MAS2203/3203 Graph Theory: Exercises

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1 Basics

1.1 Which of the following pairs of sets V and E define a graph with vertices V and edges E? Draw those which are graphs and explain why those which are not are not.

(a) $V = \{a, b, c, d\}$, $E = \{e, f, g, h\}$, with $e = \{a, a\}$, $f = \{b, b\}$, $g = \{c, c\}$, $h = \{c, d\}$;

(b) $V = \{a, b, c, d\}$, $E = \{e, f, g, h\}$, with $e = \{a, a\}$, $f = \{e, e\}$, $g = \{c, c\}$, $h = \{c, d\}$;

(c) $V = \{a, b, c, d\}$, $E = \{e, f, g, h\}$, with $e = \{a, a\}$, $f = \{a, a\}$, $g = \{a, a\}$, $h = \{a, a\}$;

(d) $V = \{e, f, g, h\}$, $E = \{a, b, c, d\}$, with $e = \{a, a\}$, $f = \{a, a\}$, $g = \{a, a\}$, $h = \{a, a\}$;

1.2 Where possible draw the graphs below. If you can't draw the graph say why not.

(a) A simple graph with 1 edge and 2 vertices.

(b) A simple graph with 2 edges and 2 vertices.

(c) A non-simple graph with no loops.

(d) A non-simple graph with no multiple edges.

(e) A graph with 6 vertices and degree sequence $(1, 2, 3, 4, 5, 5)$.

(f) A simple graph with 6 vertices and degree sequence $(1, 2, 3, 4, 5, 5)$.

(g) A simple graph with 6 vertices and degree sequence $(2, 3, 3, 4, 5, 5)$.

1.3 (a) Is it true that any two isomorphic graphs have the same number of vertices and edges?

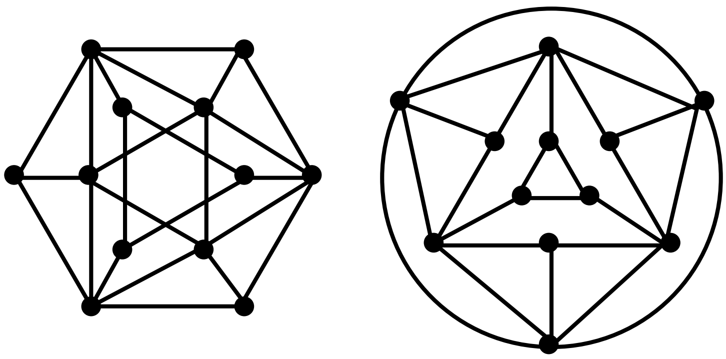
(b) Let G_1 and G_2 be graphs and let (ϕ, θ) be an isomorphism from G_1 to G_2. If v is a vertex of G_1 show that $\deg(v) = \deg(\phi(v))$.

(c) Show that isomorphic graphs have the same degree sequence.

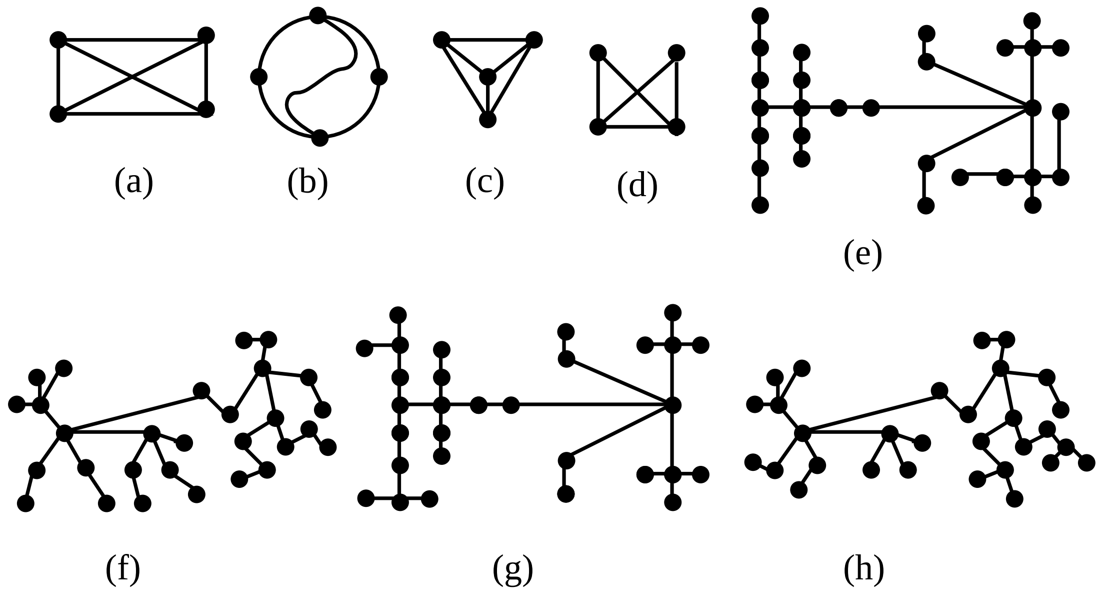
(d) If two graphs have the same degree sequence, need they be isomorphic?

1.4 There are 11 non-isomorphic simple graphs with 4 vertices. Draw all those with (a) no edges; (b) 1 edge; (c) 2 edges; (d) 3 edges; (e) 4 edges (f) 5 edges; (g) 6 edges; (h) more than 6 edges (if any).

1.5 Show, by labelling the vertices, that the graphs below are isomorphic. (You need not label the edges.)



1.6 Sort out the following into families of isomorphic graphs. (There is no need to display isomorphisms, just write down (a) is isomorphic to ... etc..)



1.7 Let G be a simple graph with $n \geq 2$ vertices. Can G have a vertex of degree 0? Can G have a vertex of degree $n-1$? Can G have a vertex of degree n or more?

(a) Show that G cannot have degree sequence $(0, 1, \ldots, n-1)$. [*Hint:* Consider the vertex of degree 0 and that of $n-1$.]

(b) Show that G must have 2 vertices of the same degree.

1.8 **The Maple package** `networks`. In this and all other Maple exercises `` `Q:` `` is used to denote the drive `` `\\campus\software\mathematics and statistics\course data` ``, which is normally mapped to the `` `Q:` `` drive. (See the instructions at `http://www.mas.ncl.ac.uk/oracle/datadrive.html`.)

(a) Start Maple open the worksheet `` `Q:/223/basic_examples.mw` `` and run through it. Note that the first thing done is to load the package `networks` by issuing the command
`> with(networks);`
A list of the functions available appears. You must load `networks` before you can do graph theory; use a ":" to suppress the list of functions. Now do the following exercise. Read the help page for the function `graph`. Define a graph G with vertices $1, \dots, 5$ and edges

$$\{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,4\}, \{3,4\}, \{4,5\}.$$

Now read the help page for the function `draw`. Draw the graph G. Use the command `degreeseq` to make Maple compute the degree sequence of G.

(b) Maple has a library of graphs. Look up the help pages for `complete`, `petersen`, `dodecahedron` and any others you can find. Now use these to define and draw the following graphs.

(i) $K16$ the complete graph K_{16}, first with all vertices around a circle, as usual, then using
`>draw(Concentric([seq(i,i=1..8)]),K16);`
so that the vertices appear in two concentric circles. Left click on the pictures that Maple draws and drag the corners out to make the drawings big enough so that the egdes can be seen separately.

(ii) Let $v10v3$, $v10v10$ and $v10v14$ be the complete bipartite graphs $K_{10,3}$, $K_{10,10}$ and $K_{10,14}$, respectively. First use the plain draw command, then use the command
`>draw(Linear([1,2,3,4,5,6,7,8,9,10]),complete(10,??));`
to draw these, so that the vertices appear in 2 lines.

(iii) Draw the Petersen graph and the dodecahedron graph in a variety of ways, using the Concentric and Linear parameters of the draw command.

(iv) Look up the help page for the `random` function. Draw a random graph R with the same number of vertices as roughly twice your age and about half as many edges.

(c) Use Maple to compute the degree sequence of the graph R of the previous part 1.8(b)iv of the question. Then use the command
`>nops(edges(R));`
to count the number of edges of this graph. Use this number and the degree sequence to verify the Handshaking Lemma for the graph R.

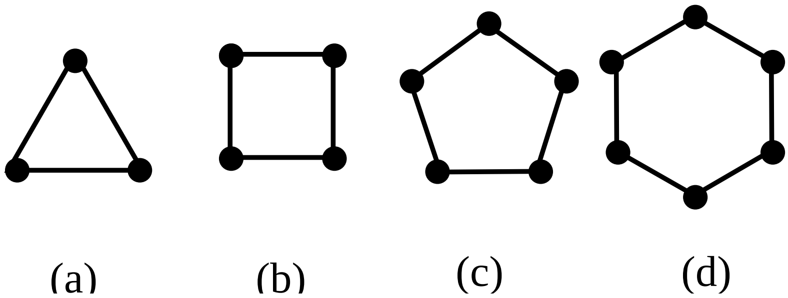
2 Subgraphs, Walks and Connectedness

2.1 Which of the following could be subgraphs of

(i) the Petersen graph?

(ii) the complete bipartite graph $K_{3,3}$?

(iii) the cube?



2.2 We define the *complement* of a simple graph G to be the simple graph $\bar{G}$ with vertex set $V(\bar{G}) = V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

What are the complements of N_d and K_d? Is the complement of a connected graph necessarily connected? Draw the following graphs

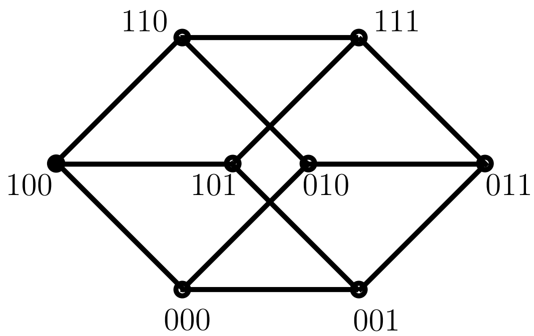
(a) the complement of the null graph N_5,

(b) the complement of the complete graph K_7,

(c) the complete bipartite graph $K_{3,5}$, and its complement,

(d) the wheel W_5, and its complement.

2.3 (a) Draw the four-cube Q_4 by hand as follows. Start with the drawing of Q_3 below.



Make two copies of this. Prefix the labels in one copy with 0 and in the other copy with 1. Now add edges joining one copy to another according to the definition of Q_4.

(b) Start maple and load the package `networks`. The `cube` function can be used to obtain the graphs Q_n. The commands

```
>Q3:=cube(3):
>draw(Linear([0,3,5,6]),Q3);
```

display the graph Q_3 as a bipartite graph. The sequence $0, 3, 5, 6$ is chosen using Lemma 4.4, as $0, 3, 5$ and 6 are all the integers between 0 and 7 which have an even number of digits equal to 1 when written in binary.

Define the graph `Q4` as the cube Q_4 using a similar command and, using Lemma 4.4, draw it in such a way that it is clear that it is bipartite, as in the case of Q_3.

(c) Repeat the previous part of the question for Q_5.

2.4 Draw all simple cubic graphs with at most eight vertices. (A regular graph of degree 3 is called *cubic*.)

2.5 If v is a vertex of a graph G then we define the graph $G - v$ to the graph obtained from G by deleting v and all edges incident to v. If e is an edge of G then $G - e$ is the graph obtained by deleting the edge e from G. Let G be a graph with n vertices and m edges. Let v be a vertex of G of degree d and let e be an edge of G. How many vertices and edges have $G - v$ and $G - e$? Justify your answers.

2.6 Start maple and load the package `networks`. `` `Q:/223/subgraph_example.mw` `` is a Maple worksheet that may help with this exercise. Define and draw a graph `G` equal to an icosahedron. Use `induce` to construct a subgraph `H` with 6 vertices containing a cycle of length 6. Draw `H`. Now use `induce` again to construct a subgraph `C` of `H` which is a cycle graph with 6 vertices (use a list of edges this time). Draw `C` with its edges arranged in circular fashion.

2.7 Classify each of the following statements as True or False. In each case either give a counterexample or a brief explanation.

(a) If G_1 and G_2 are subgraphs of a graph G with $V(G_1) \cap V(G_2) = \emptyset$ then $E(G_1) \cap E(G_2) = \emptyset$.

(b) If G_1 and G_2 are subgraphs of a graph G with $V(G_1) \cap V(G_2) = \emptyset$ then G is disconnected.

2.8 Let G be a simple graph with m edges $e_1, \cdots, e_m$. We define the *line graph* $L(G)$ of G as follows. $L(G)$ has m vertices $V_1, \cdots, V_m$ and $\{V_i, V_j\}$ is an edge of $L(G)$ if and only if the edges e_i and e_j are incident to some vertex v of G.

(a) Draw the line graphs of K_3, $K_{1,3}$, C_6 and W_5.

(b) Find an expression for the number of edges of $L(G)$ in terms of the degrees of vertices of G.

(c) Show that if G is a regular simple graph of degree k then $L(G)$ is regular of degree $2k - 2$.

(d) Show that $L(K_5)$ is the complement of the Petersen graph. (See question 2.2 for the definition of complement.)

2.9 In the Petersen graph find

(a) a trail of length 5;

(b) a path of length 9;

(c) cycles of length 5 and 9.

Do you think the Petersen graph has any circuits, of positive length, which are not cycles? If not why not?

2.10 In K_5 find circuits of lengths 6 and 10. Is there a cycle in K_5 of length more than 5? If not why not?

2.11 The *girth* of a graph is the length of its shortest cycle. Find the girths of
(a) K_9; (b) $K_{7,5}$; (c) C_8; (d) W_7; (e) Q_5;
(f) the Petersen graph; (g) the graph of the dodecahedron.

2.12 Find the girths of
(a) K_n, for $n \geq 3$; (b) $K_{r,s}$, for $r, s \geq 2$; (c) C_n, for $n \geq 3$;
(d) W_n, for $n \geq 4$; (e) Q_n, for $n \geq 2$.
(You need not justify your answers.)

2.13 Prove that if G is a simple graph then G and its complement $\bar{G}$ cannot both be disconnected. (See question 2.2 for the definition of complement.)

2.14 Let G be a graph and let u and v be vertices of G (which may or may not be the same). Suppose that G contains two distinct paths P and P' from u to v. ("Distinct" means "not equal".) Show that G contains a cycle. [**Hint:** This is like the proof of Lemma 7.2. Choose P and P' so that these paths are of minimal length (amongst all those satisfying the given conditions). Follow one path from u to v and the other back from v to u. If this isn't a cycle something can be shortened.]

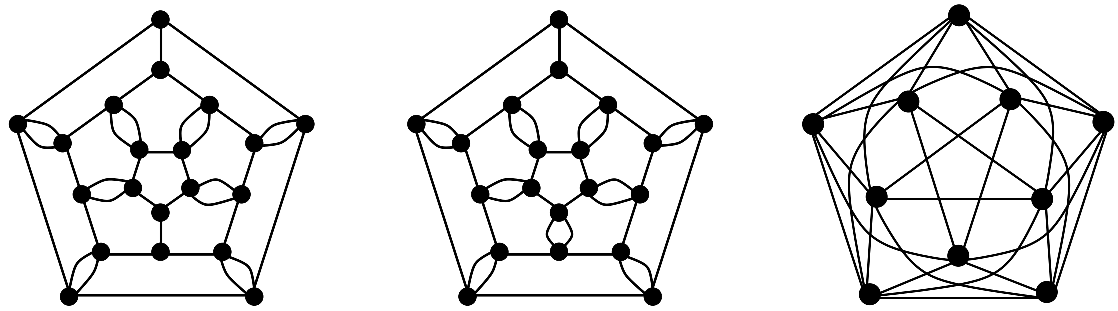
3 Eulerian and Hamiltonian graphs

3.1 Which of the following graphs are Eulerian and which are semi-Eulerian but not Eulerian?

(a) K_5, K_{2n}, K_{2n+1}, where $n \geq 1$.

(b) $K_{2,3}$, $K_{2,s}$, $K_{r,s}$, where $r, s \geq 1$.

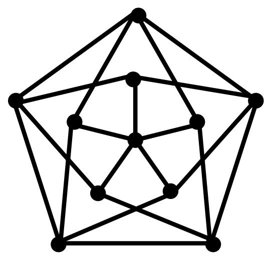
3.2 Which of the following graphs are Eulerian and which are semi-Eulerian but not Eulerian?

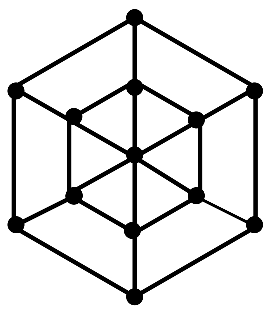


(a) (b) (c)

3.3 Find a decomposition into cycle graphs of the following graphs, showing each cycle graph on a separate diagram.
(a) $K_{4,4}$; (b) $K_{2,6}$; (c) K_7.

3.4 Find a decomposition into 3 path graphs of each of the graphs shown below.
(a) The Grötzsch graph: (b)





3.5 (a) Is it possible for a knight to travel around a chessboard in such a way that every move occurs exactly once? (A move between two squares is said to occur if it is traversed exactly once.)

(b) Repeat part 5a for a rook instead of a knight.

3.6 (a) Show that the line graph of a simple Eulerian graph is Eulerian. (See question 2.8 for the definition of line graph.)

(b) If the line graph of a simple graph G is Eulerian need G be Eulerian?

3.7 Let G be a connected graph with $2k$ vertices of odd degree (and all other vertices of even degree).

(a) Show that by adding k edges to G an Eulerian graph can be constructed.

(b) Use the previous part of the question to show that G has a decomposition into k trails.

(c) Suppose that G has a decomposition into k' trails. Explain why every vertex of odd degree of G must be an end-point of one of these k' trails. Use this fact to argue that there is no decomposition of G into k' trails, where $k' < k$.

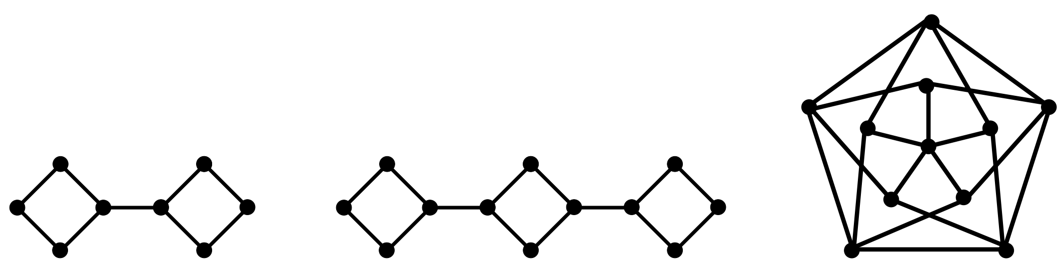
3.8 Secret service agents A, B, C, D, E and F must meet in pairs in an underground bunker to exchange vital information. For security reasons no two pairs may meet at once. To preserve the integrity of the information it is preferred that one of the participants at each meeting (except the last) is present at the next. The following pairs must meet (in some order).
A must meet B and F.
B must meet C, D and E.
C must meet E and F.
D must meet F.
E must meet F.

How can this be done so that only these pairs meet, in the shortest possible time. Justify your answer and give an appropriate ordering if possible. [**Hint:** Construct a graph with vertices corresponding to agents and two vertices joined if their agents meet. The requirement is then to list all edges in such a way that for each edge in the list one of its vertices appears in the next edge on the list. This means you need an ________ _____.]

3.9 Which of the following graphs are Hamiltonian?
(a) K_5; (b) $K_{5,3}$; (c) the graph of the octahedron; (d) W_6.

3.10 (a) For which values of r and s is $K_{r,s}$ Hamiltonian?
(b) For which values of n is Q_n Hamiltonian?

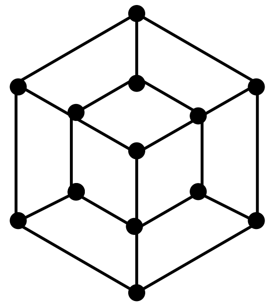
3.11 Which of the following are Hamiltonian and which are semi-Hamiltonian. Give your reasons. In particular find Hamiltonian closed paths for those which are Hamiltonian and Hamiltonian paths for those which are semi-Hamiltonian but not Hamiltonian.



(a) (b) (c)

3.12 (a) Prove that if G is bipartite with an odd number of vertices then G is non-Hamiltonian.

(b) Deduce that the graph below is non-Hamiltonian.



(c) Show that if n is odd it is not possible for a knight to visit all squares of an $n \times n$ chessboard exactly once and return to its starting point.

3.13 Which of the Platonic graphs are Hamiltonian?

3.14 King Arthur and his knights wish to sit at the round table every evening in such a way that each person has different neighbours on each occasion. If there are 10 knights (and 1 king) for how long can they do this? If King Arthur wishes to make sure this can be done for 7 evenings in a row how many knights must he have?

3.15 Decompositions into Hamiltonian closed paths of the complete graphs K_7 and K_{11} can be found by the following method. Arrange all the vertices around a regular polygon. For the first closed path choose consecutive vertices, for the second choose every second vertex, for the third every third and so on. Draw this decomposition of K_7. Does this method work for K_9 and if not why not? Make a conjecture as to which numbers this will work for.

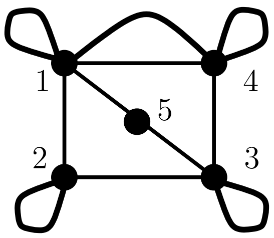
4 Trees

4.1 Draw all trees with

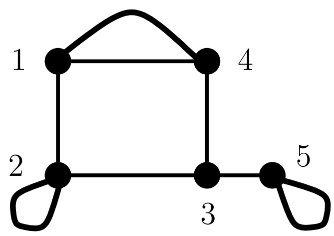
(a) 6 vertices (there are 6);

(b) 7 vertices (there are 11).

4.2 Draw all spanning trees for the graphs below.



(a)



(b)

4.3 Let G be a connected graph and let e be an edge of G.

(a) Show that if e appears in every spanning tree for G that e is a bridge.

(b) Show that if e appears in no spanning tree for G then e is a loop.

4.4 Given vertices u and v of a tree the *distance* $d(u, v)$ from u to v is the length of the path from u to v. A *leaf* of a tree is a vertex of degree 1.

(a) Let T be a tree with at least 2 vertices.

i. Show that if P is a path from u to v and $\deg(u) \geq 2$ then the path P can be lengthened.

ii. Let P be a path of maximal length in T. Show that if P is from a vertex u to a vertex v then u and v are leaves.

iii. Complete a proof that any tree with at least 2 vertices has at least 2 leaves.

(b) Let T be a tree with at least 3 vertices. Given a vertex u of T define the *range* of u in T to be

$$\text{range}_T(u) = \max\{d(u, v) : v \in V(T)\}.$$

Let T' be the tree obtained from T by removing all leaves of T and their incident edges.

i. Show that if v is a vertex of T' then $\text{range}_T(v) = \text{range}_{T'}(v) + 1$.

ii. Let c be a vertex of T such that

$$\text{range}_T(c) \leq \text{range}_T(u),$$

for all $u \in V(T)$. Show that c is not a leaf and conclude that c is a vertex of T'.

iii. Show further that

$$\text{range}_{T'}(c) \leq \text{range}_{T'}(u),$$

for all $u \in V(T')$

(c) Define a *centre* of a tree T to be a vertex of T such that $\text{range}_T(c) \leq \text{range}_T(u)$, for all $u \in V(T)$. Show that every tree T has either one or two centres.

4.5 (a) Prove that every tree with more than one vertex is a bipartite graph. (You may assume that every tree with at least two vertices has a leaf.)

(b) Which trees are complete bipartite graphs? Justify your claim.

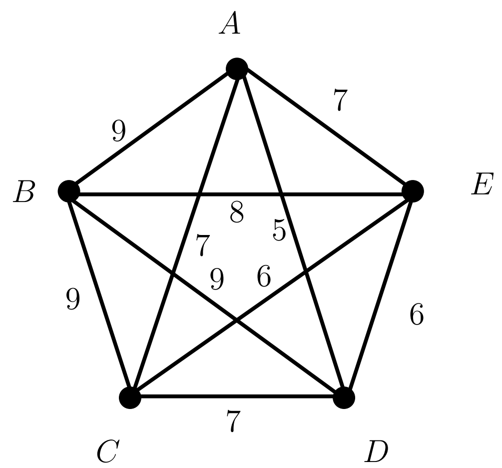
4.6 Let T_1 and T_2 be spanning trees of a connected graph G.

(a) Show that if e is an edge of T_1 then there is an edge f of T_2 such that $(T_1 - e) \cup f$ is a spanning tree for G. [**Hint:** Suppose $e = \{u, v\}$. Then $T_1 - e$ is the disjoint union of two components with u and v in different components. There is a path from u to v in T_2 some edge of which must join one component to the other.]

(b) Deduce that T_1 can be transformed into T_2 by replacing edges of T_1 with edges of T_2, one at time in such a way that a spanning tree for G is obtained at each stage.

5 Weighted graphs and Travelling Salesmen

5.1 (a) Use the Greedy algorithm to find a minimum weight spanning tree for the weighted graph below.



(b) Find lower bounds for the travelling salesman problem corresponding to the graph of part 5.1a

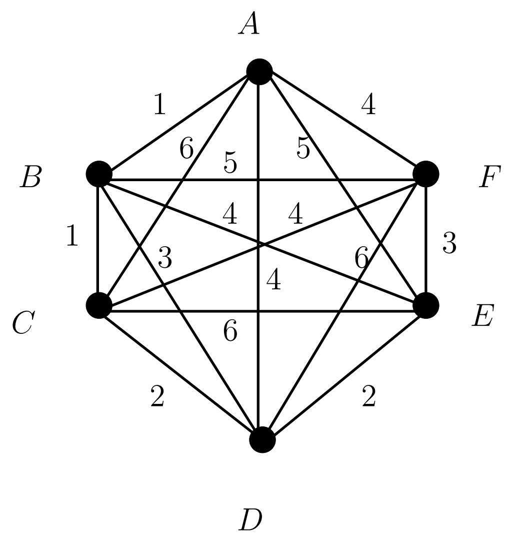
i. by removing vertex B and

ii. by removing vertex E.

(c) Find the solution to the travelling salesman problem of part 5.1b by inspection.

5.2 (a) Find lower bounds for the travelling salesman problem corresponding to the graph below by removing vertex A.

(b) Find the solution to the travelling salesman problem.



5.3 In this exercise Maple is used to try to solve a bigger travelling salesman problem. There are procedures written to do this, which you need to load first. There is also a sample solution to the problem which you will probably need to consult to see how some of the Maple functions work: follow the link to "Travelling Salesman sample run" from `www.mas.ncl.ac.uk/~najd2/teaching/mas223/`. (The sample solution uses 20 vertices as opposed to the 10 vertices used here.)

(a) Remember to load the networks package and then pick up the procedures by typing
`>read `Q:/223/travel_sales.m`;`
(If you have not mapped the `Q` drive this will not work. In this case see the instructions at `http://www.mas.ncl.ac.uk/oracle/datadrive.html`.)

(b) Open the Maple worksheet `` `Q:/223/travel_sales_sample.mw` `` and run through it to familiarise yourself with the procedures in `travel_sales.m`.

(c) Initialise the seed for the random number generator by typing
`>randomize(s);`
where `s` is the integer consisting of the last four digits of your student number.

(d) Generate a graph `R` with 10 vertices and randomly chosen edges and weights by typing
`>R:=random_G(10);`

(e) Draw the graph using the draw function.

(f) Make sure this graph has a Hamiltonian closed path (by inspection) and if not run `random_G` again. (Remember you can enlarge the graph drawing by left clicking on it and dragging the corners.) List all edges of R by typing
`>list_eweights(R);` Using this information and the `induce` function generate and draw the graph H consisting of the vertices of R and the edges of the Hamiltonian closed path you found above. Use `total_eweight` to find the sum of the weights of edges of H.

(g) For each vertex v construct the graph $R - v$ and find the weight of a minimal weight spanning tree. This can be done by typing
`>del_span(R);`
The output from `del_span()` is a table called `min_span` which has 2 columns. (You'll need to use `eval` to see it.) Row i corresponds to the graph $R - i$ obtained by removal of vertex i from the graph R. The first column of each row can be ignored for now. The second column of row i gives the weight of a minimal spanning tree of $R - i$.

(h) For each vertex v of R find the weights of the edges incident to v. (See the sample solution to do this quickly with Maple.) Use the table `min_span` and these weights to find as good a bound as you can for the travelling salesman problem in your original graph R. Compare the lower bound with the weight of the Hamiltonian closed path you found by inspection above.

(i) The first entry of row i of `min_span` is a minimal spanning tree for $R-i$. Draw the minimal spanning tree N corresponding to the lower bound you've found. Now find a lower weight Hamiltonian closed path (if possible). How good is your lower bound?

6 Planar graphs

6.1 The *girth* of a graph is the length of its shortest cycle. Let G be a simple planar connected graph with $n \geq 3$ vertices and m edges. Show that if the girth of G is 5 then

(a) $m \leq \frac{5}{3}(n-2)$.

(b) Use this to show that the Petersen graph is non-planar.

(c) Generalize 1a) to planar graphs of girth g.

6.2 Let G be a graph with n vertices and m edges.

(a) Show that if every vertex of a graph has degree at least 6 then $m \geq 3n$.

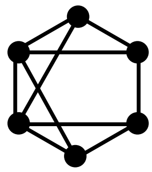
(b) Use the previous part of the question and Corollary 12.8 to show that if G is a simple connected planar graph then G has at least one vertex of degree $d \leq 5$.

6.3 Let G be a simple connected plane graph all of whose faces are pentagons or hexagons.

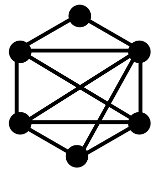
(a) Use Euler's formula to show that if $\deg(v) \geq 3$, for all $v \in V(G)$, then G must have at least 12 pentagonal faces.

(b) If in addition G is regular of degree 3 prove that G has exactly 12 pentagonal faces.

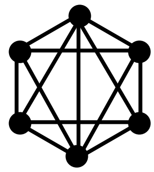
6.4 Which of the following graphs are planar? For those that are planar give a plane drawing. For those that are non-planar find subgraphs which are subdivisions of K_5 or $K_{3,3}$.



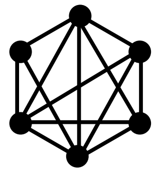
(a)



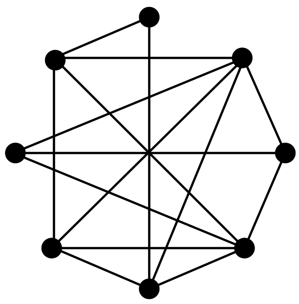
(b)



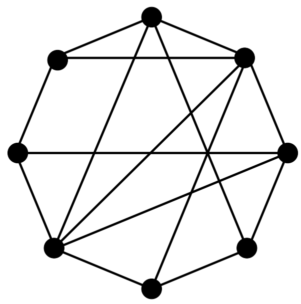
(c)



(d)



(e)



(f)

6.5 Using `networks` verify the planarity of the dodecahedron in Maple with the `isplanar` function. Verify also that K_5 and $K_{3,3}$ are non-planar. Use the function `random` generate a random graph `G` with 20 vertices and 25 edges: type
`>G:=random(20,25):`
Use `isplanar` to test it for planarity. If it fails run `random` again. Repeat until you get a planar graph and then draw it.

7 Colourings of graphs

7.1 Find the chromatic number, giving your reasons, of

(a) K_7;

(b) $K_{3,5}$;

(c) the Petersen graph;

(d) the cube Q_4;

(e) each of the Platonic graphs.

7.2 (a) Let G be a graph obtained from K_n by removing one edge. Using a theorem from the course show that G has an $n-1$ colouring. Describe a method of colouring G with $n-1$ colours and demonstrate that it works using K_5.

(b) Prove that K_d is the only graph with d vertices and chromatic number d.

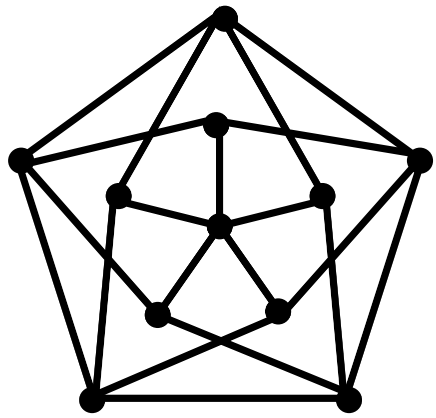
7.3 Let G be a simple graph with n vertices which is regular of degree d. Prove that $\chi(G) \geq n/(n-d)$. [**Hint**: Let v be a vertex of G. How many vertices of G can have the same colour as v?]

7.4 Find proper edge-colourings using 3 colours of the cube and the dodecahedron.

7.5 Find the edge-chromatic number, giving your reasons, of

(a) $K_{3,2}$ (b) $K_{4,3}$ (c) $K_{5,3}$ (d) $K_{111,3}$ (e) $K_{r,s}$.

7.6 Find the edge-chromatic number of the Grötzch graph.



7.7 Eleven students A, B, C, D, E, F, G, H, I, J and K are to take a variety of key-skills courses. There are 6 courses, $1, \ldots, 6$. The class lists are

1 A, B, C
2 B, C, H, K
3 C, D, H, J
4 C, D, E, G
5 E, F, G
6 A, F, H, I

What is the minimal number of periods required to schedule these courses so that no student has a clash? Justify your answer and devise a timetable. [**Hint:** courses = vertices and if two courses are attended by the same student they're joined by an edge. You must assign a time to each vertex so that no clashes occur. You're led to a k-_________ .]

7.8 A department has lecturers A, B, C and D to teach courses a, b, c, d, e and f. A course may be taught by more than one lecturer. The entries in the following table show the number of sections of each course that each lecturer gives. So for example, lecturer A gives 2 sections of course a and gives 1 section of course c.

	a	b	c	d	e	f
A	2	1	1			
B	1	1		1		1
C	1		3			
D		2		1	1	

No lecturer must have a clash and only one section of a course can be taught during a particular period. Calculate the minimum number of time periods required to timetable these courses. Justify your answer and produce a schedule. [**Hint** Make a graph with vertices A, B, C, D, a, b, c, d, e and f. Join a lecturer vertex to a course vertex with n edges if the lecturer gives n sections of the course. Allocate a period to each edge to form a representation of the timetable. You must make sure no courses are taught by two people at the same time and no lecturer has a clash. Therefore you need to find a ______ ____ - _________ .]