conic is in fact a pair of lines (which could be the same!). All other conics are **non-singular**. Note: the terms **degenerate** and **non-degenerate** are sometimes used in place of singular and non-singular.

Theorem 3.7. *A conic is singular if and only if it contains three collinear points.*

Proof.

$\square$

Definition 3.8. Recall that a square matrix is **singular** if and only if it is not invertible, i.e., if and only if its determinant is 0.

Theorem 3.9. *A conic is singular if and only if its matrix is singular.*

Proof. We don't prove this. $\square$

Remark 3.10. *The matrix of a conic depends on the co-ordinate system chosen. However if the conic is non-singular, then the matrix is always non-singular.*

Theorem 3.11. *Suppose that* $\mathcal{C}$ *is a non-singular conic, with matrix* M. *Let* P *be any point of* $\mathbb{P}_2(\mathbb{K})$. *Then the points* X *that satisfy* $XMP^T = 0$ *are all the points of a line. Conversely any line of* $\mathbb{P}_2(\mathbb{K})$ *can be described in this way.*

Proof.

□

Theorem 3.12. *If $\mathcal{C}$ is a non-singular conic with matrix M and if P, Q are distinct points of $\mathbb{P}_2(\mathbb{K})$, then the lines given by $XMP^T = 0$ and $XMQ^T = 0$ are distinct.*

Proof.

□

Definition 3.13. A line which meets a conic in precisely one point P is called a **tangent** to the conic at P.

Theorem 3.14. *There is a unique tangent at each point of a non-singular conic $\mathcal{C}$. The tangent at a point P of $\mathcal{C}$ is given by $XMP^T = 0$.*

Proof.

□

Example 3.15. Let $\mathcal{C}$ be the conic in $\mathbb{P}_2(\mathbb{C})$ given by

$$x^2 + 4y^2 + 3z^2 - 2xy - 2yz - 12zx = 0.$$

Write down the matrix for $\mathcal{C}$. Show that $\mathcal{C}$ is non-singular (i.e., non-degenerate) and find the equation of the tangent to $\mathcal{C}$ at the point $(1, 2, 1)$.

Solution.

Theorem 3.16. *Let $\mathcal{C}$ be a non-singular conic with matrix M. Let P be a point of $\mathbb{P}_2(\mathbb{K})$ that does not lie on $\mathcal{C}$. Then:*

(a) P does not lie on the line $XMP^T = 0$.

(b) The line $XMP^T = 0$ is not a tangent.

(c) A point Q of $\mathcal{C}$ lies on the line $XMP^T = 0$ if and only if P lies on the tangent to Q.

(d) The number of points of the line $XMP^T = 0$ lying on $\mathcal{C}$ is equal to the number of tangents to $\mathcal{C}$ passing through P, and this number is either 0 or 2.

(e) If $\mathbb{K} = \mathbb{C}$, then P lies on 2 tangents to $\mathcal{C}$.

Proof. (a) P does not lie on $\mathcal{C}$, so $PMP^T \neq 0$, which means that P does not lie on the line $XMP^T = 0$.

(b) If the line $XMP^T = 0$ were a tangent, at Q say, then it would have equation $XMQ^T = 0$, with Q necessarily distinct from P (since Q is on $\mathcal{C}$). But this is not possible, since P and Q are distinct points, and so (by Theorem 3.12), the lines $XMP^T = 0$ and $XMQ^T = 0$ are distinct.

(c) Q lies on the line $XMP^T = 0$ precisely when $QMP^T = 0$, i.e., precisely when $PMQ^T = 0$, i.e., precisely when P lies on the line $XMQ^T = 0$. But this last line is the tangent at Q, so the statement is proved.

(d)

(e)

□

Example 3.17. Let $\mathcal{C}$ be the (non-singular) conic $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}_2(3)$ and let P be the point $(1, 0, 0)$ of $\mathbb{P}_2(3)$. Show that the line given by $XMP^T = 0$ does not meet $\mathcal{C}$.

Solution.

Example 3.18. Find the equation of the non-singular conic in $\mathbb{P}_2(\mathbb{C})$ that contains the points $P = (1, 0, 0), Q = (0, 1, 0)$ and $R = (0, 0, 1)$ and for which the tangents at $(1, 0, 0)$ and $(0, 1, 0)$ meet at $U = (1, 1, 1)$.

Solution.

3.4 A Canonical Form

As you might expect, we can choose coordinates so that the equation of a nonsingular conic takes a very simple form.

Theorem 3.19. *Let A and B be two points on a nonsingular conic. Then we can choose coordinates so that*

(i) $A = (1, 0, 0)$*;*

(ii) $B = (0, 0, 1)$*;*

(iii) the tangents to the conic at A *and* B *meet at* $C = (0, 1, 0)$*;*

(iv) the conic has equation $y^2 = zx$*.*

Proof.

□

Remark 3.20. *This is a* ***canonical form****:* $y^2 = zx$*. In the case of* $\mathbb{P}_2(\mathbb{R})$*,* $z = 0$ *is the line at infinity. We can get all the points on the conic which lie in* $\mathbb{R}^2$ *by setting* $z = 1$*. This gives* $y^2 = x$*, a parabola.*

Example 3.21. Let A and B be two points on a non-singular conic $\mathcal{C}$ and suppose that the tangents at A and B meet at C. Let D be a third point of $\mathcal{C}$ and let ℓ be a line through C distinct from AC and BC. Suppose that ℓ meets AD at F and BD at G. Show that AG meets BF in a point of $\mathcal{C}$.

Solution.

Theorem 3.22. *The points on a non-singular conic in the canonical form:* $y^2 = zx$ *are precisely the points of the form*

$$(1,0,0) \textit{ and } (\theta^2,\theta,1)$$

where $\theta \in \mathbb{K}$*.*

Proof.

$\square$

Definition 3.23. A line meeting a non-singular conic in 2 points is called a **chord** (or sometimes a **secant line**).

Theorem 3.24. *Suppose that $\mathcal{C}$ is a non-singular conic in the canonical form $y^2 - zx = 0$.*

(a) The chord joining distinct points $(\theta^2, \theta, 1)$ and $(\phi^2, \phi, 1)$ has equation

$$x - (\theta + \phi)y + \theta\phi z = 0.$$

(b) The chord joining distinct points $(\theta^2, \theta, 1)$ and $(1, 0, 0)$ has equation

$$y = \theta z.$$

(c) The tangent at the point $(\theta^2, \theta, 1)$ has equation

$$x - 2\theta y + \theta^2 z = 0.$$

The tangent at the point $(1, 0, 0)$ has equation $z = 0$.

Proof. (a) The chord has equation

$$\begin{vmatrix} x & y & z \\ \theta^2 & \theta & 1 \\ \phi^2 & \phi & 1 \end{vmatrix} = 0,$$

i.e., $x(\theta - \phi) - y(\theta^2 - \phi^2)y + z(\theta^2\phi - \phi^2\theta) = 0$, i.e.,

$$(\theta - \phi)[x - y(\theta + \phi) + \theta\phi z] = 0.$$

Since $\theta \neq \phi$, we can cancel $\theta - \phi$ to get

$$x - (\theta + \phi)y + \theta\phi z = 0.$$

(b) Both points satisfy the equation, so it is the equation of the line joining them.

(c)

□

Example 3.25. Let A, B and C be three points on a non-singular conic $\mathcal{C}$. Suppose that the tangent at A meets BC in P, the tangent at B meets AC in Q and the tangent at C meets AB in R. Show that P, Q, R are collinear.

Solution.

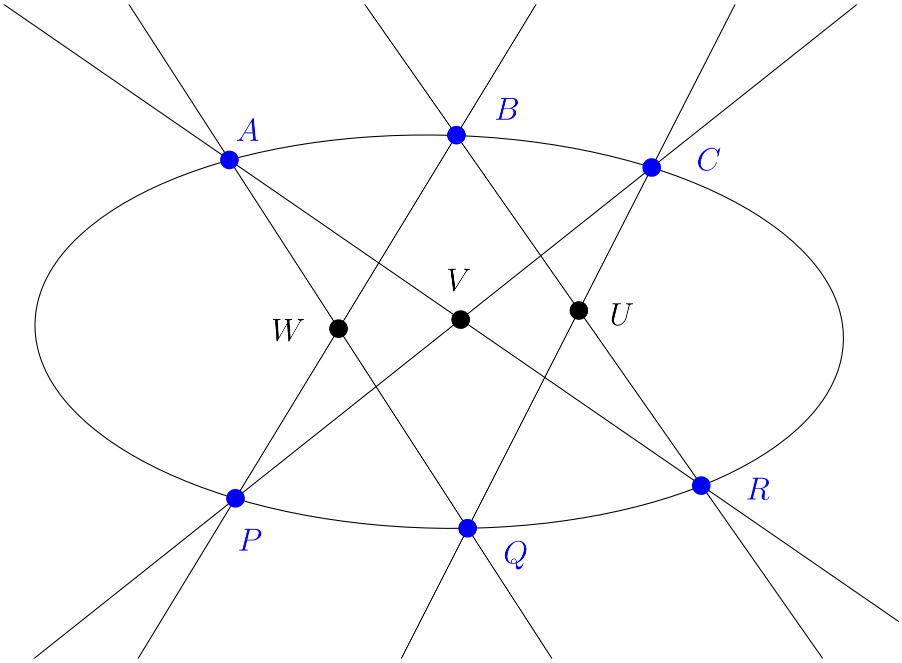
3.5 Pascal's Theorem

Theorem 3.26 (Pascal)**.** *Let A, B, C, P, Q, R be 6 (distinct) points on a non-singular conic. Then the intersection points*

$$\text{(i) } U = BR \wedge CQ; \text{ (ii) } V = AR \wedge CP; \text{ (iii) } W = AQ \wedge BP$$

are collinear.

Proof.


A
B
C
V
W
U
P
Q
R

□

Note: It might be easier to see that $U + (1-\theta)V = \psi W$.

Note: we could have computed

$$\Delta\begin{vmatrix} \theta\psi & \theta\psi & \theta+\psi-1 \\ \phi\psi & \psi & 1 \\ \theta+\phi-\theta\phi & 1 & 1 \end{vmatrix}$$

$$= \theta\psi(\psi-1) - \theta\psi(\phi\psi-(\theta+\phi-\theta\phi)) + (\theta+\psi-1)(\phi\psi-(\theta+\phi-\theta\phi)\psi)$$

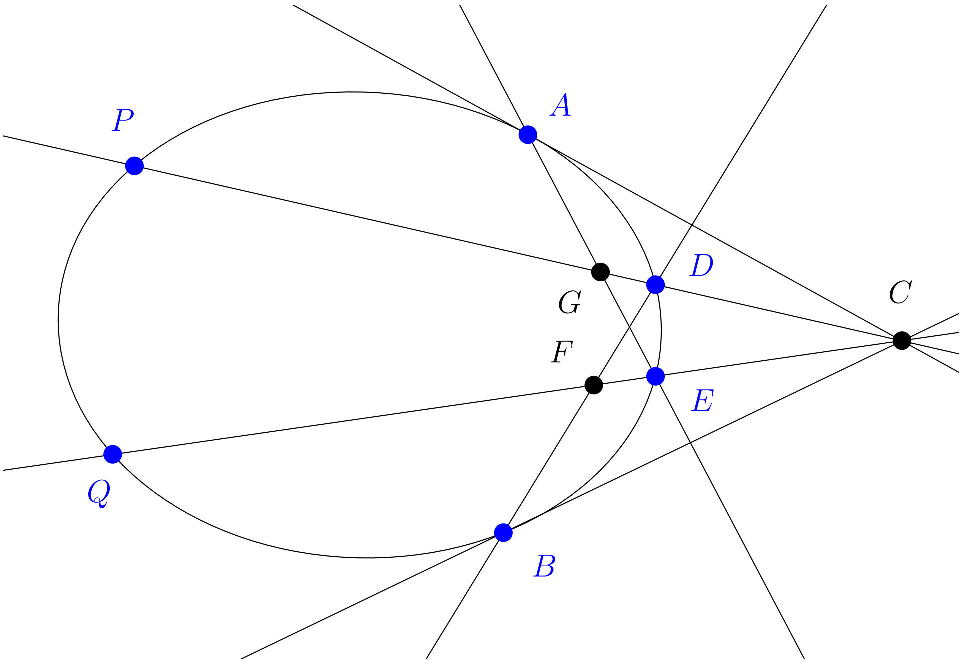
$$= \theta\psi^2 - \theta\psi - \theta\psi^2\phi + \theta^2\psi + \theta\psi\phi - \theta^2\psi\phi + \theta\phi\psi - \theta^2\psi - \theta\phi\psi$$

$$+\theta^2\phi\psi + \phi\psi^2 - \theta\psi^2 - \phi\psi^2 + \theta\phi\psi^2 - \phi\psi + \theta\psi + \phi\psi - \theta\phi\psi = 0.$$

Remark 3.27. *As we shall see soon, a conic in* $\mathbb{P}_2(p)$ *has* $p+1$ *points. Thus a conic in* $\mathbb{P}_2(3)$ *has only* 4 *points and Pascal's Theorem cannot apply.*

Example 3.28. Let $\mathcal{C}$ be a non-singular conic (in $\mathbb{P}_2(\mathbb{C})$ or $\mathbb{P}_2(p)$ with $p \geq 5$). Let A, B, D, E be distinct points on $\mathcal{C}$ and let C be the intersection of the tangents at A and B. Assume that C, D, E are not collinear. Let F be the intersection of DB and CE, let G be the intersection of EA and CD, let P be the intersection of CD with $\mathcal{C}$ ($P \neq D$) and let Q be the intersection of CE with $\mathcal{C}$ ($Q \neq E$). Prove that PQ, FG and DE are concurrent.

Solution.



3.6 5 points determine a conic

Theorem 3.29. *Suppose we are given 5 points, no three of which are collinear. Then they lie on a unique conic.*

Proof.

$\square$

Now use the fact that U and V lie on the conic:

- U lies on the conic, so $2f + 2g + 2h = 0$, i.e., $f + g + h = 0$.
- V lies on the conic, so $2fpq + 2gqr + 2hrp = 0$, i.e., $fpq + gqr + hrp = 0$.

Substituting $h = -(f + g)$ into the second equation gives

$$fpq + gqr - (f + g)rp = 0, \text{ i.e.,} fp(q - r) + gr(q - p) = 0.$$

Then $p, r, q - r, q - p \neq 0$, so

$$g = -\frac{p(q-r)}{r(q-p)} f,$$

$$h = -f - g = \left[-\frac{r(q-p)}{r(q-p)} + \frac{p(q-r)}{r(q-p)} \right] f = \frac{q(p-r)}{r(q-p)} f.$$

The conic is then

$$2fxy - 2\frac{p(q-r)}{r(q-p)} fyz + 2\frac{q(p-r)}{r(q-p)} fzx = 0.$$

We cannot have $f = 0$ (for then $g = h = 0$) so we can simplify the equation of the conic to

$$r(q-p)xy - p(q-r)yz + q(p-r)zx = 0.$$

We have shown that there is a unique conic passing through the five given points.

3.7 Non-singular Conics in $\mathbb{P}_2(p)$

In this section we consider a non-singular conic $\mathcal{C}$ in $\mathbb{P}_2(p)$, with p odd.

Theorem 3.30. *There are exactly $p+1$ points on $\mathcal{C}$.*

Proof.

$\square$

Definition 3.31. A line of $\mathbb{P}_2(p)$ is said to be **external** to $\mathcal{C}$ if it does not contain any points of $\mathcal{C}$. [Recall from Deinition 3.23 that a line meeting a non-singular conic in 2 points is called a **chord**.]

Theorem 3.32. *(a) The number of tangents to $\mathcal{C}$ is $p+1$.*

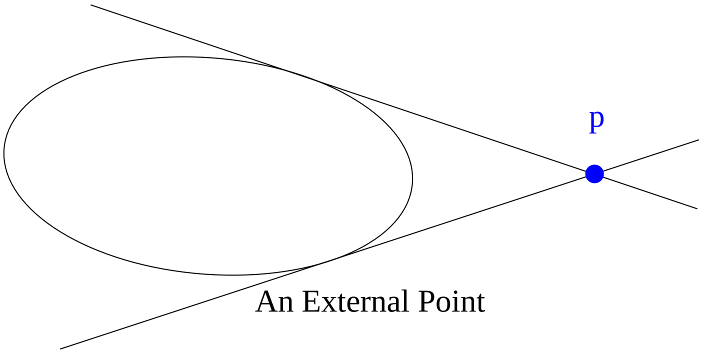
(b) The number of chords to $\mathcal{C}$ is $\Delta\frac{p(p+1)}{2}$.

(c) The number of external lines to $\mathcal{C}$ is $\Delta\frac{p(p-1)}{2}$.

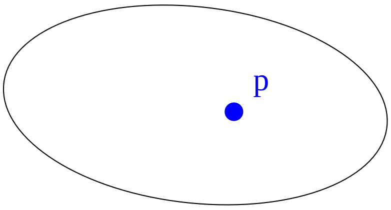
Proof.

□

Definition 3.33. A point of $\mathbb{P}_2(p)$ not on $\mathcal{C}$ is called **external** if it lies on 2 tangents to $\mathcal{C}$ and **internal** if it lies on no tangents to $\mathcal{C}$.



An External Point



An Internal Point

Theorem 3.34. *Let P be a point of $\mathbb{P}_2(p)$ not on $\mathcal{C}$. Then the line $XMP^T = 0$ is external to $\mathcal{C}$ if and only if P is an internal point.*

Proof.

□

Theorem 3.35. *(a) The number of external points is* $\Delta \frac{p(p+1)}{2}$.

(b) The number of internal points is $\Delta \frac{p(p-1)}{2}$.

Proof. This follows immediately from Theorems 3.32 and 3.34. □

Example 3.36. Given a non-singular conic $\mathcal{C}$ in $\mathbb{P}_2(p)$, prove that each external line contains $(p+1)/2$ internal points and $(p+1)/2$ external points.

Solution.

MAS3210 Geometries and Designs

Assignment Exercises

1

1.1 Show that there is no $2-(24,4,2)$ design.

1.2 Show that there is no $2-(46,26,5)$ design.

1.3 The University is setting up a new School of Meta-Psycho-Philosophy with a brand new degree programme of the same name. The School will offer 10 30-credit modules at Stage 3, of which each student must select 4. It so happens that no two students make the same module selections. Further, for each pair of modules on offer, there are exactly two students taking both. How many students are there in the class? Show that each module is taken by the same number of students and determine this number.

1.4 Show that for any $2-(v,k,\lambda)$ design, $\lambda \leq \binom{v-2}{k-2}$. Prove that the design is trivial if and only if $\lambda = \binom{v-2}{k-2}$.

1.5 Suppose that $(X, \mathfrak{B})$ is a $2-(8,3,\lambda)$ design. Prove that λ is divisible by 6. Using an earlier question, show that $\lambda = 6$ and that the design is trivial.

1.6 Suppose that $(X, \mathfrak{B})$ is a $2-(12n+3,4,\lambda)$ design for some $n \in \mathbb{N}$. Prove that λ is divisible by 6.

1.7 Show that any $2-(v,k,\lambda)$ design with $v \leq 5$ is trivial.

1.8 Prove Fisher's Inequality for a $2-(v,4,\lambda)$ design.

1.9 Suppose that $(X, \mathfrak{B})$ is a non-trivial $2-(6,k,\lambda)$ design. Prove that $k = 3$ and $\lambda = 2$.

1.10
  (a) Show that for any $2-(v,k,\lambda)$ design with $v > k$, $\lambda \geq \frac{k(k-1)}{v-1}$.
  (b) Prove that if $\lambda = 1$ and $v > k > 2$, then $v \geq 3k-2$. By considering separately the cases $k = 3$ and $k \geq 4$, show that $v \geq 4k-3$ except when $k = 3$ and $v = 7$.

1.11 A boys club competes in a number of 'tug-of-wars', supplying a team for each tug. Each boy is in 3 teams. The teams all have the same number (> 2) of boys. No two teams are the same and any two boys pull together in 1 tug. Find the number of boys and the size of each team.

2

2.1 Show that if $(X, \mathfrak{B})$ is a symmetric design with $\lambda = 1$, then it is a projective plane.

2.2 Find the value of v for:

- (a) A symmetric $2-(v, 14, 1)$ design;
- (b) A symmetric $2-(v, 5, 2)$ design.

2.3 For each of the following sets of parameters either give an example of a design with these parameters or prove that no such design exists.

- (a) $2-(52, 18, 6)$
- (b) $2-(63, 3, 1)$
- (c) $2-(21, 6, 1)$
- (d) $2-(65, 3, 1)$
- (a) $2-(183, 14, 1)$
- (b) $2-(9, 6, 5)$
- (c) $2-(225, 3, 1)$

2.4 Show that there is no projective plane of order 22.

2.5 Show that there is a $2-(8, 4, 3)$ design having the additional property that

- Every 3 points lie together in exactly 1 block.

What does this additional property say about two rows of the incidence matrix?

2.6 Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathfrak{B}$ be the set of blocks:

$B_1 = \{1, 2, 4\}, B_2 = \{1, 3, 5\}, B_3 = \{2, 4, 5\}, B_4 = \{2, 3, 4\}, B_5 = \{3, 4, 5\}, B_6 = \{1, 2, 3\}$.

Write down the incidence matrix A for $(X, \mathfrak{B})$ (with the given labelling) and calculate:

- (a) $A^T A$;
- (b) $\det(A^T A)$.

[Note: $(X, \mathfrak{B})$ is not a design, but it is still a system of points and blocks for which we can write down an incidence matrix.]

2.7 A real ale fan approaches you for advice on setting up a comparison of beers from various breweries. Her idea is as follows:

- (a) The various breweries have agreed to donate 46 barrels of different beers;
- (b) The members of the Real Ale Society will each drink 10 pints of beer (each pint a different brew), and given any two beers there will be exactly two students drinking both;
- (c) No two members will drink the same selection of beers.

What advice can you offer?

Suppose that, regardless of your mathematical advice, you persuade the real ale fan that 10 pints is too much to drink (in the light of concern over student alcohol abuse). If she now suggests that there be 11 barrels of beer and that each student drinks 5 pints, will her idea work? [Assume that there will now be fewer members taking part.]

2.8 Suppose we are given numbers v, k, λ, with $r = \frac{\lambda(v-1)}{k-1}$, $b = \frac{\lambda v(v-1)}{k(k-1)}$ both integers, and that $k' = v - k$, $\lambda' = b - 2r + \lambda$. Show that:

(a) $\lambda' = \frac{\lambda(v-k)(v-k-1)}{k(k-1)}$

(b) $v - k' = k$

Assuming that there exists a $2-(v, k', \lambda')$ design with b' blocks and with each point in r' blocks, show that:

(a) $b' = b$

(b) $r' = \frac{\lambda(v-k)(v-1)}{k(k-1)} = b - r$

(c) $b' - 2r' + \lambda' = \lambda$

3

3.1 Without using the fact that two distinct points of $\mathbb{E}$ lie on a unique line, prove that:

(a) A pair of distinct non-parallel lines meets in a unique point of $\mathbb{E}$.

(b) A pair of distinct parallel lines has no point of $\mathbb{E}$ in common.

3.2 (a) Determine whether or not $(-2, 1+3i, 6+i)$ and $(-2-2i, -2+4i, 5+7i)$ are the same point in $\mathbb{P}_2(\mathbb{C})$.

(b) Determine whether or not $(5, 2, 4)$ and $(4, 6, 7)$ are the same point in $\mathbb{P}_2(11)$.

(c) Find all the primes p such that $(1, 3, 9)$ and $(9, 1, 3)$ are the same point in $\mathbb{P}_2(p)$.

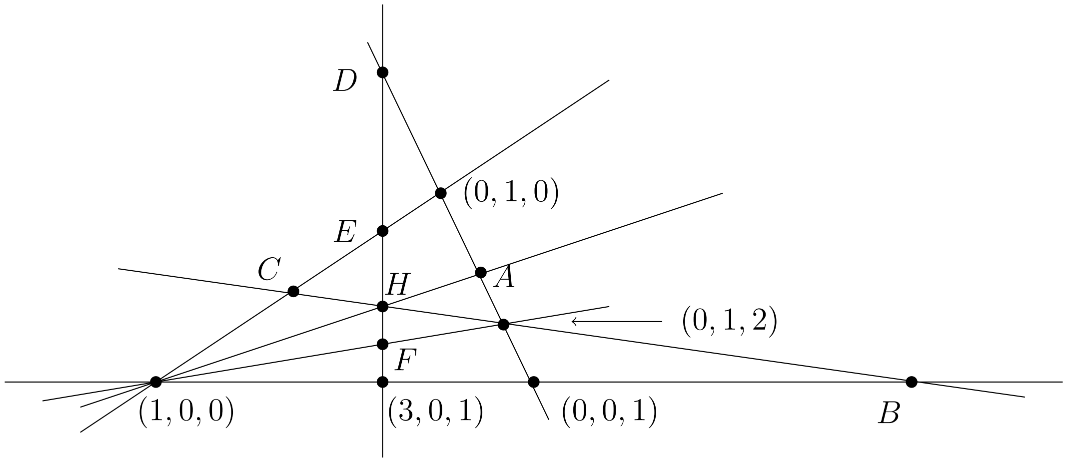
Justify your answers.

3.3 (a) Find the equation of the line joining the points $(1, 3, 5)$ and $(-2, 1, 6)$ in $\mathbb{P}_2(\mathbb{R})$. What are the co-ordinates of the points where this line meets the lines $x = 0, y = 0$ and $z = 0$.

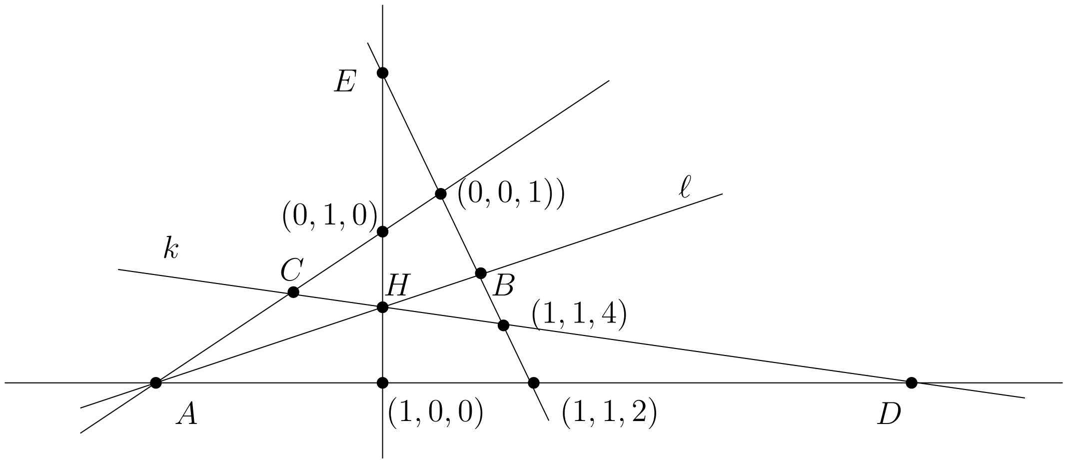
(b) Find the equation of the line in $\mathbb{P}_2(\mathbb{C})$ that passes through the intersection of the lines $x+2y+3z = 0$ and $x+4y+6z = 0$ and also contains the point $(1, 2, 3)$.

3.4 Find the number of triangles and quadrangles in $\mathbb{P}_2(5)$.

3.5 The diagram shows (some) distinct points and lines in $\mathbb{P}_2(7)$. The point marked H is the point $(1,1,1)$. Find the co-ordinates of the points marked $A - F$.



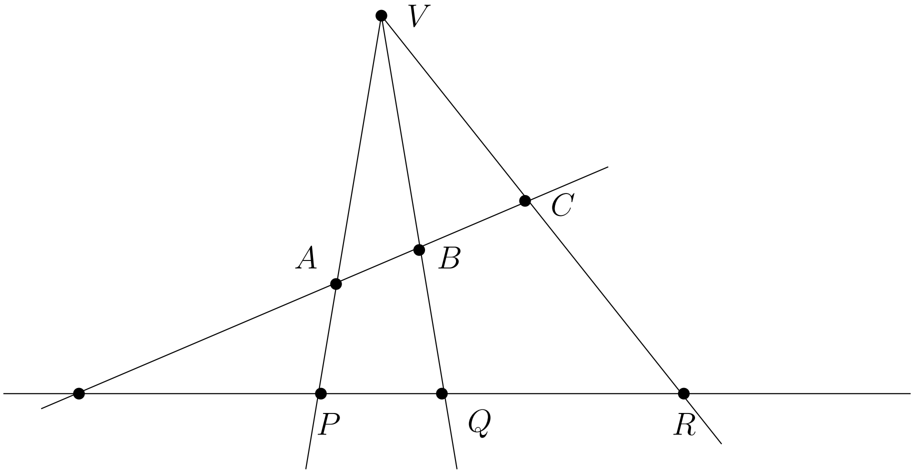
3.6 The diagram shows (some) distinct points and lines in $\mathbb{P}_2(7)$. The point marked H has co-ordinates $(1,2,0)$. Find the co-ordinates of the points marked $A - E$ and the equations of the lines marked k and ℓ.



3.7 A, B, C, D are distinct points with no three on a line in $\mathbb{P}_2(\mathbb{K})$ with $\mathbb{K} \neq \mathbb{Z}_2$. The lines AB and CD meet at U; AC and BD meet at V; UV meets AD and BC in F and G respectively; BF meets AC at L. Prove that LG and CF meet at a point of AU.

3.8 A line $lx + my + nz = 0$ (with $lmn \neq 0$) meets the sides $x = 0$, $y = 0$, $z = 0$ of the triangle of reference XYZ in points A, B and C respectively. YB meets ZC in P, ZC meets XA in Q and XA meets YB in R. Prove that XP, YQ and ZR meet in a point K and find its co-ordinates.

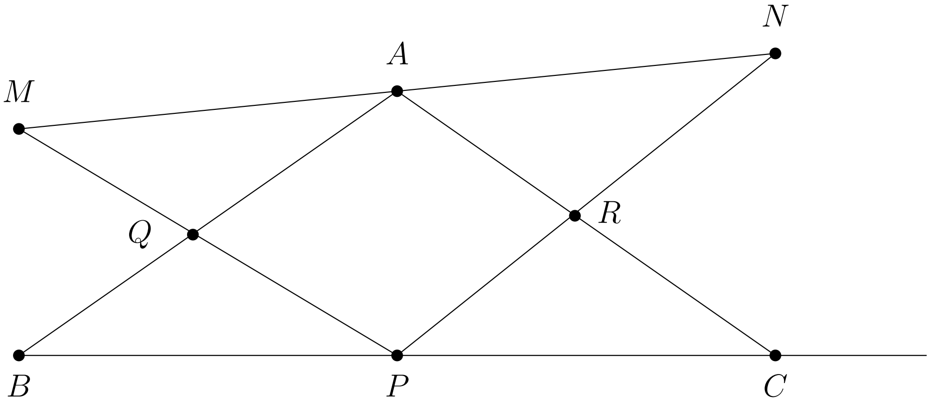
3.9 In this figure, what result is obtained by applying Desargues' Theorem to the triangles



AQC and PBR?

3.10 State and prove the converse of Pappus' Theorem without invoking the Principle of Duality.

3.11 In this question we assume that the points are as shown in the diagram; that is, they are distinct from eachother and also from the intersection of lines MAN and BPC. It is further assumed that the field is $\mathbb{C}$. Suppose that ABC is a fixed triangle and that M and N are fixed points on a line through A. Let P be any point on the line BC. The line MP meets the AB in Q. The line NP meets AC in R. Show that, as P varies on BC the corresponding lines QR all pass through a certain fixed point. Identify this point.



3.12 Triangles ABC and PQR are in perspective from O_1; triangles PQR and UVW are in perspective from O_2; triangles UVW and ABC are in perspective from O_3. assume that O_1, O_2 and O_3 are collinear. Show that any pair of the triangles APU, BQV, CRW are in perspective with the centres of perspective lying on a line ℓ that is the comon axis of perspective of any of the pair of the triangles ABC, PQR, UVW.

3.13 Let V, A, B, C, P, Q, R be the points in the statement of Desargues' Theorem (and that ABC and PQR are triangles in perspective from V). Assume that they are distinct points and that AP, BQ and CR are distinct lines meeting at V. Suppose that the co-ordinates are chosen such that the points have co-ordinates (in the given order):

$$(1,1,1),(1,0,0),(0,1,0),(0,0,1),(\alpha,1,1),(1,\beta,1),(1,1,\gamma)$$

with $\alpha, \beta, \gamma \neq 1$. Prove that AQ, BR, CP meet in a point if and only if AR, BP, CQ meet in a point. [Hint: three distinct lines meet in a point if and only if the intersection of two of them lies on the third.]

3.14 Suppose that V, A, B, C, P, Q, R are points as in Q **??**. Suppose that AQ, BR, CP meet in a point U and that AR, BP, CQ meet in a point W. Prove that P, Q, R, V, U, W are distinct and that PV, QU, RW meet in a point.

3.15 Suppose that we are considering points in $\mathbb{P}_2(\mathbb{C})$. Let $\omega = e^{2\pi i/3}$; so $\omega^3 = 1$ and the cube roots of unity are $1, \omega, \omega^2$. Let $\mathcal{S}$ be the set of points $(\omega^a, -1, 0)$, $(0, \omega^b, -1)$, $(-1, 0, \omega^c)$ where a, b, c can each have one of the values $0, 1, 2$.

(a) What is $|\mathcal{S}|$?

(b) Show that $(\omega^a, -1, 0)$ lies on the line joining $(0, \omega^b, -1)$ to $(-1, 0, \omega^c)$ if and only if $a + b + c$ is $0, 3$ or 6.

(c) Show that the line joining any two lines of $\mathcal{S}$ contains a third point of $\mathcal{S}$.

(d) Show that there 12 such lines.

(e) What kind of design do we get if we take $\mathcal{S}$ as the set of points and these 12 lines as blocks?

3.16 Use Theorem 2.30 to show that there exists an STS of order 9.

4

4.1 Let $\mathcal{C}$ be the conic in $\mathbb{P}_2(\mathbb{C})$ given by

$$x^2 + y^2 + z^2 - 4xy - 6yz + 4zx = 0.$$

Write down the matrix for $\mathcal{C}$. Show that $\mathcal{C}$ is non-singular and find the equation of the tangent to $\mathcal{C}$ at the point $(2, 1, 1)$.

4.2 Suppose that $\mathcal{C}$ is a non-singular conic in $\mathbb{P}_2(\mathbb{K})$ with $\mathbb{K} \neq \mathbb{Z}_3$ containing the points $P = (1, 0, 0)$, $Q = (1, 1, 0)$ and $R = (0, 1, 1)$. Moreover the tangents at Q and R meet at the point $U = (0, 0, 1)$. Find the equation of $\mathcal{C}$.

4.3 Let $\mathcal{C}$ be the conic in $\mathbb{P}_2(3)$ given by

$$x^2 + y^2 + z^2 = 0.$$

Determine the co-ordinates of all the points A, B, C, D of $\mathcal{C}$. Find also the co-ordinates of the points L, M, N, where L is the intersection of AB, CD, M is the intersection of AC, BD and N is the intersection of AD, BD. Let $P = (1, 1, 0)$ (not on $\mathcal{C}$). Determine the number of points of $\mathcal{C}$ that lie on the line $XMP^T = 0$, where M is the matrix of $\mathcal{C}$.

4.4 Suppose that $\mathcal{C}$ is a non-singular conic with a given co-ordinate system and with matrix M. Given any point A not on $\mathcal{C}$, the line $XMA^T = 0$ is called the **polar line** of A. Conversely, a line ℓ that is not a tangent can be written $XMA^T = 0$ for some A not on $\mathcal{C}$: the point A is called the **pole** of ℓ (and ℓ is then the polar line of A).

Suppose that $P = (1, -1, 0)$ and $Q = (0, 1, -1)$ are points of $\mathbb{P}_2(\mathbb{K})$ that do not lie on $\mathcal{C}$. Suppose further that the polar line of P meets $\mathcal{C}$ at $(0, 0, 1)$ and one other point, and that the polar line of Q meets $\mathcal{C}$ at $(1, 0, 0)$ and one other point. Assuming that PQ is not a tangent and that its pole is $(0, 1, 0)$, find the equation of $\mathcal{C}$.

4.5 The tangents to a nonsingular conic $\mathcal{C}$ at points A and B meet at a point C. PQ is a chord of the conic which passes through C. Draw the diagram. Prove that the tangents at P and Q meet at a point on the line AB.

4.6 P, Q, R are 3 points on a nonsingular conic $\mathcal{C}$. The tangents at Q and R meet at a point U; the tangents at R and P meet at V; and the tangents at P and Q meet at W. Draw the diagram. Prove that the lines UP, VQ and WR are concurrent.

4.7 Given a non-singular conic $\mathcal{C}$ in $\mathbb{P}_2(p)$ (p odd), prove that each chord contains $(p-1)/2$ internal points and $(p-1)/2$ external points together with 2 points of $\mathcal{C}$.

4.8 P, Q, R, S are 4 points on a nonsingular conic $\mathcal{C}$. Prove that the point of intersection of the tangents at P and Q, the point of intersection of lines PR and QS and the point of intersection of the lines PS andd QR are collinear. Draw the diagram.

4.9 A, B, U, V are 4 points on a nonsingular conic $\mathcal{C}$. The tangents at A and B meet at the point C; the tangents at U and V meet at the point W. Show that the 6 points A, B, C, U, V, W lie on a conic. Draw the diagram.

MAS3210 Geometries and Designs

Problems Classes

1

1.1 Show that there is no $2-(63,7,1)$ design.

1.2 Show that there is no $2-(81,17,4)$ design.

1.3 Prove Fisher's Inequality for a $2-(v,5,\lambda)$ design: If $v>5$, then $b\geq v$.

1.4 Show that if $k=v-1$, then the only $2-(v,k,\lambda)$ design is the trivial one. [Hint: use the expression for r to get a lower bound for λ and use this to show that $b\geq v$; explain why in the case of a $2-(v,v-1,\lambda)$ design it must happen that $b\leq v$.]

1.5 A wine merchant sends his price list to a number (>1) of customers. On his list there are 16 varieties of wine. Each customer orders the same number of bottles of wine and in each order no two bottles are the same variety. No two customers select the same selection of varieties; each pair of varieties is ordered by just one customer. Find the number of customers, given that there are fewer than 100. [You may assume that if $v>k$, then $b\geq v$.]

2

2.1 For each of the following sets of parameters either give an example of a design with these parameters or prove that no such design exists.

(a) $2-(1023,3,1)$.	(a) $2-(13,9,6)$
(b) $2-(34,12,4)$.	(b) $2-(1953,3,1)$
(c) $2-(133,12,1)$.	(c) $2-(15,12,22)$
(d) $2-(91,21,2)$.	

2.2 Show that there is a $2-(11,5,2)$ design. In this question (but not in general) you may assume that each pair of blocks have exactly two points in common. An incidence matrix has been partially created:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ \hline
0 & 1 & 1 & 0 & 0 & & & & & & \\ \hline
0 & 1 & 0 & 1 & 0 & & & & & & \\ \hline
0 & 1 & 0 & 0 & 1 & & & & & & \\ \hline
0 & 0 & 1 & 1 & 0 & & & & & & \\ \hline
0 & 0 & 1 & 0 & 1 & & & & & & \\ \hline
0 & 0 & 0 & 1 & 1 & & & & & &
\end{array}\right]$$

3

3.1 (a) Determine whether or not $(3, 2+i, 5-i)$ and $(6-3i, 5, 11-7i)$ are the same point in $\mathbb{P}_2(\mathbb{C})$.

(b) Determine whether or not $(4, 2, 6)$ and $(5, 6, 4)$ are the same point in $\mathbb{P}_2(7)$.

Justify your answer.

3.2 (a) Find the equation of the line joining the points $(2, -3, 4)$ and $(1, 7, -5)$ in $\mathbb{P}_2(\mathbb{R})$. What are the co-ordinates of the points where this line meets the lines $x = y$, $y = z$ and $z = x$?

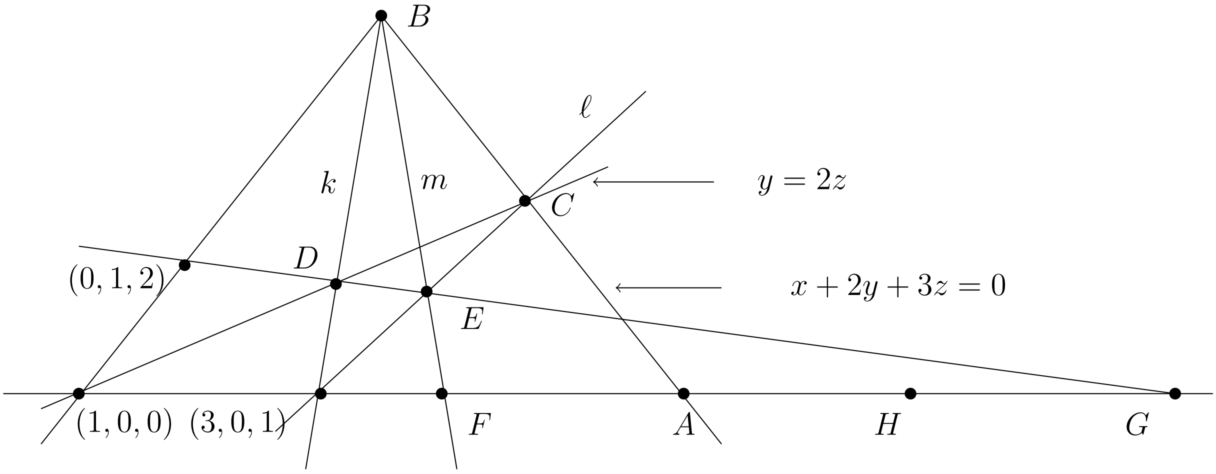
(b) Find the equation of the line in $\mathbb{P}_2(7)$ that passes through the intersection of the lines $x + 3y + z = 0$ and $x + 6y + 3z = 0$ and also contains the point $(2, 2, 5)$.

3.3 Find the number of triangles in $\mathbb{P}_2(7)$ (a *triangle* being a set of 3 non-collinear points).

3.4 Determine all the lines in $\mathbb{P}_2(5)$ passing through the point $(1, 1, 1)$.

4

4.1 The diagram shows (some) distinct points and lines in $\mathbb{P}_2(5)$. Find the co-ordinates of the points marked $A - H$ and the lines k, ℓ, m.



4.2 D is a point not on any side of a triangle ABC. The lines AD, BD, CD meet BC, CA, AB in points P, Q, R respectively. A line ℓ, distinct from any side of the triangle PQR meets its sides QR, RP, PQ in points L, M, N respectively. AL meets BC in L^*, BM meets CA in M^*, and CN meets AB in N^*. Prove that L^*, M^*, N^* lie on a line: draw a diagram.

4.3 Let A, B, C be distinct points on a line ℓ of $\mathbb{P}_2(\mathbb{K})$ with $\mathbb{K} \neq \mathbb{Z}_2$. Let P be any point not on ℓ and let Q be a point on PC with $Q \neq P, C$.

(a) (i) Show that if ℓ is the line $z = 0$ and if C lies also on $x = y$, then $C = (1, 1, 0)$.
 (ii) Show that a co-ordinate system may be chosen so that $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (1, 1, 0)$ without specifying co-ordinates for P and Q.

(b) Let S be the intersection of AQ and BP. Let T be the intersection of AP and BQ. Let D be the intersection of ST and ℓ. Show that D is independent of the choices of P and Q. The point D is called the *harmonic conjugate of* C *with respect to* A *and* B, it does not depend on the co-ordinate system.

(c) Let G be the point $(1, \theta, 0)$ with $\theta \neq 0$. Find the harmonic conjugate of G.

(d) Find the harmonic conjugate of A with respect to C and D.

5

5.1 Let $\mathcal{C}$ be the conic in $\mathbb{P}_2(\mathbb{C})$ given by

$$x^2 - y^2 + z^2 + 2yz + 2zx = 0.$$

Write down the matrix for $\mathcal{C}$. Show that $\mathcal{C}$ is non-singular (i.e., non-degenerate) and find the equation of the tangent to $\mathcal{C}$ at the point $(1, 1, 0)$.

5.2 Suppose that $\mathcal{C}$ is a non-singular conic containing the points $P = (1, 1, 0)$, $Q = (0, 1, 1)$ and $R = (0, 1, -1)$. Moreover the tangents at P and Q meet at the point $U = (1, 1, 1)$. Find the equation of $\mathcal{C}$.

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5.3 Suppose that A, B are distinct points on a non-singular conic $\mathcal{C}$ in $\mathbb{P}_2(\mathbb{C})$ and that the tangents at A and B meet at C. Suppose that D is a point on AC distinct from A, C and that E is a point on BC distinct from B, C. Let F be the second point of BD on $\mathcal{C}$ (B being the first) and let G be the second point of AE on $\mathcal{C}$ (A being the first). Let X be the intersection of AE and BD. Let Y be the intersection of AF and BG. Show that C, X, Y are collinear. If $F \neq G$, show that the lines AB, DE, FG are concurrent.

5.4 Suppose that $\mathcal{C}$ is a non-singular conic in $\mathbb{P}_2(\mathbb{K})$. Suppose that A is the intersection of tangents to $\mathcal{C}$ at points M_1 and M_2. Let N_1 be a third point on $\mathcal{C}$ and suppose that B is the intersection of M_1M_2 with the tangent at N_1. Let N_2 be the other point of $\mathcal{C}$ whose tangent passes through B. Prove that N_1N_2 passes through A.

Suppose that C is the intersection of M_1M_2 and N_1N_2. Finally suppose that D is the intersection of AM_1 and BN_1. Take A, B, C as the triangle of reference and D as the unit point. Find the equation of $\mathcal{C}$.

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1 2007/2008

MAS3210

Module Title

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are TWO questions in Section A and THREE questions in Section B.

Answers to questions in Section A should be entered directly on this question paper in the spaces provided. Rough work should be done on the blank sides of the pages. The rough work will not be marked. This question paper must be handed in, attached inside an anonymised cover sheet, at the end of the examination.

SECTION A

A1. For each of the following sets of parameters either give an example of a design with these parameters or prove that no such design exists.

(a) $2-(361, 57, 7)$.

(b) $2-(307, 18, 1)$.

(c) $2-(70, 24, 10)$.

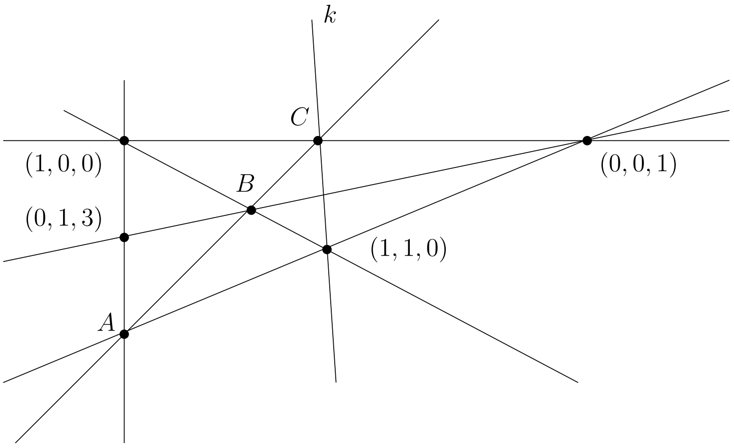
(d) $2-(106, 15, 2)$.

[18 marks]

A2. Find all the primes p such that $(1, 7, 11)$ and $(7, 11, 1)$ are the same point in $\mathbb{P}_2(p)$.

[4 marks]

A3. The following diagram shows some points and lines in $\mathbb{P}_2(5)$. Find the co-ordinates of the points marked A, B and C and the equation of the line marked k.



[8 marks]

A4. Suppose that $\mathcal{C}$ is a non-singular conic in $\mathbb{P}_2(p)$ for some odd prime p, containing the points $P = (1, 0, 0)$, $Q = (0, 0, 1)$ and $R = (1, -1, 0)$. Moreover the tangents at P and Q meet at the point $T = (0, 1, 1)$. Find the equation of $\mathcal{C}$.

[**10 marks**]

SECTION B

B5. (a) In the context of designs, what is meant by a *projective plane*? Show that a projective plane is a symmetric design with an odd number of points.

(b) Suppose that the design $(X, \mathcal{B})$ is a projective plane. Select a particular block $B \in \mathcal{B}$ and let

$$Y = X \setminus B, \text{ and } \mathcal{D} = \{B' \cap Y : \ B' \in \mathcal{B}, \ B' \neq B\}.$$

Prove that $(Y, \mathcal{D})$ is a $2-(n^2, n, 1)$ design for some n.

(c) Prove that there is no projective plane of order 22. [You may assume that if there exists a symmetric $2-(v, k, \lambda)$ design with v odd, then the equation

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

has a solution in integers x, y, z not all 0.]

[30 marks]

B6. (a) Suppose that $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are distinct points of $\mathbb{P}_2(p)$.

(i) Consider the points Q and $P+\lambda Q = (x_1+\lambda x_2, y_1+\lambda y_2, z_1+\lambda z_2)$ arising from all possible choices of $\lambda \in \mathbb{Z}_p$. Prove that these points lie on the line PQ and that they are distinct.

(ii) Show that each point on the line PQ can be written in the form Q or $P + \lambda Q$ for some $\lambda \in \mathbb{Z}_p$.

(iii) Prove that PQ consists of all points of the form $\alpha P + \beta Q$ with $(\alpha, \beta) \neq (0, 0)$.

(b) Suppose that $V = (1, 1, 1)$, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, and $D = (1, -1, 1)$ are points in $\mathbb{P}_2(p)$ (with p odd), and that $T = (1, \alpha, 1)$ for some fixed number α distinct from 1 and -1. Let P and Q be the intersections of VC with AD and AT respectively. Let R and S be the intersections of VA with CD and CT respectively. Prove that PR, QS and AC are concurrent.

[30 marks]

B7. (a) Let $\mathcal{C}$ be the non-singular conic $y^2 = zx$ in $\mathbb{P}_2(\mathbb{K})$. Prove that the points on $\mathcal{C}$ are precisely the points of the form

$$(1, 0, 0) \text{ and } (\theta^2, \theta, 1)$$

where $\theta \in \mathbb{K}$.

(b) Prove that the tangent to $\mathcal{C}$ at the point $(\theta^2, \theta, 1)$ has equation

$$x - 2\theta y + \theta^2 z = 0$$

and that the tangent at the point $(1, 0, 0)$ has equation $z = 0$.

(c) Now assume that $\mathbb{K} = \mathbb{Z}_5$ (i.e., $\mathcal{C}$ is the non-singular conic $y^2 = zx$ in $\mathbb{P}_2(5)$). Let A, B, C and F be the points $(1, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 1)$ respectively. Find the co-ordinates of the points P and Q on $\mathcal{C}$ whose tangents pass through F. Let ℓ be the tangent to $\mathcal{C}$ at P and m the tangent at Q, and let S and T be the intersections of ℓ and m with the tangents at A and B respectively. Show that AQ, BP, CF and ST are concurrent. [Note: when finding the two points on $\mathcal{C}$ whose tangents pass through F, it does not matter which way you label them as P and Q.]

[**30 marks**]

THE END

MAS3210 Geometries and Designs

2008 Exam Solutions

Section A

A1. For each of the following sets of parameters either give an example of a design with these parameters or prove that no such design exists.

(a) $2-(361,57,7)$.
ANSWER: [**6 MARKS**] If such a design exists, then it has b blocks and each point lies in r blocks, where

$$b=\frac{\lambda v(v-1)}{k(k-1)}, \quad r=\lambda\frac{(v-1)}{(k-1)}.$$

Here

$$b=\frac{7\times 361\times 360}{57\times 56}=285$$

and

$$r=\frac{7\times 360}{56}=45.$$

Such a design satisfies $v>k$, so by Fisher's Inequality it would follow that $b\geq v$. The fact that here $b<v$ shows that no design with these parameters can exist.

(b) $2-(307,18,1)$.
ANSWER: [**3 MARKS**] Such a design would be a $2-(17^2+17+1,17+1,1)$ design. Since 17 is a prime, such a design does exist (a projective plane of order 17).

(c) $2-(70,24,10)$.
ANSWER: [**3 MARKS**] If such a design exists, then it has b blocks, where

$$b=\frac{\lambda v(v-1)}{k(k-1)}.$$

Here

$$b=\frac{10\times 70\times 69}{24\times 23}=\frac{175}{2}$$

and $\frac{135}{2}\notin\mathbb{N}$.We conclude that no design with these parameters can exist.

(d) $2-(106,15,2)$.
ANSWER: [**6 MARKS**] If such a design exists, then it has b blocks and each point lies in r blocks, where

$$b = \frac{\lambda v(v-1)}{k(k-1)}, \quad r = \lambda \frac{(v-1)}{(k-1)}.$$

Here

$$b = \frac{2 \times 106 \times 105}{15 \times 14} = 106$$

and

$$r = \frac{2 \times 105}{14} = 15.$$

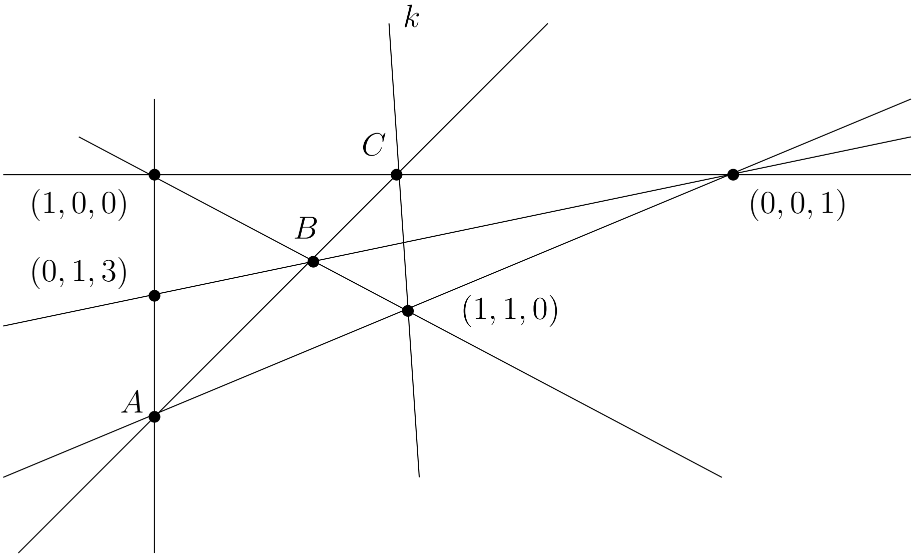
Such a design satisfies $b = v$ so would be symmetric. It also has v even so that necessarily $k - \lambda$ is a perfect square. Here, however, $k - \lambda = 13$ which is not a perfect square. We conclude that no design with these parameters can exist.

A2. Find all the primes p such that $(1,7,11)$ and $(7,11,1)$ are the same point in $\mathbb{P}_2(p)$.
ANSWER: If $(1,7,11)$ and $(7,11,1)$ are the same point in $\mathbb{P}_2(p)$, then $(7,11,1) = \lambda(1,7,11)$ for some $\lambda \in \mathbb{Z}_p$. This happens precisely when there exists a solution in λ to the simultaneous equations:

$$7 = \lambda,\ 11 = 7\lambda,\ 1 = 11\lambda,$$

i.e., if $\lambda = 7$ and $11 = 49$, $1 = 77$. This happens precisely when $38 = 76 = 0$ in $\mathbb{Z}_p$, i.e., when $p = 2$ or 19.

A3. The following diagram shows some points and lines in $\mathbb{P}_2(5)$. Find the co-ordinates of the points marked A, B and C and the equation of the line marked k.



ANSWER: The points $(1, 0, 0)$ and $(0, 1, 3)$ lie on the line $z = 3y$. The points $(0, 0, 1)$ and $(1, 1, 0)$ lie on the line $x = y$. A is the intersection of these two lines so has $z = 3y = 3x$, so $A = (1, 1, 3)$.

The points $(1, 0, 0)$ and $1, 1, 0)$ lie on the line $z = 0$. The points $(0, 1, 3)$ and $(0, 0, 1)$ lie on the line $x = 0$. B is the intersection of these two lines so has co-ordinates $(0, 1, 0)$.

The line AB has equation $z = 3x$. The points $(1, 0, 0)$ and $(0, 0, 1)$ lie on the line $y = 0$. The point C is the intersection of these two lines, so has co-ordinates $(1, 0, 3)$.

The equation of k can be found by the determinental formula

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 0,$$

i.e., $x.3 - y.3 + z.(-1) = 0$, i.e., $3x - 3y - z = 0$, or by inspection.

A4. Suppose that $\mathcal{C}$ is a non-singular conic in $\mathbb{P}_2(p)$ for some odd prime p, containing the points $P = (1, 0, 0)$, $Q = (0, 0, 1)$ and $R = (1, -1, 0)$. Moreover the tangents at P and Q meet at the point $T = (0, 1, 1)$. Find the equation of $\mathcal{C}$.

ANSWER: Let the conic have equation $ax^2+by^2+cz^2+2fxy+2gyz+2hzx=0$.

- $P=(1,0,0)$ lies on the conic so $a=0$.
- $Q=(0,0,1)$ lies on the conic so $c=0$.
- $R=(1,-1,0)$ lies on the conic so $b-2f=0$.

We deduce that $b=2f$. Hence the equation of the conic reduces to $2fy^2+2fxy+2gyz+2hzx=0$ and the matrix is $M=\begin{pmatrix} 0 & f & h \\ f & 2f & g \\ h & g & 0 \end{pmatrix}$. The tangent at P is given by $XMP^T=0$, i.e., $(x,y,z)(0,f,h)^T=0$, i.e., $fy+hz=0$. The tangent at Q is given by $XMQ^T=0$, i.e., $(x,y,z)(h,g,0)^T=0$, i.e., $hx+gy=0$. If T lies on both of these, then

$$f+h=0 \text{ and } g=0.$$

Therefore $a=b=g=0$, $h=-f$, $b=2f$ and the conic is $2fy^2+2fxy-2fzx=0$, i.e., $y^2+xy-zx=0$. (Non-singular implies $f\neq 0$.)

Section B

B1. (a) In the context of designs, what is meant by a *projective plane*? Show that a projective plane is a symmetric design with an odd number of points.

ANSWER: [6 MARKS, *bookwork*] A projective plane is a $2-(n^2+n+1, n+1, 1)$ design for some n. We calculate

$$b=\frac{\lambda v(v-1)}{k(k-1)}=\frac{(n^2+n+1)(n^2+n)}{(n+1)n}=n^2+n+1$$

so a projective plane is symmetric. Furthermore $n^2+n+1=(n+1)n+1$ with $(n+1)n$ necessarily even. Therefore a projective plane has an odd number of points.

(b) Suppose that the design $(X,\mathcal{B})$ is a projective plane. Select a particular block $B\in\mathcal{B}$ and let

$$Y=X\setminus B, \text{ and } \mathcal{D}=\{B'\cap Y:\ B'\in\mathcal{B},\ B'\neq B\}.$$

Prove that $(Y,\mathcal{D})$ is a $2-(n^2,n,1)$ design for some n.

ANSWER: [9 MARKS, *bookwork*] X is a $2-(n^2+n+1, n+1, 1)$

design for some n. Given that we have a symmetric design, any two distinct blocks meet in $\lambda = 1$ points. Thus for any $B' \in \mathcal{B}$ with $B' \neq B$ we have $|B' \cap B| = 1$. It follows that $|B' \cap Y| = (n+1) - 1 = n$. Thus $\mathcal{D}$ is a non-empty collection of $n-$subsets of Y. Any 2-subset T of Y is a 2-subset of X, so lies in a unique member B' of $\mathcal{B}$. Now $B' \neq B$ (since B contains no points of Y) and T lies in $B' \cap Y$. Hence each 2-subset of Y lies in exactly 1 member of $\mathcal{D}$. Y has $(n^2 + n + 1) - (n + 1) = n^2$ points, so $(Y, \mathcal{D})$ is a $2 - (n^2, n, 1)$ design.

(c) Prove that there is no projective plane of order 22. [You may assume that if there exists a symmetric $2 - (v, k, \lambda)$ design with v odd, then the equation

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

has a solution in integers x, y, z not all 0.]

ANSWER: [15 MARKS, *similar to homework example*] A projective plane of order 22 would be a (symmetric) $2-(507, 23, 1)$ design. By the given Theorem, if there exists a $2 - (507, 23, 1)$ design, then the equation

$$x^2 = (23 - 1)y^2 + (-1)^{(507-1)/2} \times 1 \times z^2$$

has a solution in integers x, y, z not all 0. The equation simplifies to $x^2 = 22y^2 - z^2$. *(5 marks)*

Assuming that there is a solution with x, y, z not all 0, x cannot be 0 (because if $x = 0$, then we would have a solution in integers y, z, not both 0, to the equation $z^2 = 22y^2$, which is impossible since $\sqrt{22}$ is irrational), so we can select a solution: $x = x_0, y = y_0, z = z_0$ not all 0, for which $|x|$ has its smallest possible value. Then $x_0^2 + z_0^2 = 22y_0^2 \equiv 0 \pmod{11}$. If we consider the squares (mod 11):

$$0^2 = 0, 1^2 = 1, 2^2 = 4, 3^3 = 9, 4^2 = 16 = 5, 5^2 = 25 = 3$$

$$6^2 = 36 = 3, 7^2 = 49 = 5, 8^2 = 64 = 9, 9^2 = 81 = 4, 10^2 = 100 = 1$$

we see that the only way in which $x_0^2 + z_0^2$ can be 0 (mod 11) is if $11 \Big| x_0$ and $11 \Big| z_0$. Then $11^2 \Big| x_0^2 + z_0^2 = 22y_0^2$ so $11 \Big| 2y_0^2$ and (given that 11 is prime) $11 \Big| y_0$. Now $(x_0/11)^2 + (z_0/11)^2 = 22(y_0/11)^2$, so

$x = x_0/11, y = y_0/11, z = z_0/11$ is also a solution in integers, not all 0. But $|x_0/11| < |x_0|$ which is a contradiction to the choice of x_0 with $|x_0|$ as small as possible. This contradiction tells us that there can be no $2 - (507, 23, 1)$ design. *(10 marks)*

B2. (a) Suppose that $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are distinct points of $\mathbb{P}_2(p)$.

i. Consider the points Q and $P+\lambda Q = (x_1+\lambda x_2, y_1+\lambda y_2, z_1+\lambda z_2)$ arising from all possible choices of $\lambda \in \mathbb{Z}_p$. Prove that these points lie on the line PQ and that they are distinct.

ii. Show that each point on the line PQ can be written in the form Q or $P + \lambda Q$ for some $\lambda \in \mathbb{Z}_p$.

iii. Prove that PQ consists of all points of the form $\alpha P + \beta Q$ with $(\alpha, \beta) \neq (0, 0)$.

ANSWER: [15 MARKS, *bookwork slightly reformulated*]

i. Since P and Q are distinct, neither of (x_1, y_1, z_1), (x_2, y_2, z_2) is a scalar multiple of the other. Observe that

$$\begin{vmatrix} x_1+\lambda x_2 & y_1+\lambda y_2 & z_1+\lambda z_2 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

(first subtracting λ Row 3 from Row 1, then Row 2 from Row 1). Thus each of the $p + 1$ points $P + \lambda Q$ together with Q lies on the line PQ. Next observe that these points are distinct, for if

$$(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2) = \alpha(x_1 + \mu x_2, y_1 + \mu y_2, z_1 + \mu z_2),$$

then

$$(1 - \alpha)(x_1, y_1, z_1) = (\alpha\mu - \lambda)(x_2, y_2, z_2)$$

which can only happen if $\alpha = 1$ and $\mu = \lambda$. Similarly if

$$(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2) = \alpha(x_2, y_2, z_2),$$

then

$$(x_1, y_1, z_1) = (\alpha - \lambda)(x_2, y_2, z_2)$$

which can't happen at all. *(10 marks)*

ii. There are exactly $p+1$ points on PQ so these must be precisely the p points $P + \lambda Q$ together with Q. *(2 marks)*

iii. Now scalar multiples of $P+\lambda Q$ have the form $\alpha P+\beta Q$. Moreover $\alpha P+\beta Q$ is the same point as $P+(\beta/\alpha)Q$ if $\alpha \neq 0$ and as Q if $\alpha = 0$. Hence PQ consists of all $\alpha P+\beta Q$ (with repetitions). *(3 marks)*

(b) Suppose that $V = (1,1,1)$, $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, and $D = (1,-1,1)$ are points in $\mathbb{P}_2(p)$ (with p odd), and that $T = (1,\alpha,1)$ for some fixed number α distinct from 1 and -1. Let P and Q be the intersections of VC with AD and AT respectively. Let R and S be the intersections of VA with CD and CT respectively. Prove that PR, QS and AC are concurrent.

ANSWER: [15 MARKS, *Unseen, though similar examples will have been seen*] VC has equation $x = y$ and AD has equation $z = -y$. Therefore $P = (1,1,-1)$.

AT has equation $y = \alpha z$ so meets VC at $Q = (\alpha,\alpha,1)$.

Similarly VA has equation $y = z$ and CD has equation $x = -y$. Therefore $R = (-1,1,1)$. CT has equation $y = \alpha x$ so meets VA at $S = (1,\alpha,\alpha)$. *(6 marks)*

PR has equation

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 0,$$

i.e., $x.2 - y.0 + z.2 = 0$, i.e., $2x + 2z = 0$, i.e., $z = -x$ (or by inspection).

QS has equation

$$\begin{vmatrix} x & y & z \\ \alpha & \alpha & 1 \\ 1 & \alpha & \alpha \end{vmatrix} = 0,$$

i.e., $x.(\alpha^2-\alpha) - y.(\alpha^2-1) + z.(\alpha^2-\alpha) = 0$. Given that $\alpha \neq 1$, this equation may be simplified to $\alpha x - (\alpha+1)y + \alpha z = 0$.

The lines $z = -x$ and $\alpha x-(\alpha+1)y+\alpha z = 0$ meet where $(\alpha+1)y = 0$, so $y = 0$ and $z = -x$. Hence the intersection is $(1,0,-1)$. This point lies on AC because it can be expressed $(1,0,0) - (0,0,1)$. Hence PR, QS and AC are concurrent. *(9 marks)*

B3. (a) Let $\mathcal{C}$ be the non-singular conic $y^2 = zx$ in $\mathbb{P}_2(\mathbb{K})$. Prove that the points on $\mathcal{C}$ are precisely the points of the form

$$(1, 0, 0) \text{ and } (\theta^2, \theta, 1)$$

where $\theta \in \mathbb{K}$.

ANSWER: [7 MARKS (changed from 8), *bookwork*] Observe that all the given points lie on the conic. Suppose that $P = (p, q, r)$ lies on the conic, so $q^2 = pr$. If $r = 0$, then $q^2 = p \times 0$ so $q = 0$ and the point is $(p, 0, 0)$, equivalently $(1, 0, 0)$.

If $r \neq 0$, then we can write $P = (\phi, \theta, 1)$, where $\phi = p/r, \theta = q/r$. Then

$$\theta^2 = \frac{q^2}{r^2} = \frac{pr}{r^2} = \frac{p}{r} = \phi.$$

Hence $P = (\theta^2, \theta, 1)$. Therefore all points on the conic have the given form.

(b) Prove that the tangent to $\mathcal{C}$ at the point $(\theta^2, \theta, 1)$ has equation

$$x - 2\theta y + \theta^2 z = 0$$

and that the tangent at the point $(1, 0, 0)$ has equation $z = 0$.

ANSWER: [8 MARKS (changed from 9), *bookwork*] For the sake of tidiness, note that we can write the equation of $\mathcal{C}$ as $2y^2 - 2zx = 0$. The matrix M of $\mathcal{C}$ can then be written $M = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The tangent at $(1, 0, 0)$ has equation

$$(x, y, z) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0, \text{ i.e., } (x, y, z) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0,$$

in other words, $z = 0$.

The tangent at $(\theta^2, \theta, 1)$ has equation

$$(x, y, z) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta^2 \\ \theta \\ 1 \end{pmatrix} = 0, \text{ i.e., } (x, y, z) \begin{pmatrix} -1 \\ 2\theta \\ -\theta^2 \end{pmatrix} = 0,$$

in other words, $-x + 2\theta y - \theta^2 z = 0$, equivalently $x - 2\theta y + \theta^2 z = 0$.

(c) Now assume that $\mathbb{K} = \mathbb{Z}_5$ (i.e., $\mathcal{C}$ is the non-singular conic $y^2 = zx$ in $\mathbb{P}_2(5)$). Let A, B, C and F be the points $(1, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 1)$ respectively. Find the co-ordinates of the points P and Q on $\mathcal{C}$ whose tangents pass through F. Let ℓ be the tangent to $\mathcal{C}$ at P and m the tangent at Q, and let S and T be the intersections of ℓ and m with the tangents at A and B respectively. Show that AQ, BP, CF and ST are concurrent. [Note: when finding the two points on $\mathcal{C}$ whose tangents pass through F, it does not matter which way you label them as P and Q.]

ANSWER: [15 MARKS (changed from 13), ***Unseen, although problems of this nature will have been encountered***]The tangent at A has equation $z = 0$. Clearly this line does not pass through F. The tangent at the point $(\theta^2, \theta, 1)$ has equation $x - 2\theta y + \theta^2 z = 0$ and this passes through F precisely when $1 + \theta^2 = 0$, i.e., $\theta = 2$ or 3. Hence P and Q are the points $(-1, 2, 1)$ and $(-1, 3, 1)$ (in some order). We take $P = (-1, 2, 1)$ and $Q = (-1, 3, 1)$. *(4 marks)*

ℓ has equation $x + y - z = 0$ and m has equation $x - y - z = 0$. The tangents at A and B have equations $z = 0$ and $x = 0$ respectively. Thus $S = (1, -1, 0)$ and $T = (0, 1, -1)$. *(4 marks, changed from 3)*

By inspection AQ has equation $y = 3z$ and BP has equation $y = -2x = 3x$. These lines meet at the point $(1, 3, 1)$. The line CF has equation $x = z$ so also passes through $(1, 3, 1)$. *(4 marks, changed from 3)*

The line ST includes the point given by $(1, -1, 0) - (0, 1, -1) = (1, -2, 1) = (1, 3, 1)$. Hence AQ, BP, CF and ST are concurrent. *(3 marks)*

[ALTERNATIVE SOLUTION. If a candidate chooses $P = (-1, 3, 1)$ and $Q = (-1, 2, 1)$, then:
ℓ has equation $x - y - z = 0$ and m has equation $x + y - z = 0$. The tangents at A and B have equations $z = 0$ and $x = 0$ respectively. Thus $S = (1, 1, 0)$ and $T = (0, 1, 1)$.
By inspection AQ has equation $y = 2z$ and BP has equation $y = -3x = 2x$. These lines meet at the point $(1, 2, 1)$. The line CF has equation $x = z$ so also passes through $(1, 2, 1)$.

The line ST includes the point given by $(1, 1, 0)+(0, 1, 1) = (1, 2, 1)$. Hence AQ, BP, CF and ST are concurrent.]

AJD July 2009