Tutorials at 2–3, 3–4 and 4–5 in M413b, on Tuesdays, in odd weeks. i.e. on the following dates. Tuesday the 28th January, 11th February, 25th February, 11th March, 25th March, 6th May.

Assignments are handed in on Tuesdays in even weeks. Deadlines are:

No.	Due	Questions
1	4th February	1.2, 1.3, 1.4, 1.5
2	18th February	
3	4th March	
4	18th March	
5	29th April	
6	13th May	

Introduction

Example 1.1. Systems of objects with connections.

Definition 1.2. A graph G consists of

(i) a finite non–empty set $V(G)$ of vertices and

(ii) a set $E(G)$ of edges

such that every edge $e \in E(G)$ is an unordered pair $\{a, b\}$ of vertices $a, b \in V(G)$.

Example 1.3. The definition illustrated.

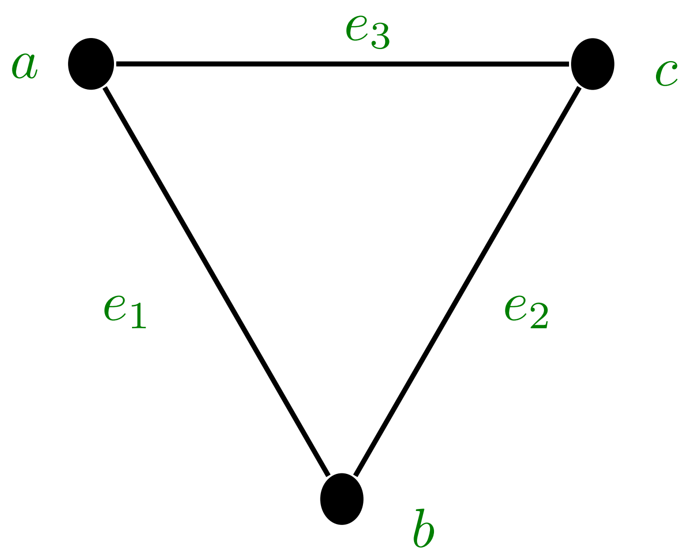
Example 1.3

1. $G_1 = (V_1, E_1)$ where

$$V_1 = \{a, b, c\} \quad \text{and} \quad E_1 = \{e_1, e_2, e_3\}$$

with

$$e_1 = \{a, b\}, e_2 = \{b, c\}, e_3 = \{c, a\}.$$

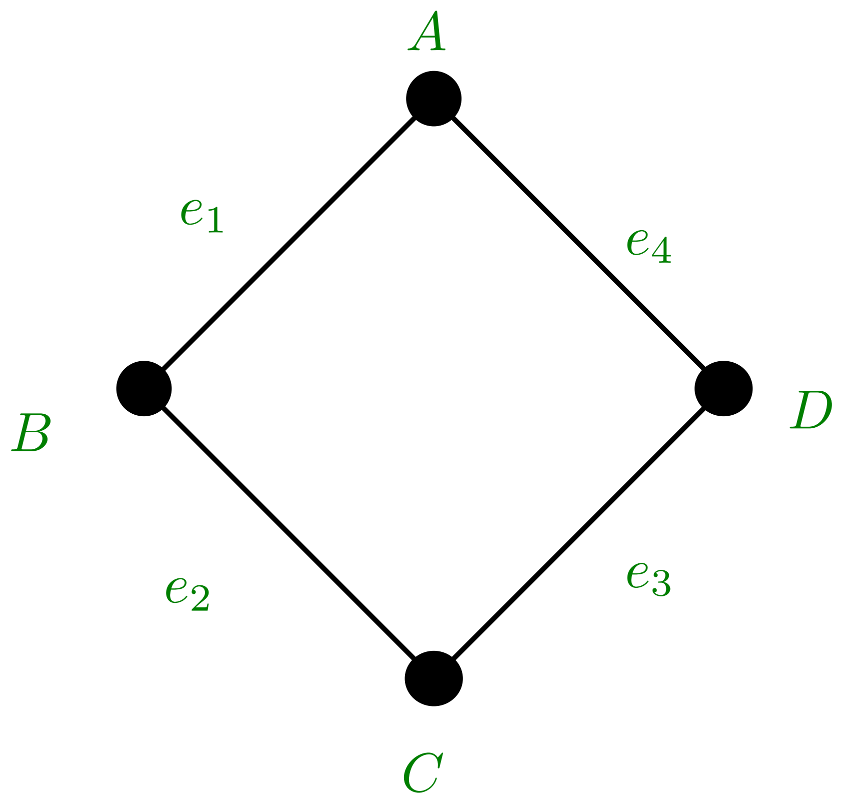


a
e_3
c
e_1
e_2
b

2. $G_2 = (V_2, E_2)$ where

$$V_2 = \{A, B, C, D\} \quad \text{and} \quad E_2 = \{e_1, \dots, e_4\} \quad \text{with}$$

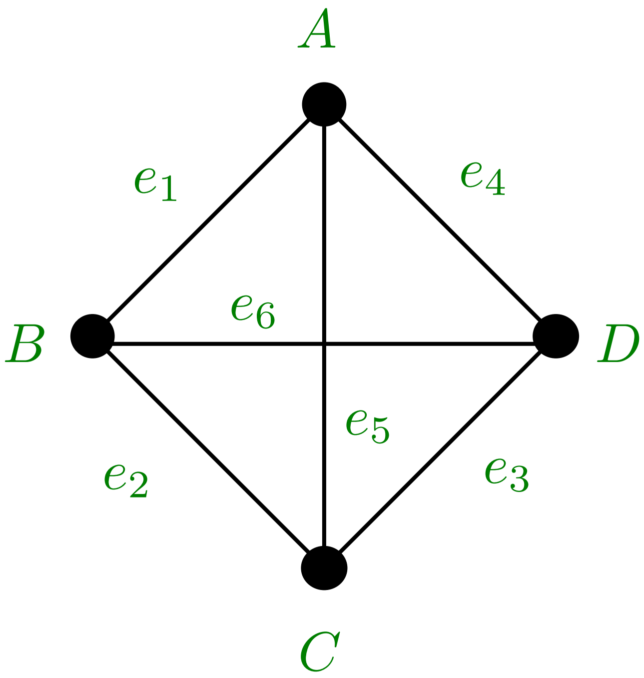
$$e_1 = \{A, B\}, e_2 = \{B, C\}, e_3 = \{C, D\}, e_4 = \{D, A\}.$$



3. $G_3 = (V_3, E_3)$ where

$$V_3 = \{A, B, C, D\} \quad \text{and} \quad E_3 = \{e_1, \ldots, e_6\} \quad \text{with}$$

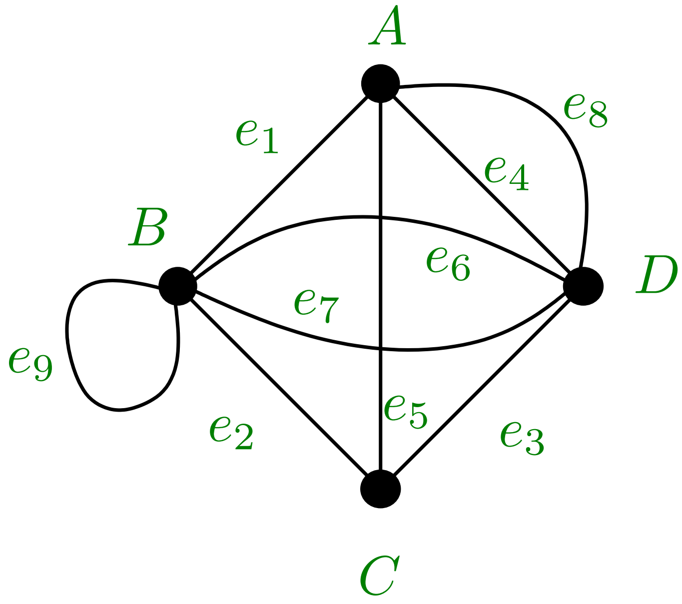
$$e_1 = \{A, B\}, e_2 = \{B, C\}, e_3 = \{C, D\},$$
$$e_4 = \{D, A\}, e_5 = \{A, C\}, e_6 = \{B, D\}.$$



4. $G_4 = (V_4, E_4)$ where

$$V_4 = \{A, B, C, D\} \quad \text{and} \quad E_4 = \{e_1, \ldots, e_9\} \quad \text{with}$$

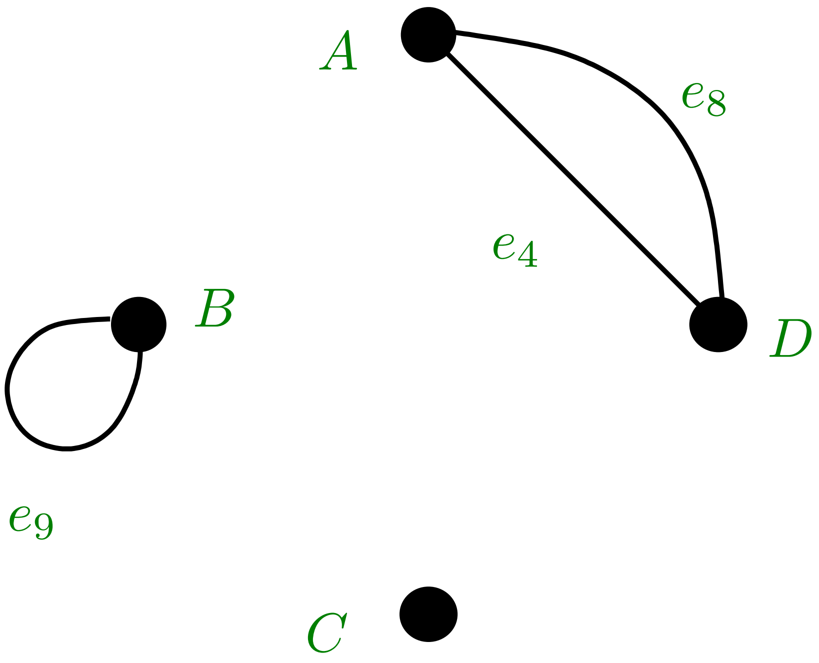
$$e_1 = \{A, B\}, e_2 = \{B, C\}, e_3 = \{C, D\},$$
$$e_4 = \{D, A\}, e_5 = \{A, C\}, e_6 = \{B, D\},$$
$$e_7 = \{B, D\}, e_8 = \{D, A\}, e_9 = \{B, B\}.$$


A
B
D
C
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9

5. $G_5 = (V_5, E_5)$ where

$$V_5 = \{A, B, C, D\} \quad \text{and} \quad E_5 = \{e_4, e_8, e_9\} \quad \text{with}$$

$$e_4 = \{D, A\}, e_8 = \{D, A\}, e_9 = \{B, B\}.$$



6. $G_6 = (V_6, E_6)$ where $V_6 = \{A, B\}$ and $E_6 = \emptyset$.

7. $G_7 = (V_7, E_7)$ where $V_7 = \{A\}$ and $E_7 = \emptyset$.

A graph must have at least one vertex but need not have any edges.

Terminology

Definition 2.1. Let $G = (V, E)$ be a graph.

(i) Vertices a and b are adjacent if there exists an edge $e \in E$ with $e = \{a, b\}$.

(ii) Edges e and f are adjacent if there exists a vertex $v \in V$ with $e = \{v, a\}$ and $f = \{v, b\}$, for some $a, b \in V$.

(iii) If $e \in E$ and $e = \{c, d\}$ then e is said to be incident to c and to d and to join c and d.

(iv) If a and b are vertices joined by edges $e_1, \ldots e_k$, where $k > 1$, then $e_1, \ldots e_k$ are called multiple edges.

(v) An edge of the form $\{a, a\}$ is called a loop.

More terminology

Definition 2.2. A graph which has no multiple edges and no loops is called a simple graph.

(In Example 1.3 the graphs in 1–3 and 6 and 7 are simple whereas those in 4 and 5 are not.)

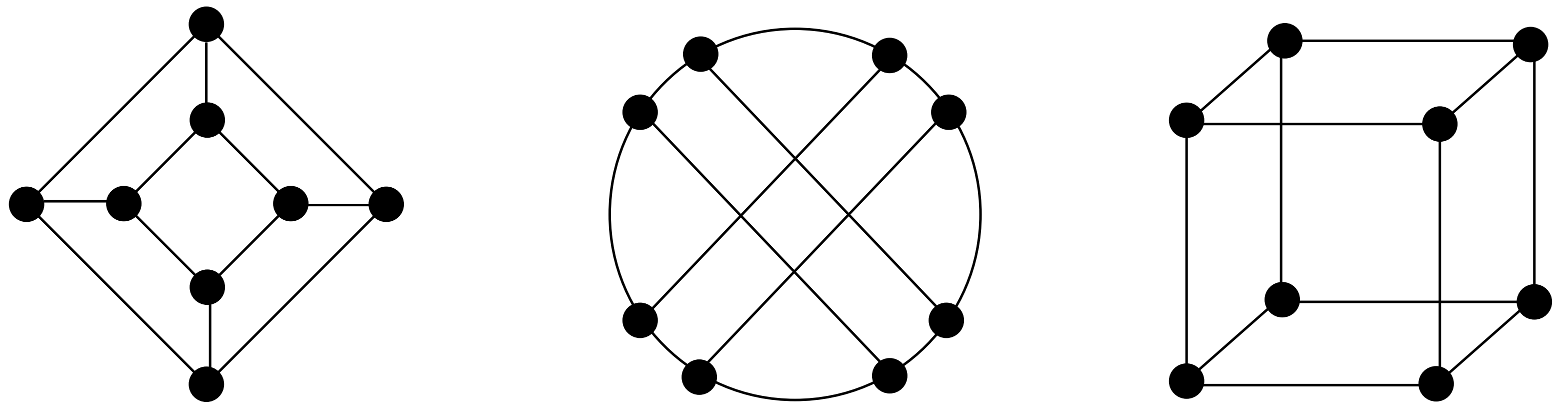
Drawing graphs

From the definition we can easily prove that any graph can be represented by a diagram in $\mathbb{R}^3$. (see Notes)

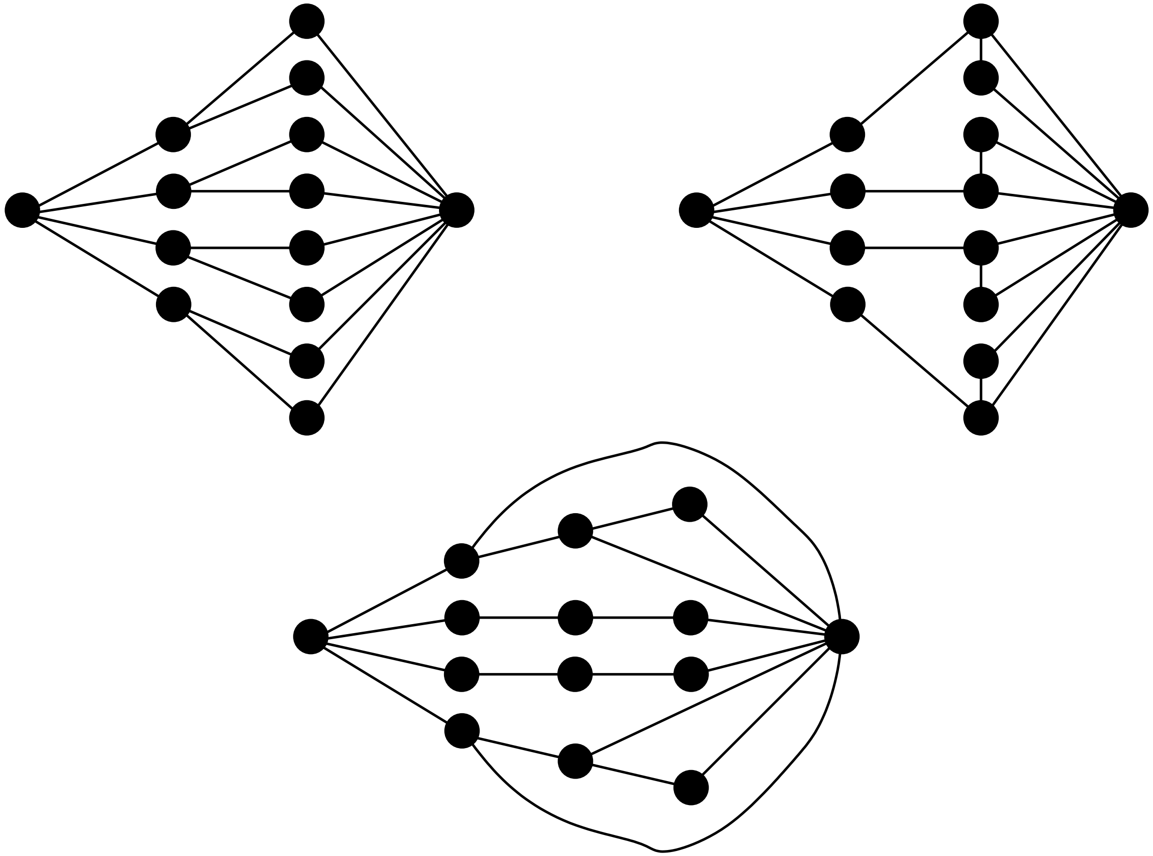
Drawings of graphs are projections of diagrams in $\mathbb{R}^3$ into the plane $\mathbb{R}^2$. In some cases it is impossible to draw a graph without edges crossing.

A graph may have many different drawings

Example 2.4. Are these three graphs the same?



Example 2.5. What about these?



Isomorphism

Definition 2.6. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exist bijections

$$\phi : V_1 \longrightarrow V_2 \quad \text{and} \quad \theta : E_1 \longrightarrow E_2$$

which preserve incidence.

That is, such that if

$$e = \{a, b\} \in E_1$$

then

$$\theta(e) = \{\phi(a), \phi(b)\} \in E_2.$$

The pair (ϕ, θ) is called an isomorphism from G_1 to G_2.

Degree of a vertex

Definition 2.7. The degree of a vertex u is the number of edges incident to u and is denoted $\deg(u)$ or $\text{degree}(u)$.

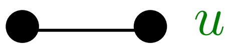
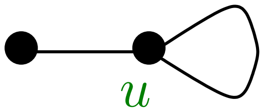
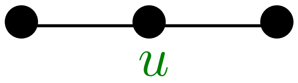
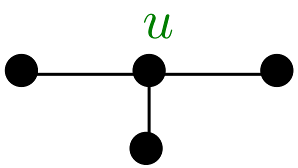
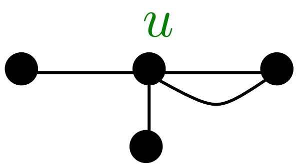
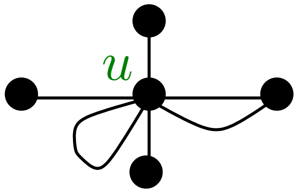
Example 2.8. – is on the next slide.

Definition 2.9. Let G be a graph with n vertices. Order the vertices $v_1, \ldots, v_n$ so that $\deg(v_i) \leq \deg(v_{i+1})$. Then G has degree sequence

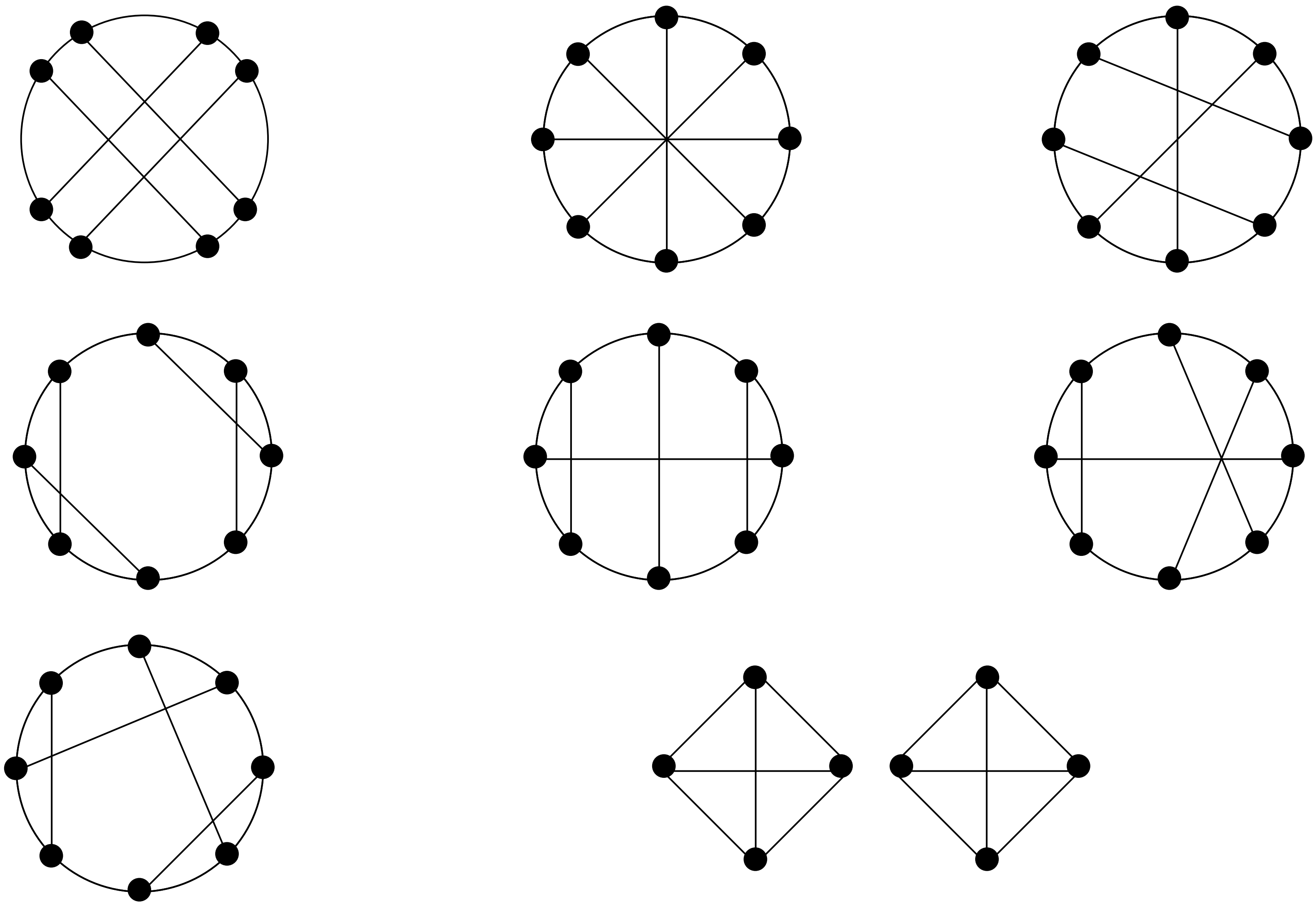
$$\langle \deg(v_1), \ldots, \deg(v_n) \rangle.$$

Definition 2.10. A graph is regular if every vertex has degree d, for some fixed $d \in \mathbb{Z}$.

In this case we say the graph is regular of degree d.

Graph G	degree of u	degree sequence of G
	0	$\langle 0 \rangle$
	1	$\langle 1, 1 \rangle$
	3	$\langle 1, 3 \rangle$
	2	$\langle 1, 1, 2 \rangle$
	3	$\langle 1, 1, 1, 3 \rangle$
	4	$\langle 1, 1, 2, 4 \rangle$
	7	$\langle 1, 1, 1, 2, 7 \rangle$

Example 2.11. Graphs which are simple, have 8 vertices, 12 edges and are regular of degree 3. Are any two of these isomorphic? What are their degree sequences?



Counting edges and vertices

G is a graph with vertices V and edges E, that is $G = (V, E)$.

Lemma 3.1. *[The Handshaking Lemma]*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

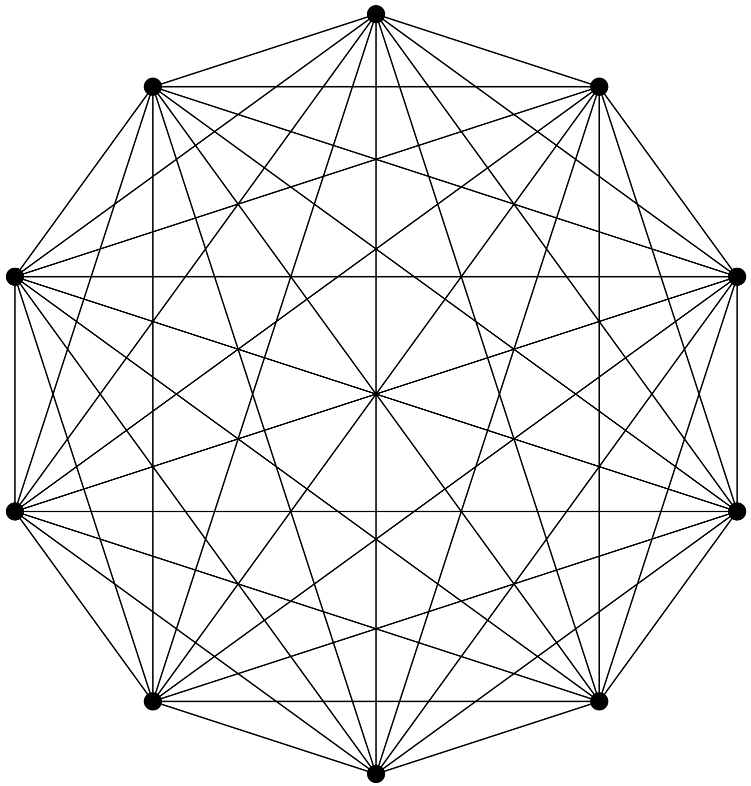
Lemma 3.2. *Suppose that G has q vertices of odd degree. Then q is even.*

Corollary 3.3. *If G has n vertices and is regular of degree d then G has $nd/2$ edges.*

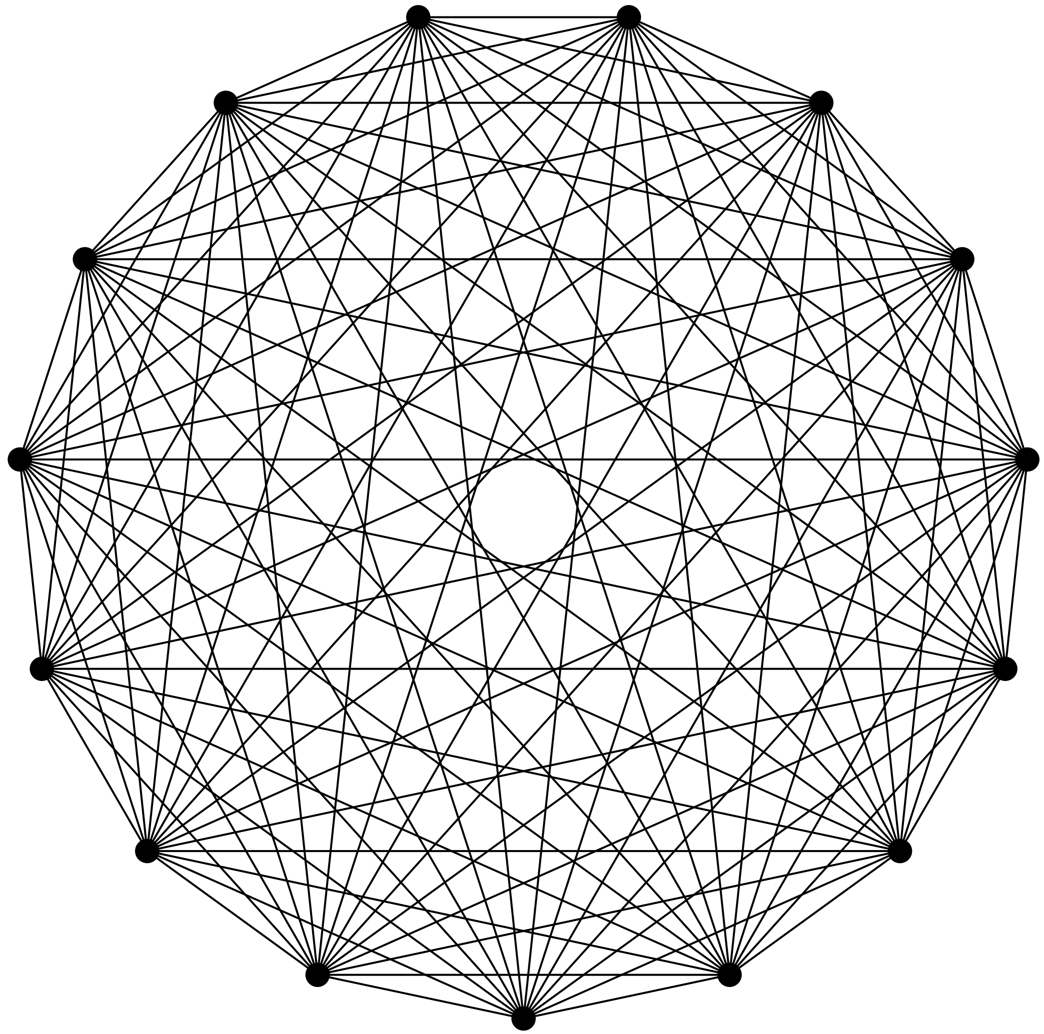
Examples of graphs

Example 3.4. The Null graph N_d, for $d \geq 1$.

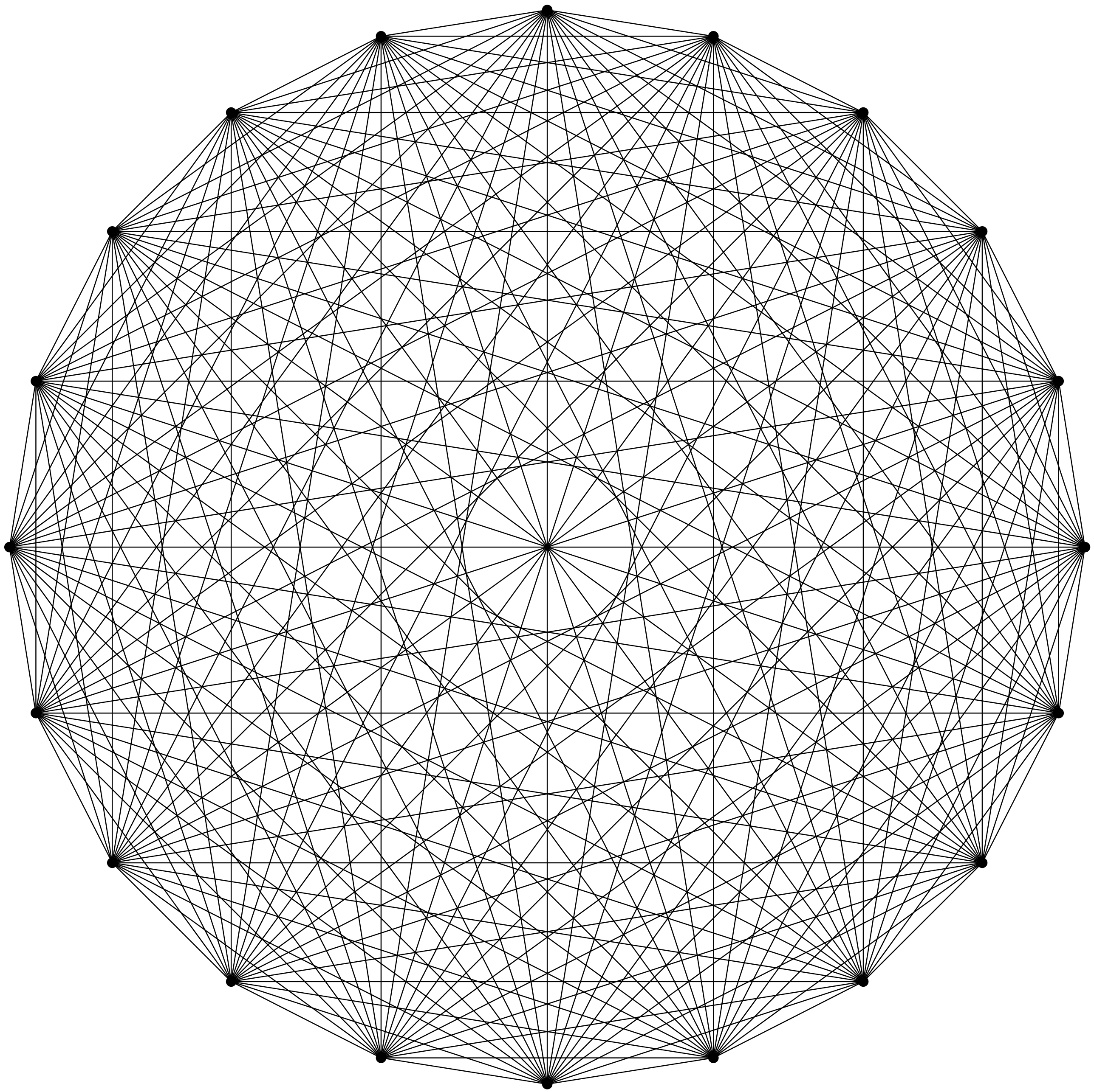
Example 3.5. The Complete graph K_d, for $d \geq 1$.



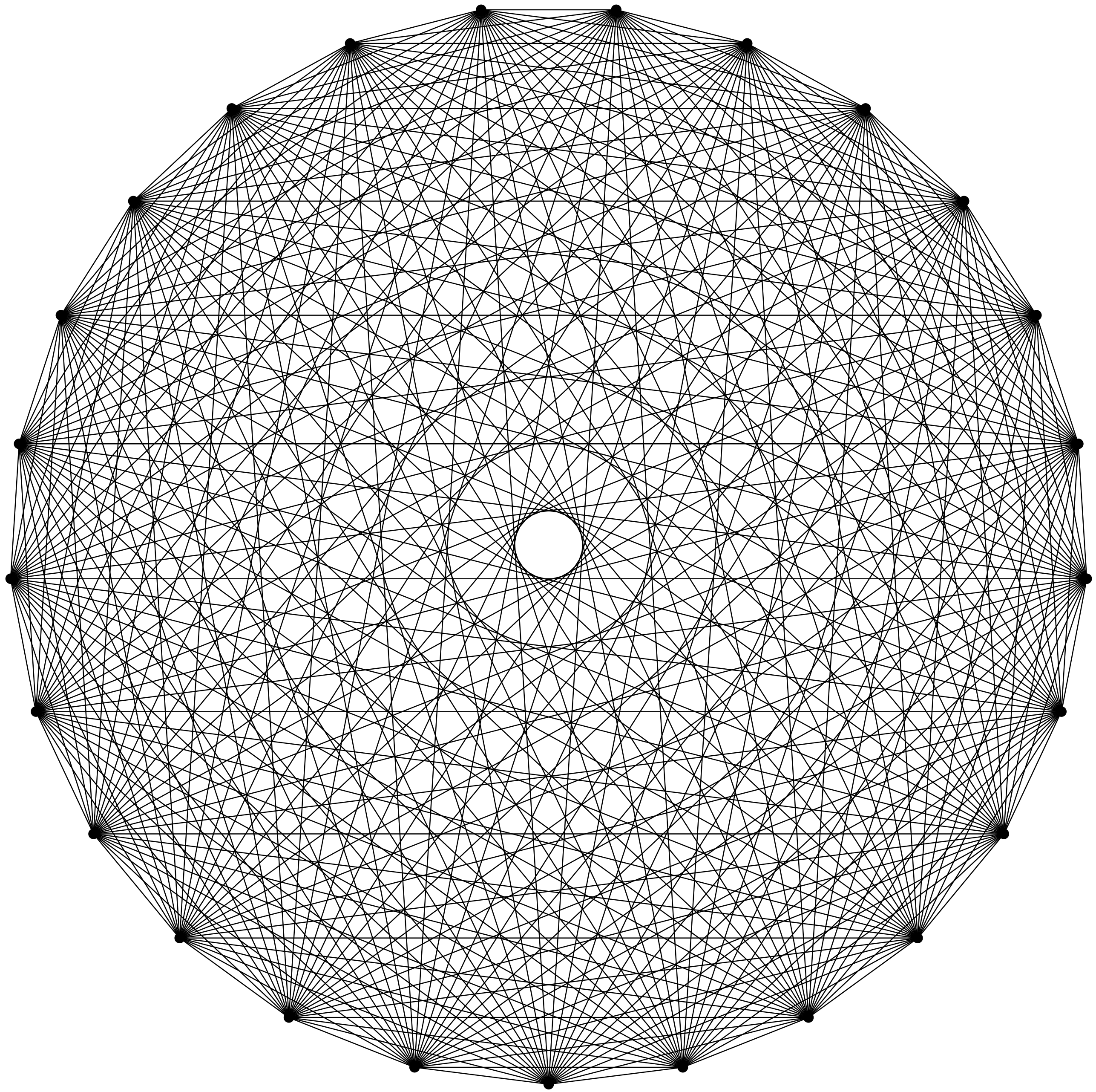
K_{10}



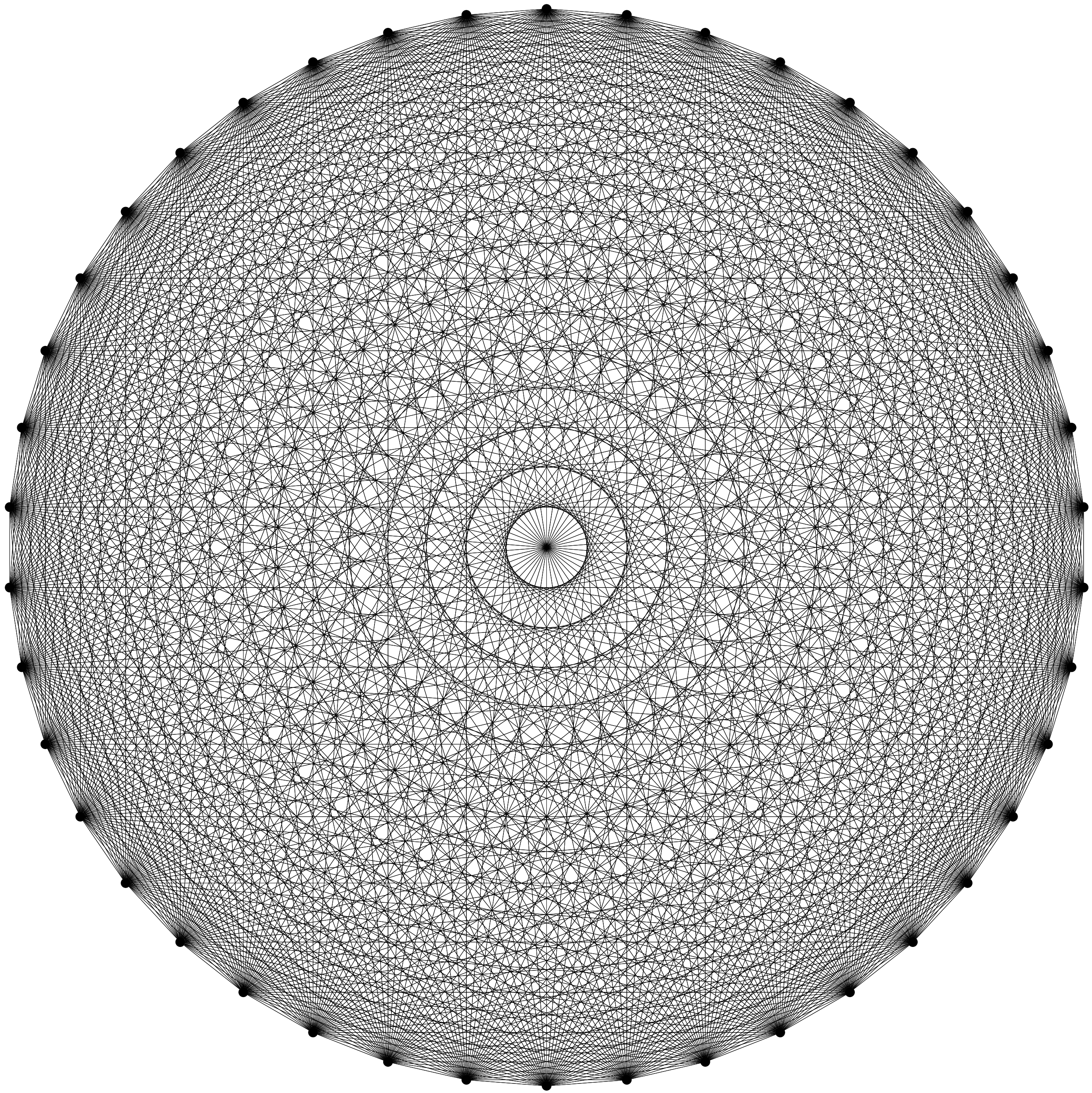
K_{15}



K_{20}



K_{25}

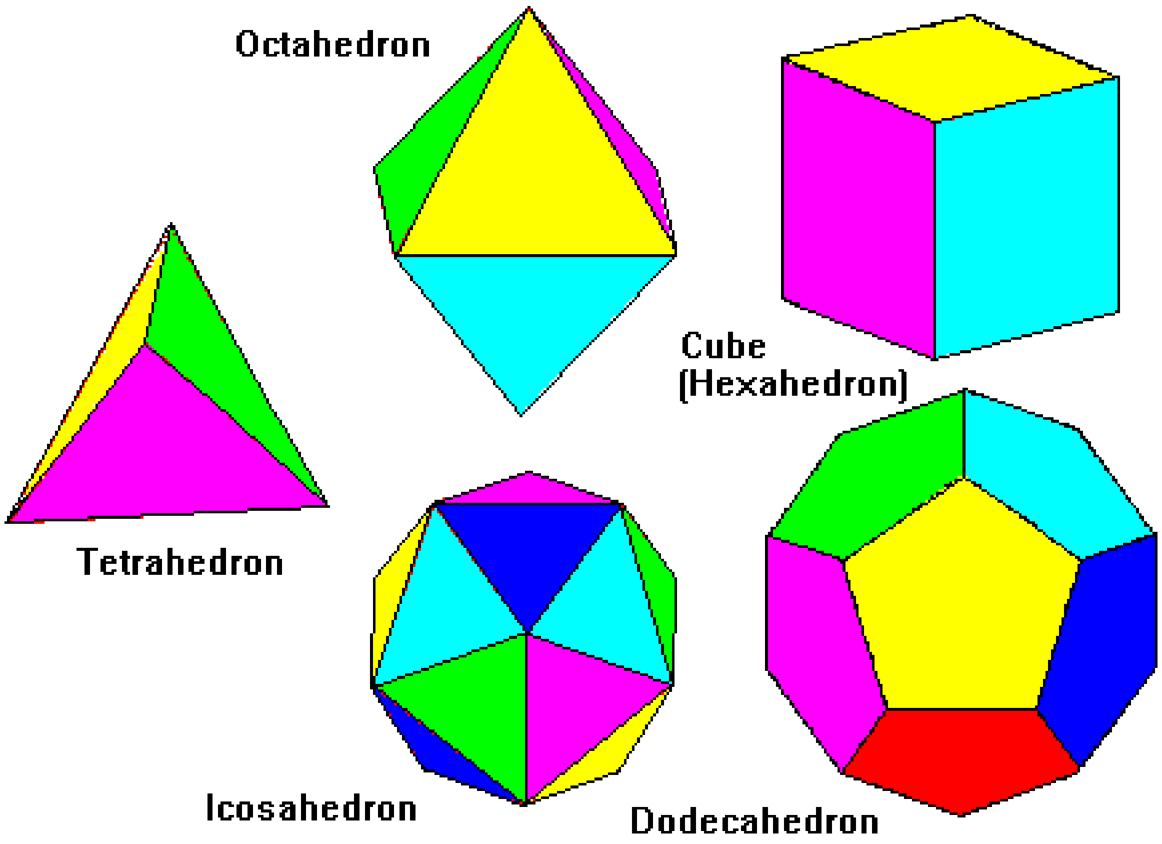


K_{42}

Lemma 3.6. *The complete graph K_d is regular of degree $d - 1$ and has $d(d-1)/2$ edges.*

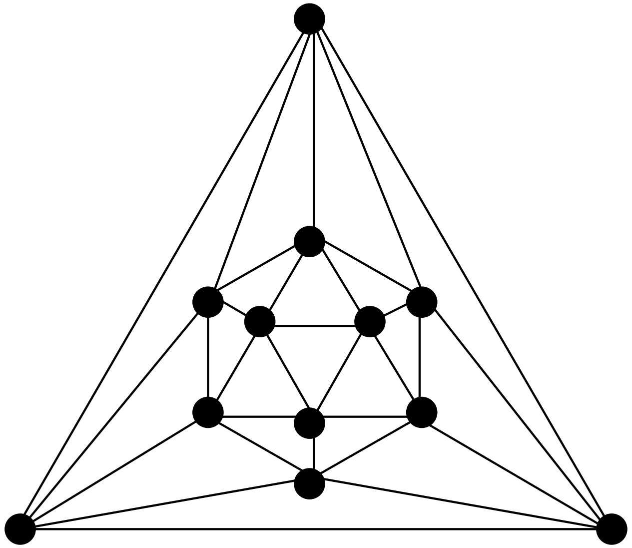
Example 3.7. The Petersen graph.

Example 3.8.
The Platonic solids are convex polyhedra with regular polygon faces. All faces are identical. There are only five of them, shown below.

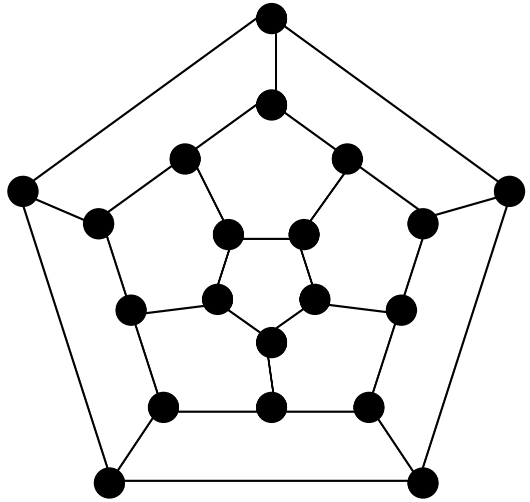


(Courtesy of Steven Dutch, Natural and Applied Sciences, University of

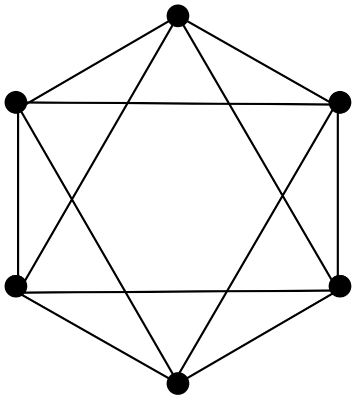
Wisconsin - Green Bay: www.uwgb.edu/dutchs/symmetry/platonic.htm.)



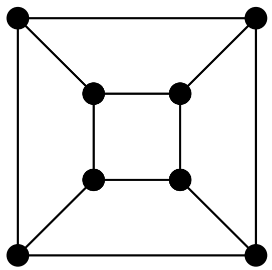
Icosahedron



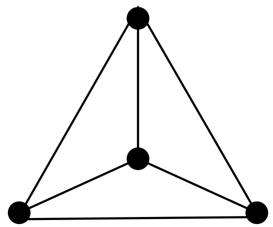
Dodecahedron



Octahedron



Cube



Tetrahedron

Bipartite graphs

Definition 4.1. Let $G = (V, E)$ be a graph such that

(i) $V = V_1 \cup V_2$, where V_1 and V_2 are non–empty subsets of V and

(ii) $V_1 \cap V_2 = \emptyset$ and

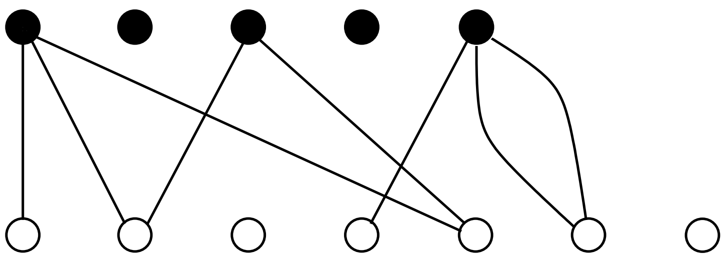
(iii) for $i = 1$ and 2, no edge of G joins a vertex of V_i to a vertex of V_i.

Then G is called a bipartite graph with bipartition (V_1, V_2).

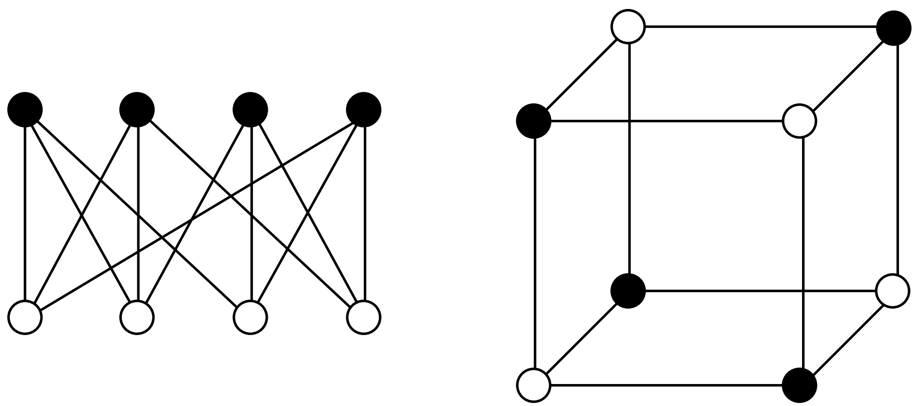
Example 4.2. 1. The Null graph N_d, where $d \geq 2$ is bipartite. (Colour one vertex blue, one vertex red and all the rest red or blue as you please.)

Example 4.2

2. A bipartite graph. (Note that not every vertex of V_1 need be incident to a vertex of V_2.)



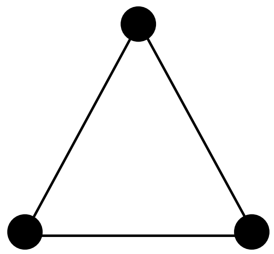
3. The cube is bipartite.



Example 4.2 continued

4. The octahedron is not bipartite.

5. Any graph which contains the following configuration is not bipartite.



The Gray code:

Example 4.3. Let k be any integer greater than 0.

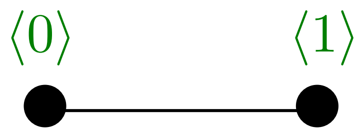
The **k**–cube Q_k is a graph whose vertex set is the set of sequences of length k of the symbols 1 and 0.

That is

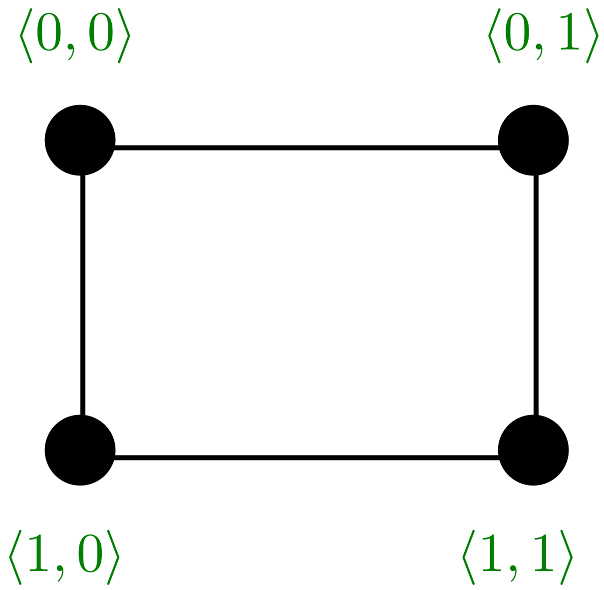
$$V(Q_k) = \{\langle a_1, \ldots, a_k \rangle : a_i = 0 \text{ or } 1\}.$$

Two vertices $\langle a_1, \ldots, a_k \rangle$ and $\langle b_1, \ldots, b_k \rangle$ are joined by and edge if and only if these sequences differ in exactly one term.

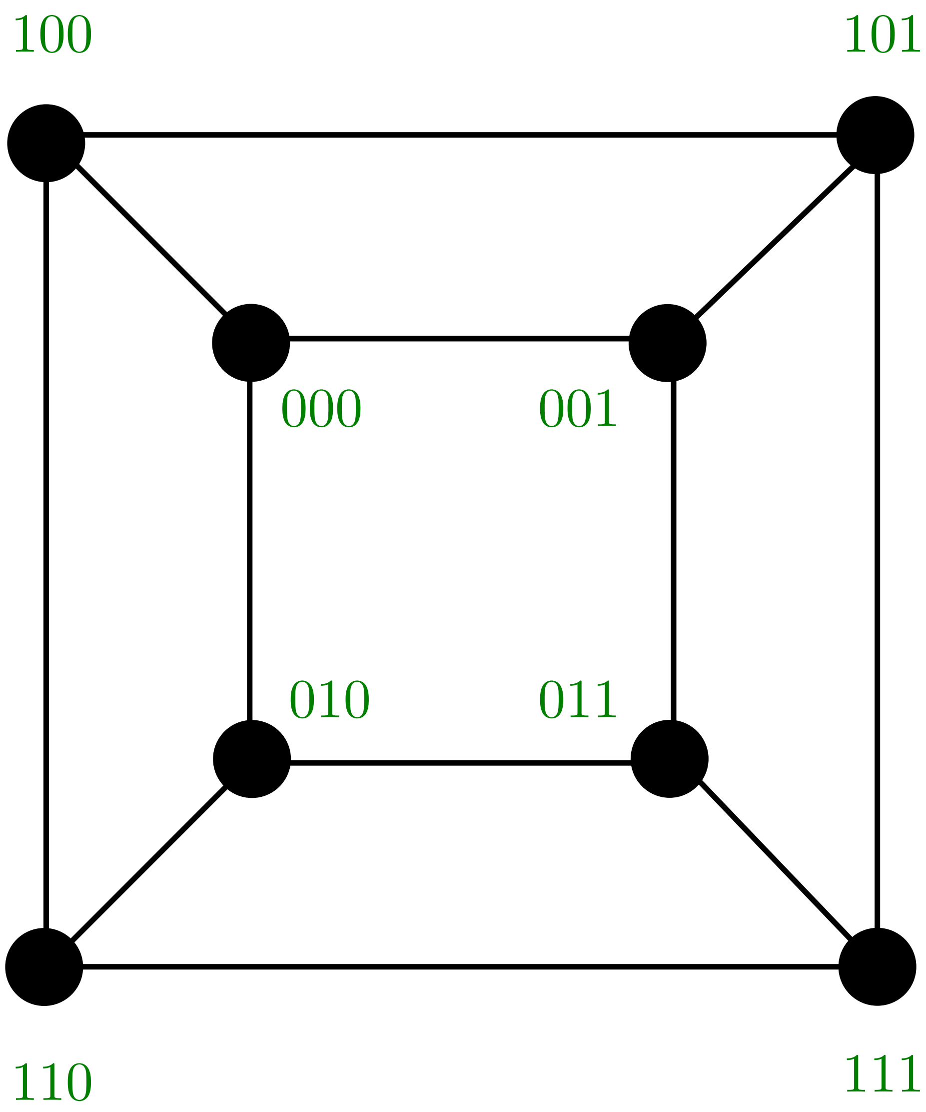
$\mathbf{Q_1}$**:** $V(Q_1) = \{\langle 0 \rangle, \langle 1 \rangle\}$.



$\mathbf{Q_2}$: $V(Q_2) = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$.



$\mathbf{Q_3}$: $V(Q_3) = \{000, 001, 010, 011, 100, 101, 110, 111\}$.



Lemma 4.4. *1. Q_k is regular of degree k.*

2. $|E(Q_k)| = k2^{k-1}$.

3. Q_k is bipartite on

$$V_1 = \{\langle a_1, \ldots, a_k \rangle : \sum a_i \equiv 0 \mod (2)\}$$

and

$$V_2 = \{\langle a_1, \ldots, a_k \rangle : \sum a_i \equiv 1 \mod (2)\}.$$

Complete Bipartite Graphs

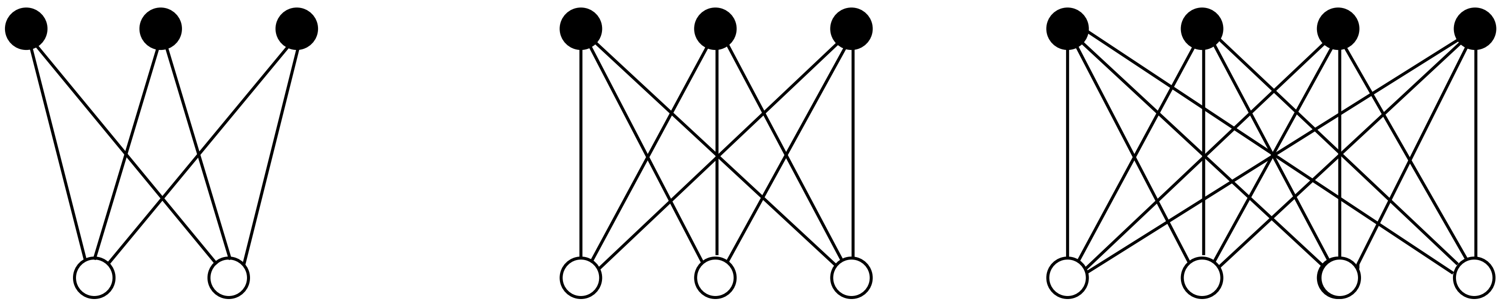
Definition 4.5. Let $r, s \in \mathbb{Z}$ with $r, s \geq 1$.

The complete bipartite graph $K_{r,s}$ is the simple graph on V_1 and V_2, where

1. $|V_1| = r$ and $|V_2| = s$ and

2. every vertex of V_1 is joined to every vertex of V_2.

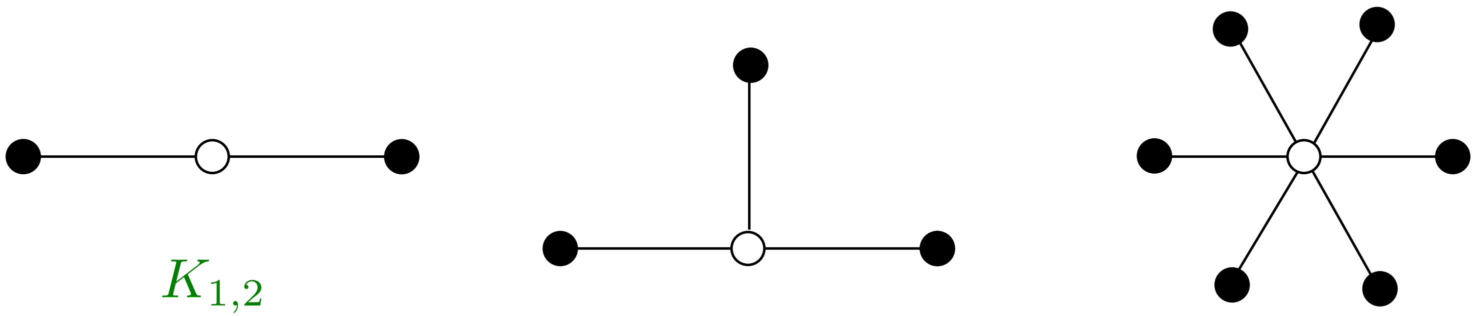
Example 4.6. 1. Some complete bipartite graphs:



$K_{2,3}$ $K_{3,3}$ $K_{4,4}$

Example 4.6 continued

2. As a special case of the complete bipartite graphs we have the family of star graphs which are the graphs $K_{1,s}$, $s \geq 1$.



$K_{1,2}$ $K_{1,3}$ $K_{1,6}$

Subgraphs

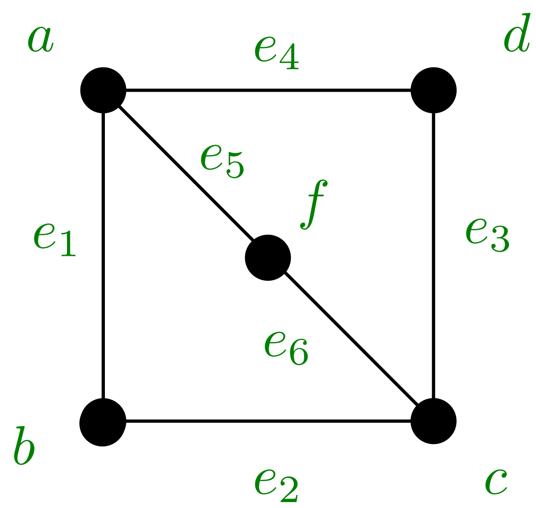
Definition 5.1. A subgraph of a graph $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

Example 5.2. Let $G = (V, E)$, where

$$V = \{a, b, c, d, f\}, \quad E = \{e_1, e_2, e_3, e_4, e_5, e_6\} \quad \text{and}$$

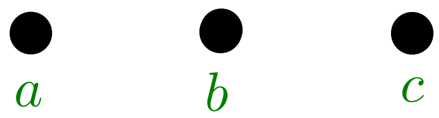
$$e_1 = \{a, b\}, e_2 = \{b, c\}, e_3 = \{c, d\}, e_4 = \{d, a\},$$

$$e_5 = \{a, f\}, e_6 = \{f, c\}.$$

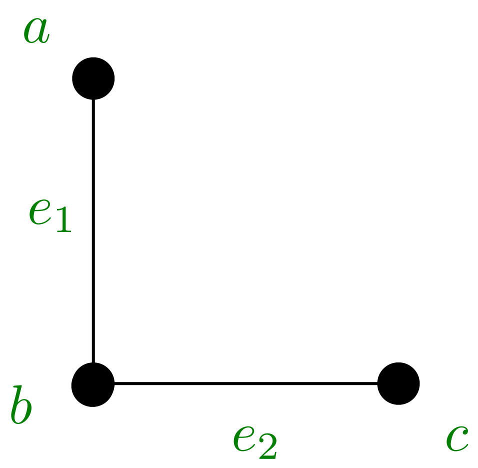


Example 5.2

1. $H_1 = (V_1, E_1)$, where $V_1 = \{a, b, c\}$ and $E_1 = \emptyset$ is a subgraph of G.

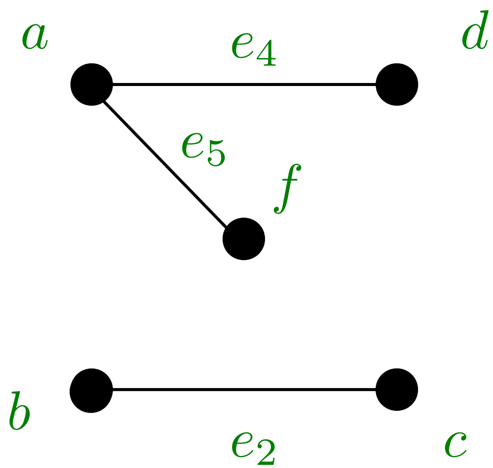


2. $H_2 = (V_2, E_2)$, where $V_2 = \{a, b, c\}$ and $E_1 = \{e_1, e_2\}$ is a subgraph of G.



Example 5.2

3. $H_3 = (V_3, E_3)$, where $V_3 = \{a, b, c, d, f\}$ and $E_1 = \{e_2, e_4, e_5\}$ is a subgraph of G.



4. G is a subgraph of G.

Example 5.2

5. $H_4 = (V_4, E_4)$, where $V_4 = \{a, b, c, x, y\}$ and $E_1 = \{e_2, e_4, e_5\}$ is not a subgraph of G, because V_4 is not a subset of V.

6. $H_5 = (V_5, E_5)$, where $V_5 = \{a, b, c, d\}$ and $E_5 = \{e_7, e_8\}$ with $e_7 = \{b, f\}, e_8 = \{a, c\}$ is not a subgraph of G, because E_5 is not a subset of E.

7. $H_6 = (V_6, E_6)$, where $V_6 = \{a, b, c, d\}$ and $E_6 = \{e_5\}$ is not a subgraph of G, because it is not a graph

($e_5 = \{a, f\}$ is not a pair of elements of V_6).

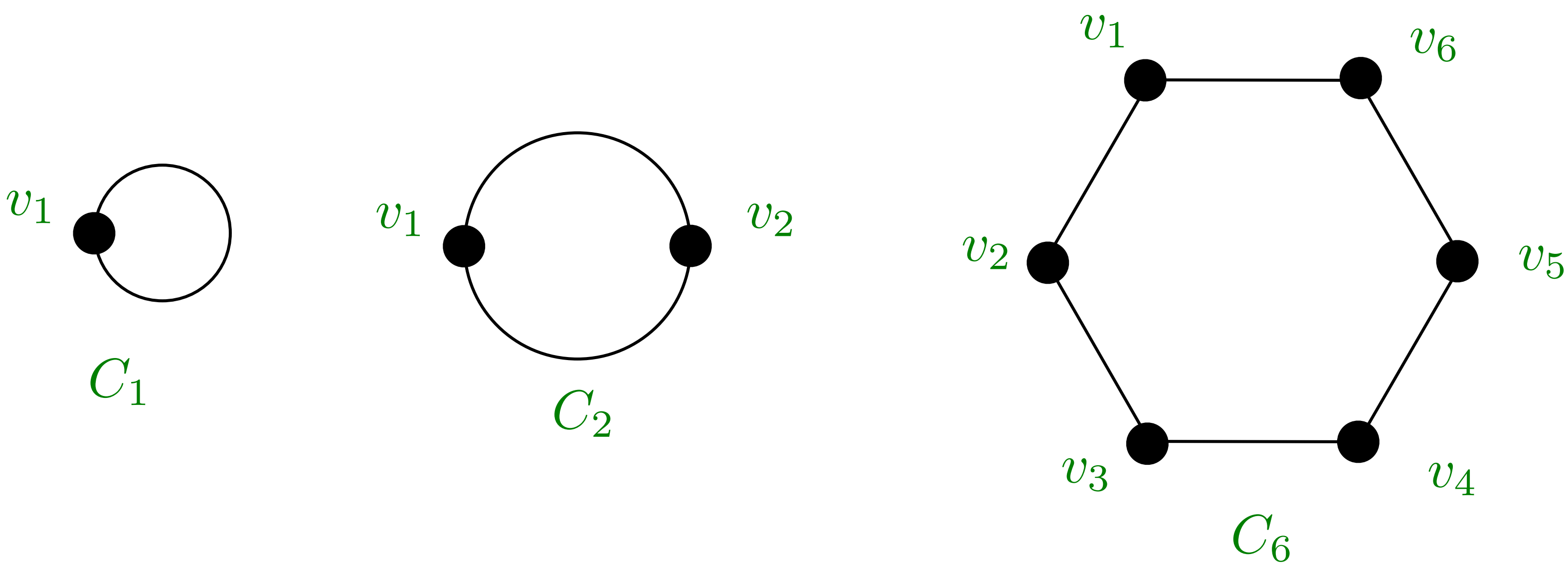
The last 3 examples illustrate the 3 possible ways in which H may fail to be a subgraph of G.

Example 5.3.

1. For $d \geq 1$ we define the cycle graph C_d to be the graph with d vertices $v_1, \ldots, v_d$ and d edges $\{v_1, v_2\}, \ldots, \{v_{d-1}, v_d\}, \{v_d, v_1\}$.

(C_1 has one vertex v_1 and one edge $\{v_1, v_1\}$.)

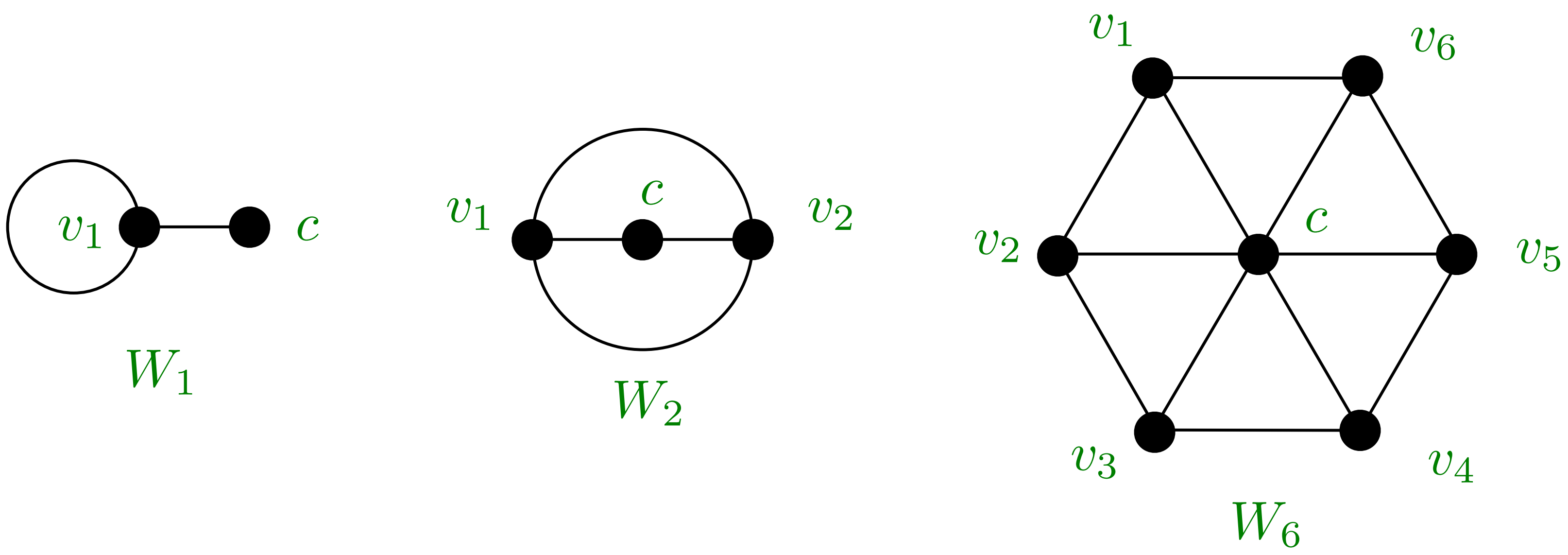
The cycle graph is regular of degree 2 and simple if $d \geq 3$.



2. For $d \geq 1$ we define the wheel graph W_d to be the graph with

$$d + 1 \text{ vertices } c, v_1, \ldots, v_d \text{ and}$$

$2d$ edges $\{v_1, v_2\}, \ldots, \{v_{d-1}, v_d\}, \{v_d, v_1\}, \quad \text{and} \quad \{c, v_1\}, \ldots \{c, v_d\}$.



The wheel graph has a subgraph isomorphic to the cycle graph C_d and a subgraph isomorphic to the star graph $K_{1,d}$.

It is simple when $d \geq 3$.

$G - e$:

$G - v$:

Walks, paths, trails, circuits and cycles

$G = (V, E)$ a graph

Definition 6.1. A sequence

$$v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n,$$

where

(i) $n \geq 0$ and

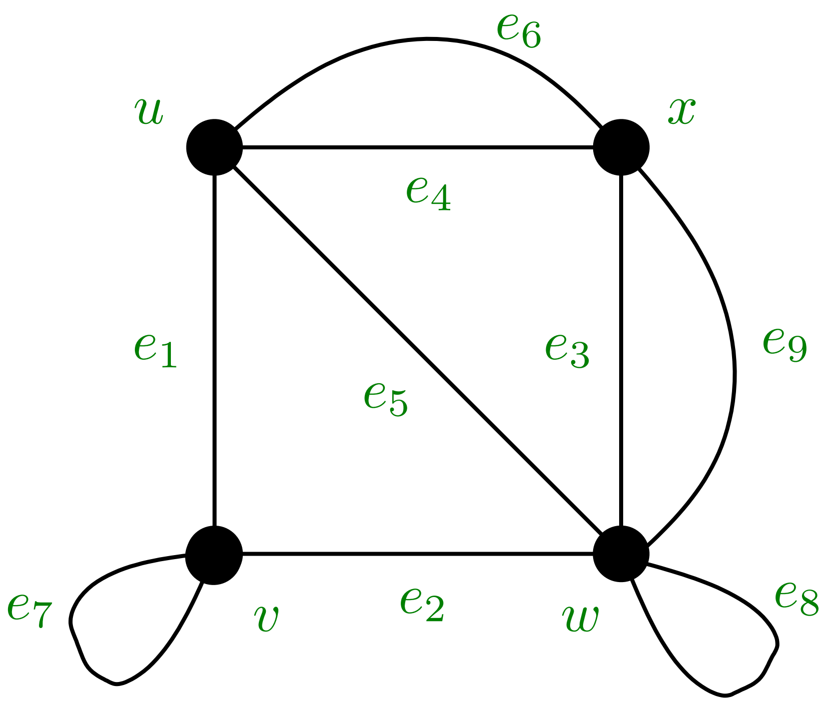
(ii) $v_i \in V$ and $e_i \in E$ and

(iii) $e_i = \{v_{i-1}, v_i\}$, for $i = 1, \ldots, n$,

is called a walk of length n.

The walk is from its initial vertex v_0 and to its terminal vertex v_n.

Example 6.2. G is the graph shown.



Definition 6.3. Let $W = v_0, e_1, v_1, \ldots, e_n, v_n$ be a walk in a graph.

1. If no edges of the walk are repeated (that is $e_i \neq e_j$ when $i \neq j$) then W is called a trail.

2. If no vertices of the walk are repeated (that is $v_i \neq v_j$ when $i \neq j$) then W is called an open path.

3. If $v_0 = v_n$ then W is a closed walk.

4. A closed trail is called a circuit.

5. If W is a closed trail and no two of the vertices $v_0, \ldots, v_{n-1}$ are the same then W is called a closed path. [Note that from the definition it follows that $v_n \neq v_i$, for $1 \leq i \leq n-1$.]

6. We refer to both open and closed paths as paths.

7. A closed path of length at least 1 is called a cycle.

Example 6.4. 7. the walk $v, e_1, u, e_4, x, e_6, u, e_5, w,$

8. the walk $w, e_3, x, e_4, u, e_1, v,$

9. the walk $u, e_4, x, e_9, w, e_3, x, e_6, u.$

	walk	**trail**	**open path**	**closed walk**	**circuit**	**closed path**	**cycle**	**length**
1	Y	N	N	N	N	N	N	4
2	Y	Y	N	Y	Y	Y	Y	2
3	Y	Y	Y	Y	Y	Y	N	0
4	N							
5	Y	Y	N	N	N	N	N	4
6	Y	Y	Y	N	N	N	N	3
7	Y	Y	N	Y	Y	N	N	9

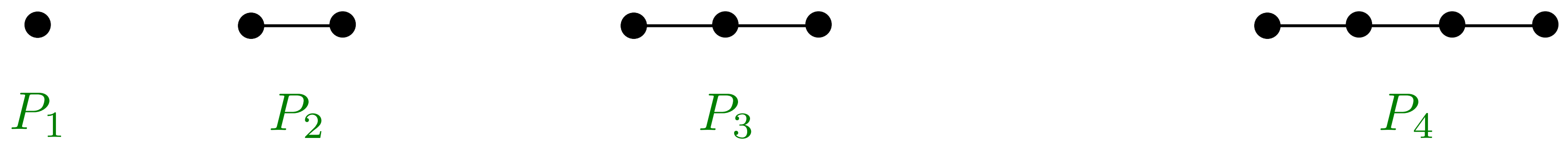
Note that an open path may be a closed walk, as in Example 6.2 3 above, but only when it has length 0.

Example 6.5. 1. The cycle graph C_d consists of a cycle of length d.

2. The path graph P_n, for $n \geq 1$, is the graph with n vertices $v_1, \ldots, v_n$ and $n-1$ edges $e_2, \ldots, e_n$, with

$$e_i = \{v_{i-1}, v_i\}, \quad \text{for} \quad i = 2, \ldots, n.$$

The path graph P_n consists of an open path of length n.



Walks in simple graphs

In a simple graph we may write only the sequence of vertices, which we call the vertex sequence of a walk.

For example the sequence

$$v_1, c, v_5, v_4, c, v_2$$

is the vertex sequence of a unique walk in the wheel graph W_6 shown above.

Connectedness

Definition 7.1. A graph is connected if, for any two vertices a and b there is an open path from a to b.

A graph which is not connected is called disconnected.

Lemma 7.2. *There is an open path from a to b if and only if there is a walk from a to b.*

Example 7.3.

Connected Component

Definition 7.4. A connected component of a graph G is a subgraph H of G such that

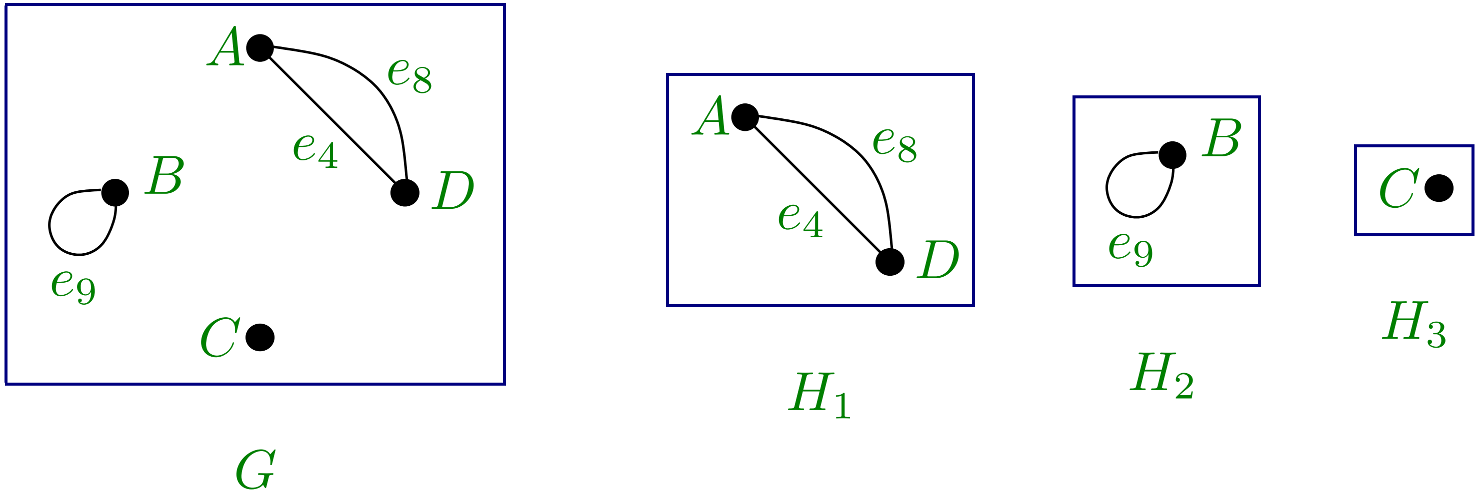
1. H is a connected subgraph of G and

2. H is not contained in any larger connected subgraph of G.

Example 7.5. 1. A connected graph has only one connected component – itself.

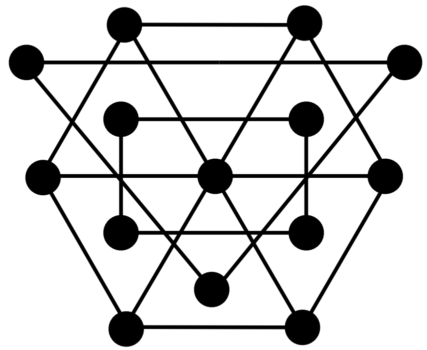
2. The graph of Example 1.3.5has 3 connected components:

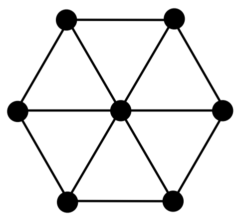
H_1**:** the graph with vertices A, D and edges e_4, e_8;
H_2**:** the graph with vertex B and edge e_9;
H_3**:** the graph with vertex C and no edges.



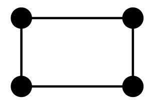
3. The null graph N_d has d connected components, each with 1 vertex.

4. The graph G below has 3 connected components A, B and C, as shown.

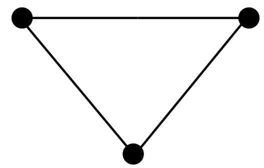




A



B



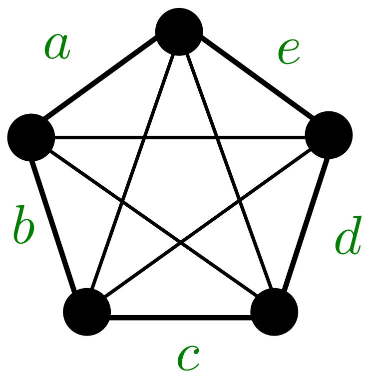
C

Deletion of edges

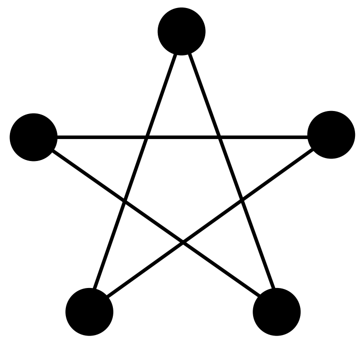
Definition 7.6. Let $G = (V, E)$ be a graph and let E' be a subset of E. The graph with vertex set V and edge set $E - E'$ is called the graph obtained from G by deleting E', denoted $G - E'$.

When E' consists of only one element we write $G - e$ instead of $G - \{e\}$.

Example 7.7. 1.

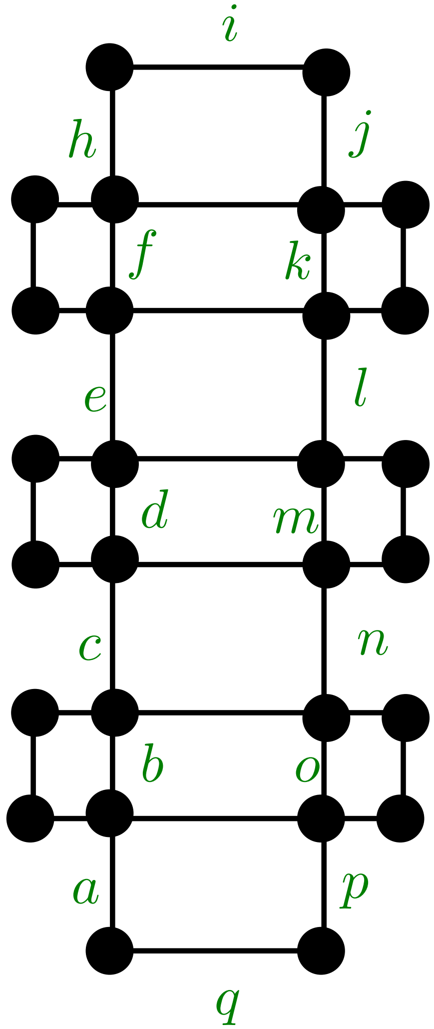


G_1

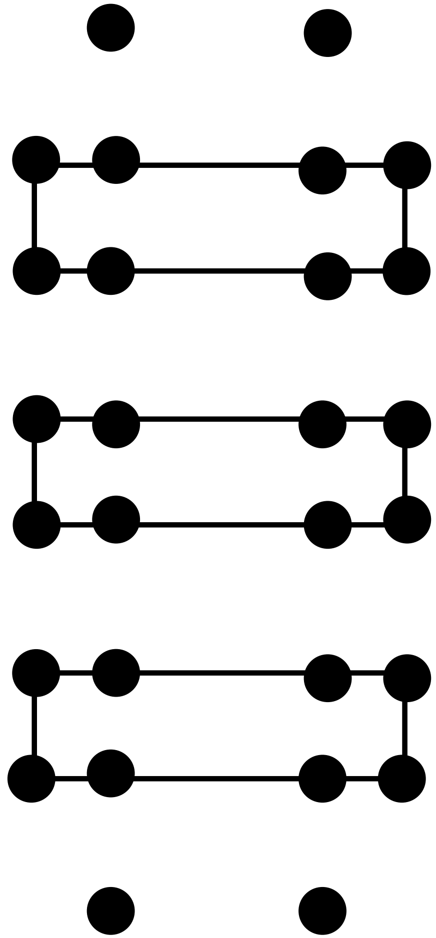


$G_1 - \{a, b, c, d, e\}$

2.

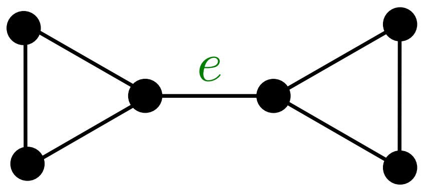


G_2

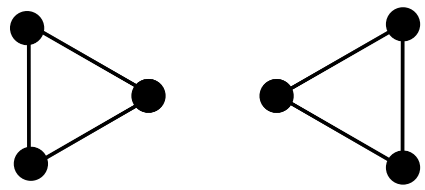


$G_2 - \{a, \ldots, q\}$

3.



G_3



$G_3 - e$

Lemma 7.8. *Let* G *be a connected graph,* C *a circuit in* G *and* e *an edge of* C*. Then* $G - e$ *is connected.*

Eulerian graphs

Definition 8.1. 1. A trail containing every edge of a graph is called an Eulerian trail.

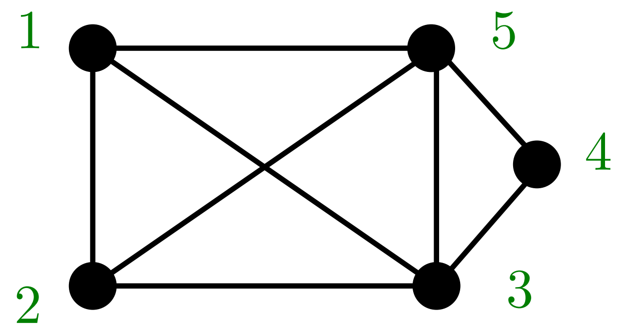
2. A circuit containing every edge of a graph is called an Eulerian circuit.

3. A graph is called semi–Eulerian if it is connected and has an Eulerian trail.

4. A graph is called Eulerian if it is connected and has an Eulerian circuit.

Note: Every Eulerian graph is semi-Eulerian.

Example 8.2. 1. The walk $1, 2, 3, 1, 5, 4, 3, 5, 2$ is a semi–Eulerian trail in the graph G_1 below.

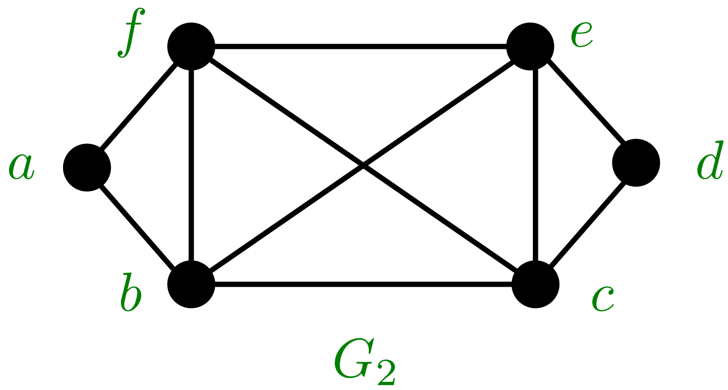


G_1

Therefore G_1 is semi–Eulerian.

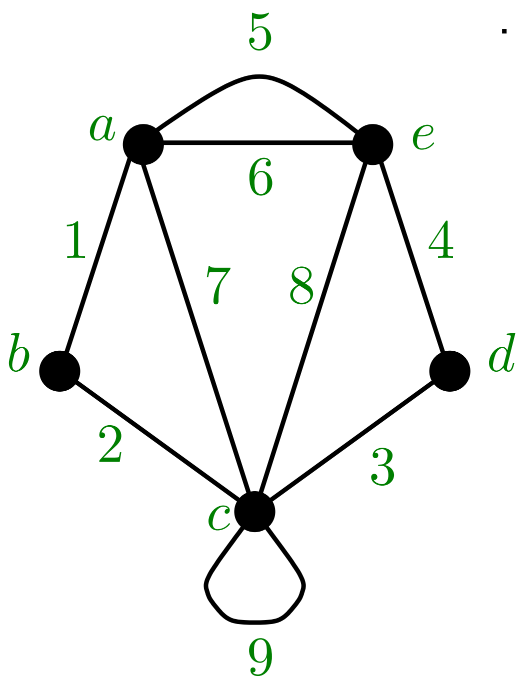
Does G_1 have an Eulerian circuit?

2. The walk $a, b, c, d, e, f, b, e, c, f, a$ is an Eulerian circuit in the graph G_2 below.



Therefore G_2 is Eulerian.

3. The walk $a, 1, b, 2, c, 3, d, 4, e, 5, a, 7, c, 9, c, 8, e, 6, a$ is an Eulerian circuit in the graph G_3 below.



G_3

Therefore G_3 is Eulerian.

Consider the vertex c in Example 8.2.3.

Each occurence of c appears between two edges of the sequence defining the walk.

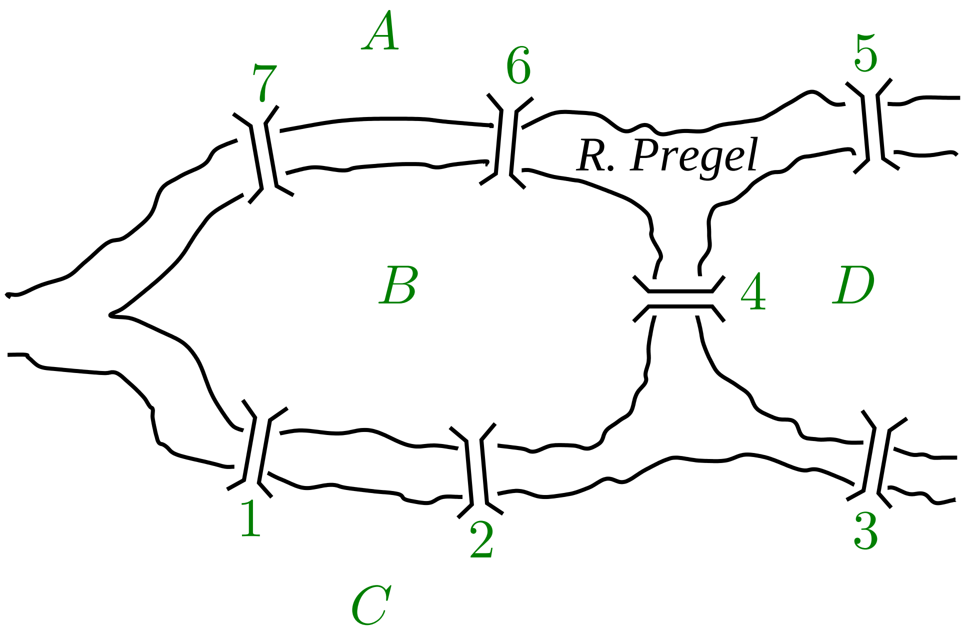
All edges of the graph appear in the Eulerian circuit and each edge appears only once.

We can compute the degree of c as twice the number of times it occurs in the sequence.

Theorem 8.3. *[Euler, 1736] If G is an Eulerian graph then every vertex of G has even degree.*

Example 8.4. 1. There is no Eulerian circuit for the graph of Example 8.2.1 above: this graph has verices of odd degree.

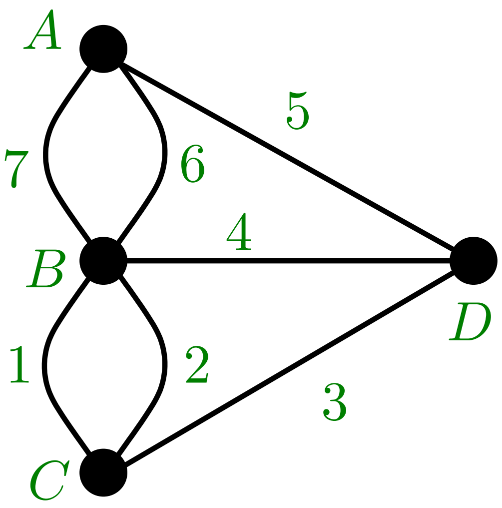
2. The Königsberg bridge problem.



The River Pregel in Königsberg

The Königsberg bridge problem is: starting at any bridge, cross all bridges exactly once and return to the start.

Euler proved the theorem above in response to this question, showing that it is not possible to do so.



A graph of the Königsberg bridges

This graph is not Eulerian as it has vertices of odd degree.

3. Of the Platonic graphs only the Octahedron can be Eulerian. Can you find an Eulerian circuit for the Octahedron?

4. The graph K_d is not Eulerian if d is even.

Lemma 8.5. *Let G be a graph such that every vertex of G has even degree. If $v \in V(G)$ with $\deg(v) > 0$ then v lies in a circuit of positive length.*

Theorem 8.6. *Let G be a connected graph. Then G is Eulerian if and only if every vertex of G has even degree.*

Proof. We have already seen (Theorem 8.3) that if G is Eulerian then every vertex of G is of even degree.

Suppose then that every vertex of G has even degree.

Let C be a circuit of maximal length in G.

If C contains every edge of G then it's an Eulerian circuit and so G is Eulerian as required.

We assume that C does not contain all edges of G and derive a contradiction.

Assume that E' is the set of edges of C and that there are some edges of G not in E'.

Consider the graph $G - E'$.

First of all, every vertex of $G - E'$ has even degree (as C is a circuit).

Also, although $G - E'$ may be a disconnected graph it does have a connected component, H say, with at least one edge.

Furthermore, as G is connected it follows that C and H have a vertex, a say, in common.

Now a has positive degree and so, by Lemma 8.5, is contained in a circuit D, of positive length, in H.

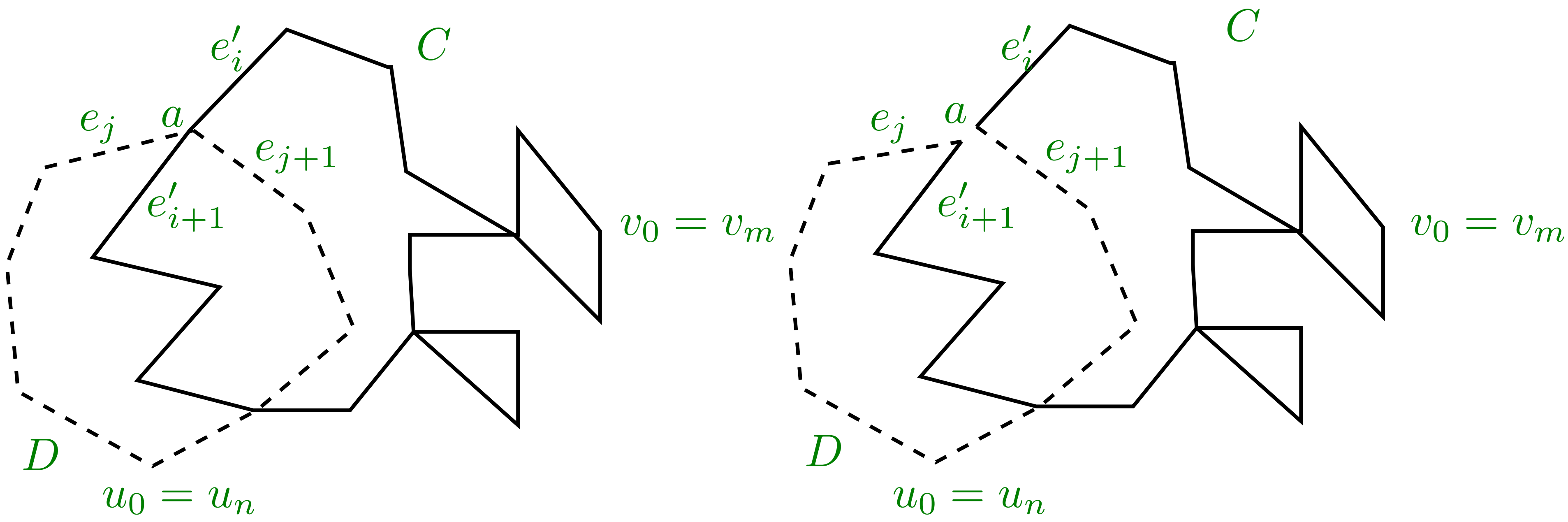
Let C be

$$v_0, e'_1, \ldots, e'_i, a, e'_{i+1}, \ldots, v_m \quad \text{and let } D \text{ be} \quad u_0, e_1, \ldots, e_j, a, e_{j+1}, \ldots, u_n.$$

Then

$$v_0, \ldots, e'_i, a, e_{j+1}, \ldots, u_n = u_0, \ldots, e_j, a, e'_{i+1}, \ldots, v_m$$

is a circuit in G of length greater than that of C.

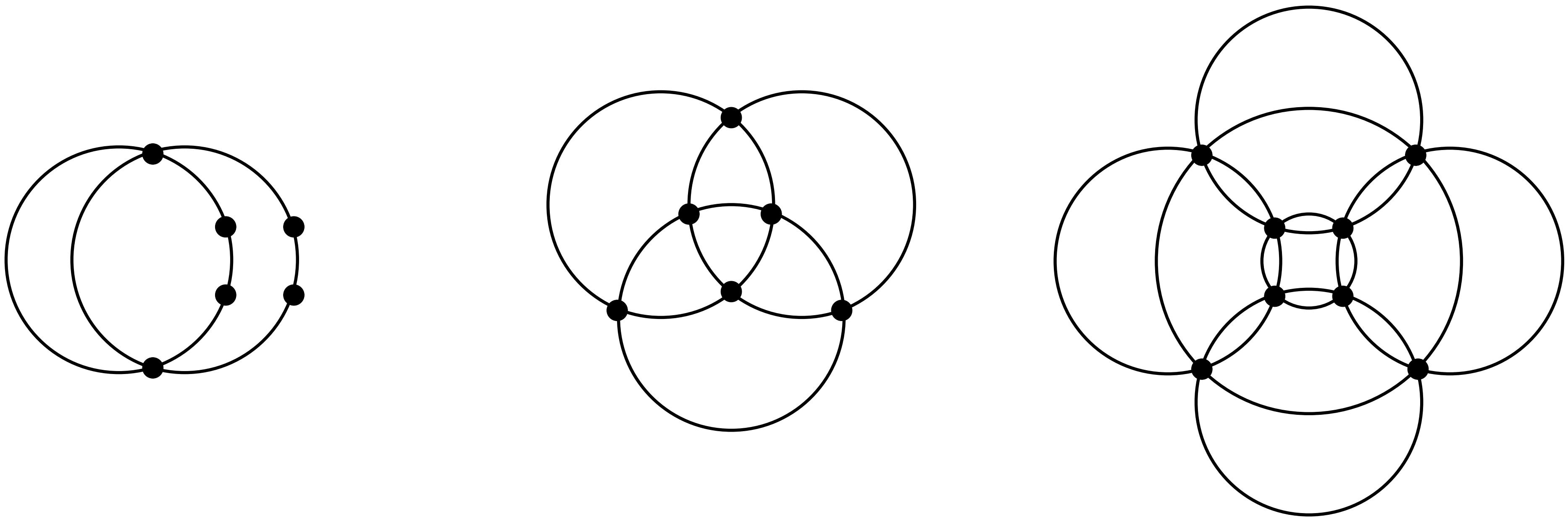


However this contradicts the choice of C as a circuit of maximal length in G.

Therefore C must contain all edges of G as required.

Example 8.7.

5.



Decompositions

Definition 8.8. A graph $G = (V, E)$ has a decomposition into subgraphs

$$H_1 = (V_1, E_1), \ldots, H_n = (V_n, E_n) \quad \text{if}$$

1. $E = E_1 \cup \cdots \cup E_n$ and

2. $E_i \cap E_j = \emptyset$, when $i \neq j$.

If G has a decomposition into subgraphs $H_1, \ldots, H_n$, where H_i is a cycle graph or a null graph, for all i, we say G has a decomposition into closed paths.

Example 8.9.

1. The Octahedron,

the 4–cube Q_4,

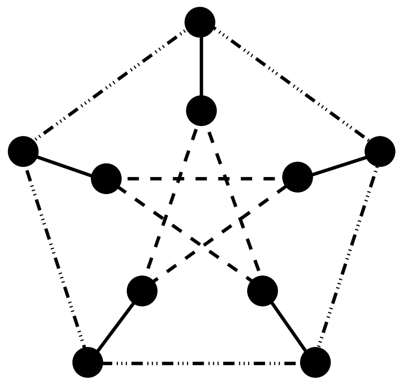
the graphs of Example 7.7.1 and 7.7.2

and all the graphs of Example 8.7.5 have decompositions into cycles.

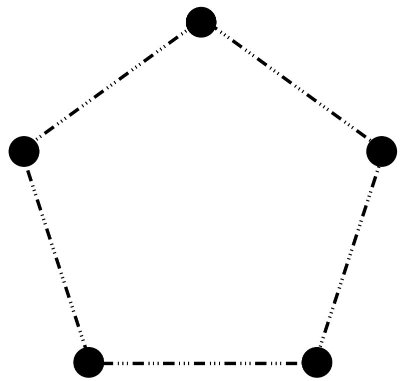
Note that, as in the right hand graph of Example 7.7.2, isolated vertices may occur in a decomposition into closed paths.

In particular the Null graph N_d has a decomposition into closed paths.

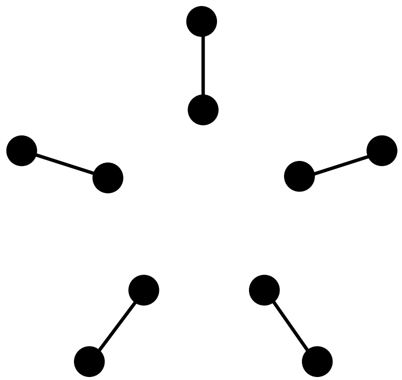
2. A decomposition of the Petersen graph into subgraphs H_1, H_2 and H_3.



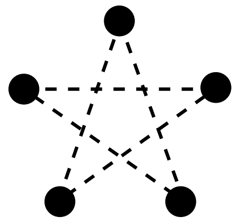
The Petersen graph



H_1

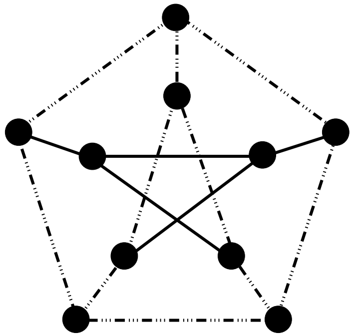


H_2

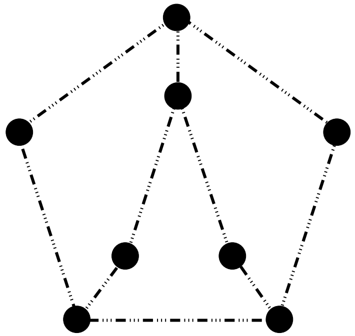


H_3

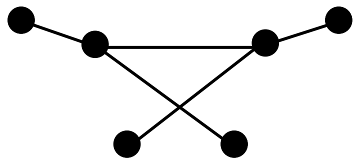
3. A different decomposition of the Petersen graph into subgraphs A_1 and A_2.



The Petersen graph

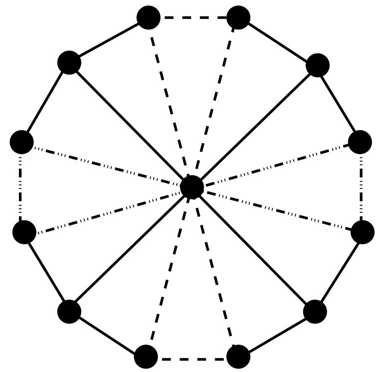


A_1

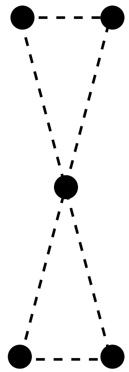


A_2

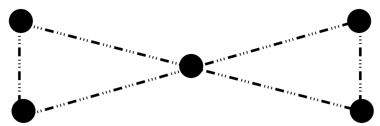
4. Another example of a decomposition into 3 subgraphs B_1, B_2 and B_3.



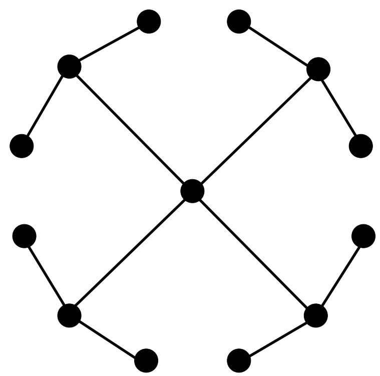
The wheel graph W_{12}



B_1



B_2



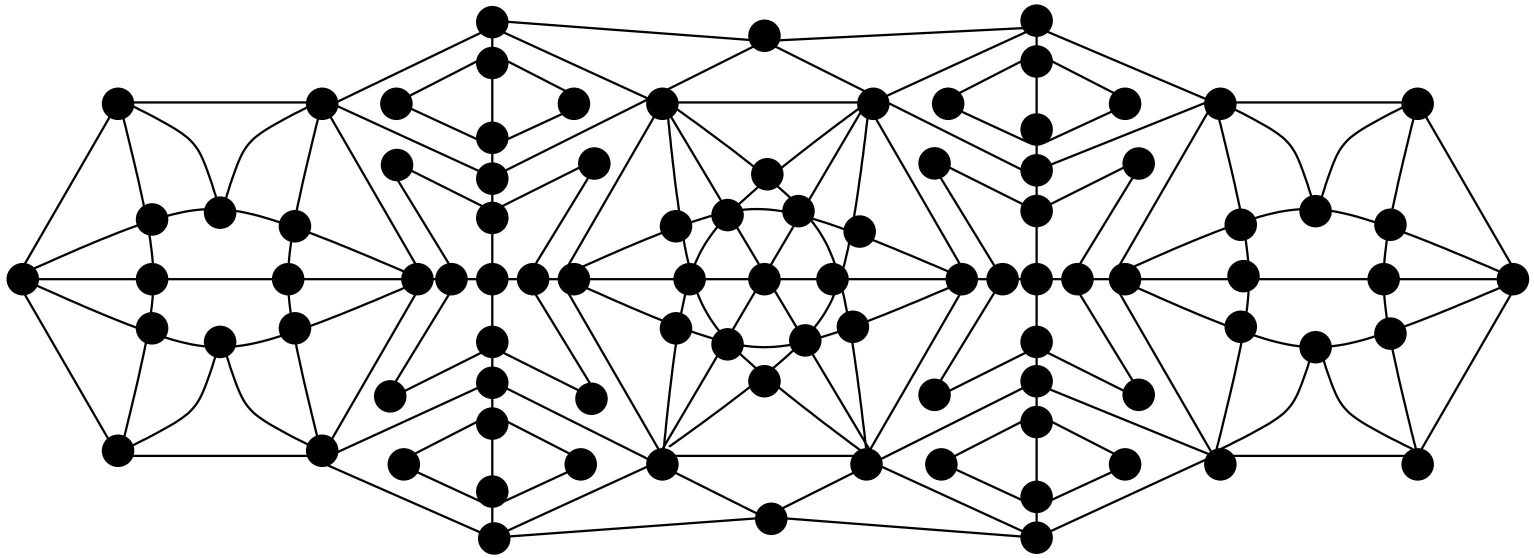
B_3

Theorem 8.10. *A graph G has a decomposition into closed paths if and only if every vertex of G has even degree.*

Corollary 8.11. *A connected graph is Eulerian if and only if it has a decomposition into closed paths.*

Theorem 8.12. *A connected graph is semi–Eulerian but not Eulerian if and only if precisely 2 of its vertices have odd degree.*

Example 8.13. The following graph has exactly 2 vertices of odd degree and is therefore semi–Eulerian.



Hamiltonian Graphs

Definition 9.1.

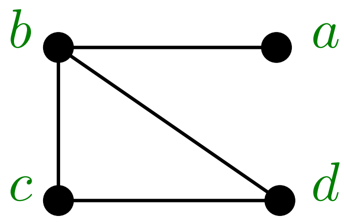
1. A path containing every vertex of a graph is called a Hamiltonian path.

2. A closed path containing every vertex of a graph is called a Hamiltonian closed path.

3. A graph is called semi–Hamiltonian if it has a Hamiltonian path and Hamiltonian if it has a Hamiltonian closed path.

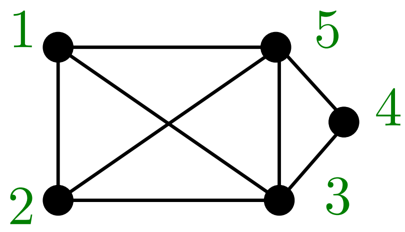
Example 9.2.

1. The walk a, b, c, d is a Hamiltonian path in the graph G_1 below.

   G_1: 

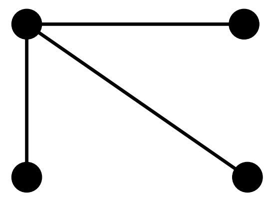
   Therefore G_1 is semi–Hamiltonian.

2. The walk $1, 2, 3, 4, 5, 1$ is a Hamiltonian closed path in the graph G_2 below.

   G_2: 

   Therefore G_2 is Hamiltonian.

3. The graph G_3 below is not semi–Hamiltonian (and therefore not Hamiltonian).

G_3: 

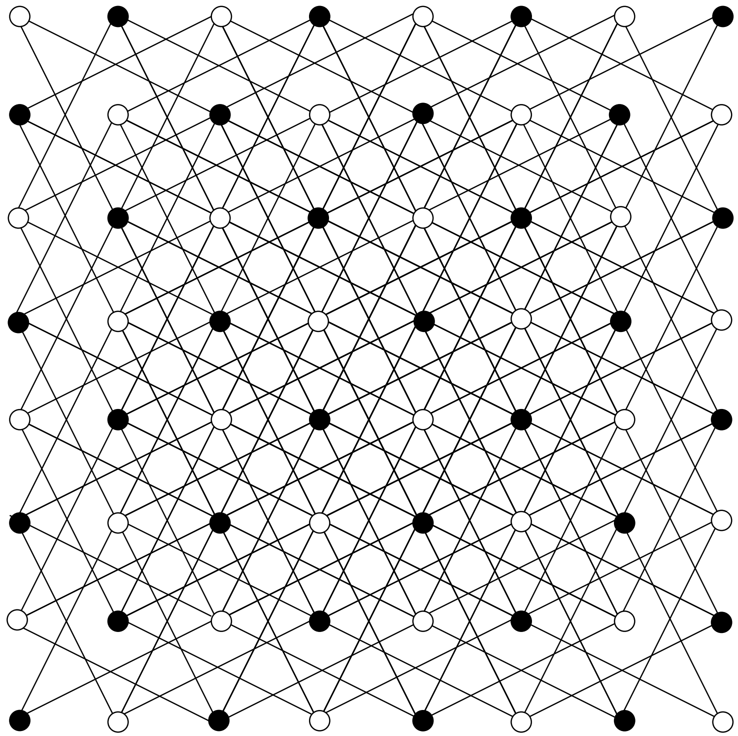
4. The complete graph K_2 is semi–Hamiltonian but not Hamiltonian.

For $d \neq 2$ the graphs K_d are Hamiltonian.

5. The cycle graphs are Hamiltonian for $d \geq 1$.

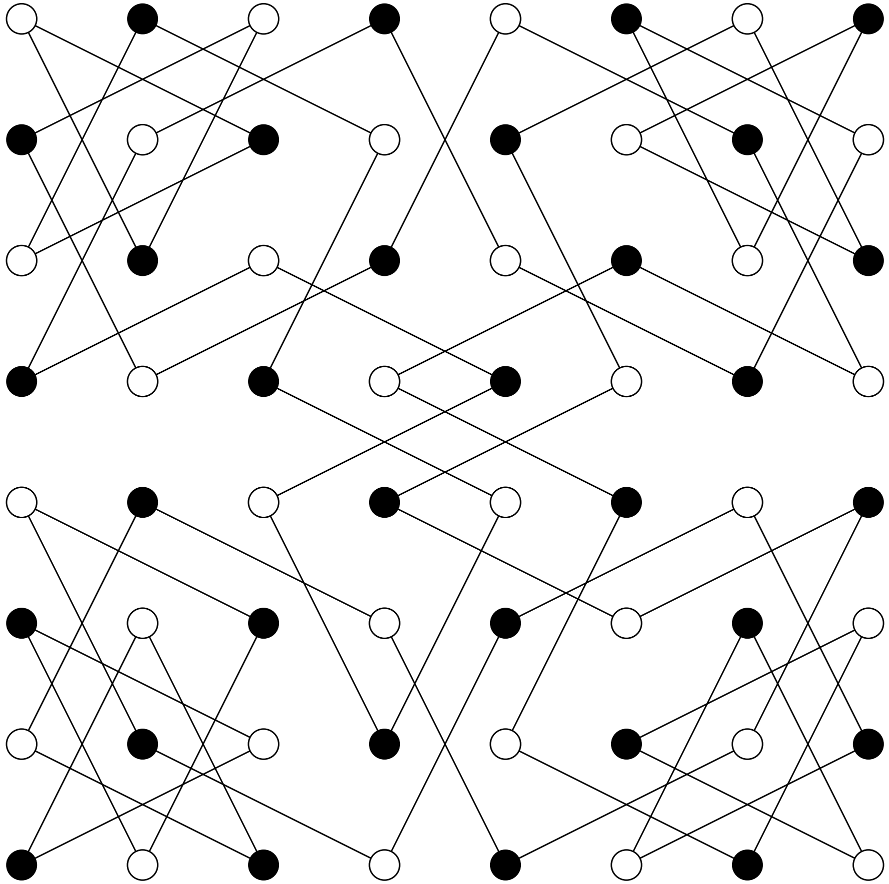
6. The wheel graph W_d is Hamiltonian for $d \geq 2$.

7. Construct a graph with one vertex corresponding to each square of a chessboard and an edge joining two vertices if a knight can move from one to the other. We call this the knight's move graph.



The knight's move graph

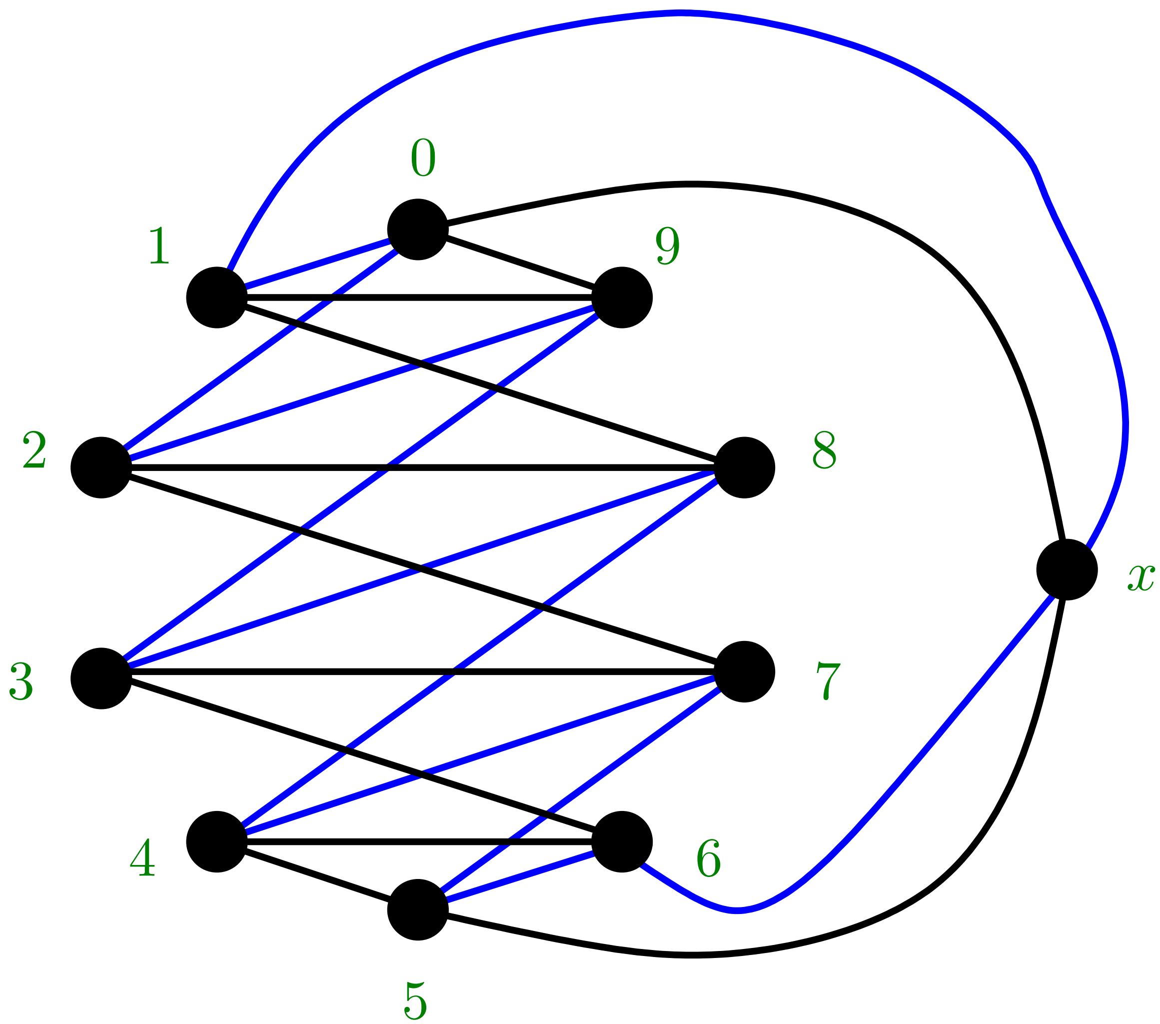
A Hamiltonian closed path for the knight's move graph



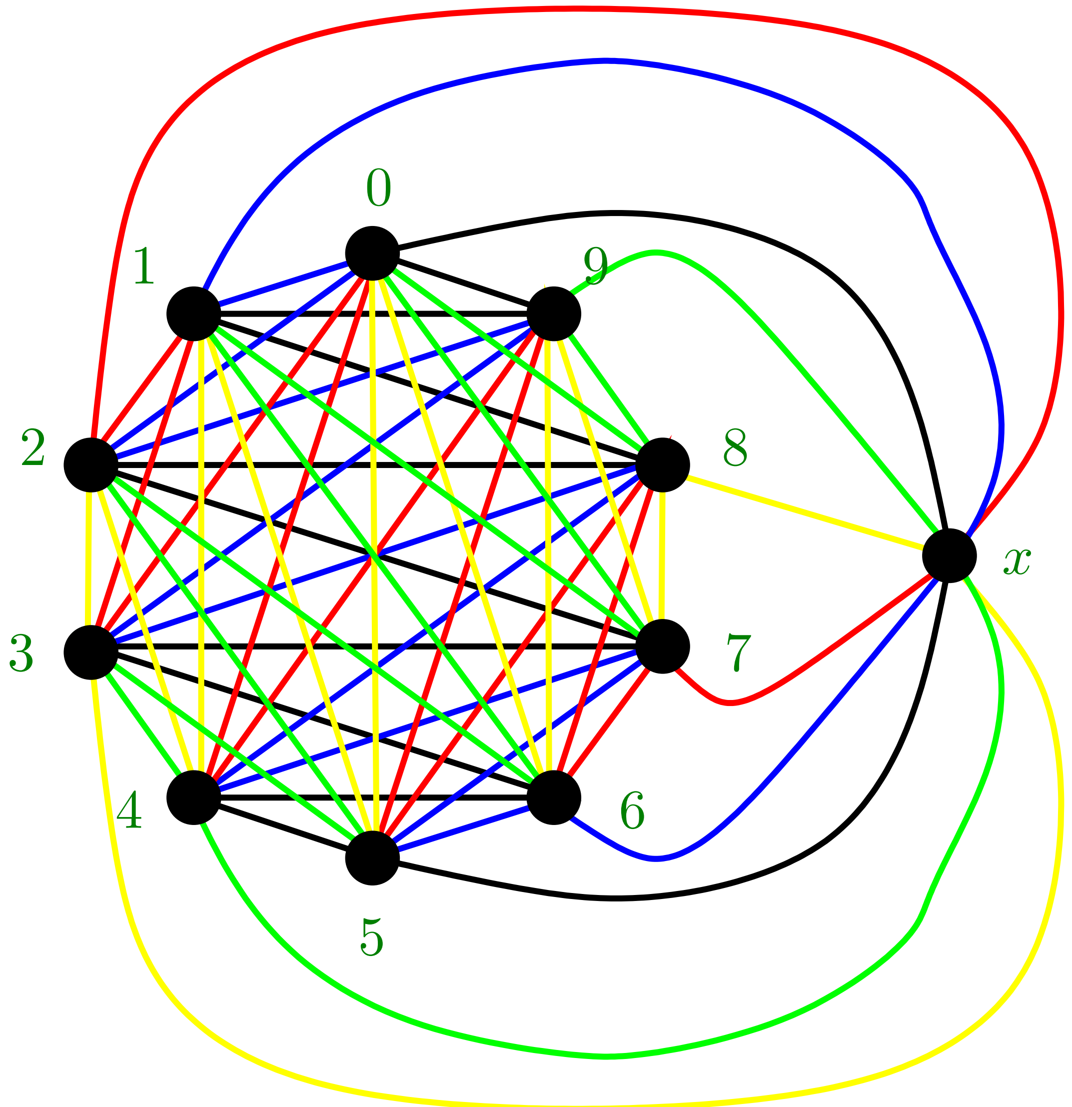
Decomposition into Hamiltonian paths

We say a graph G has a decomposition into Hamiltonian closed paths if it has a decomposition into subgraphs each of which forms a Hamiltonian closed path in G.

Example 9.3. Consider the complete graph K_{11}.


0
1
9
2
8
x
3
7
4
6
5

We continue this process, turning the zig–zag one position clockwise each time to obtain 3 further closed paths



$$C_3 = 2, 1, 3, 0, 4, 9, 5, 8, 6, 7, x, 2,$$

$$C_4 = 3, 2, 4, 1, 5, 0, 6, 9, 7, 8, x, 3 \quad \text{and}$$

$$C_5 = 4, 3, 5, 2, 6, 1, 7, 0, 8, 9, x, 4.$$

We now have the required decomposition of K_{11} into 5 Hamiltonian closed paths.

Theorem 9.4. *The complete graph K_{2d+1} has a decomposition into d Hamiltonian closed paths.*

Proof. The method of the above example (which is called the turning trick) can be used to prove the general case.

Note that K_{2d+1} has $(2d+1)2d/2 = (2d+1)d$ edges.

A Hamiltonian decomposition, if it exists, must therefore involve d Hamiltonian closed paths.

We label the vertices $0, 1, \ldots, 2d-1, x$.

Using the turning trick we obtain d closed paths with no edges in common, as in the previous example.

Decomposition of K_{2d+1} into closed paths

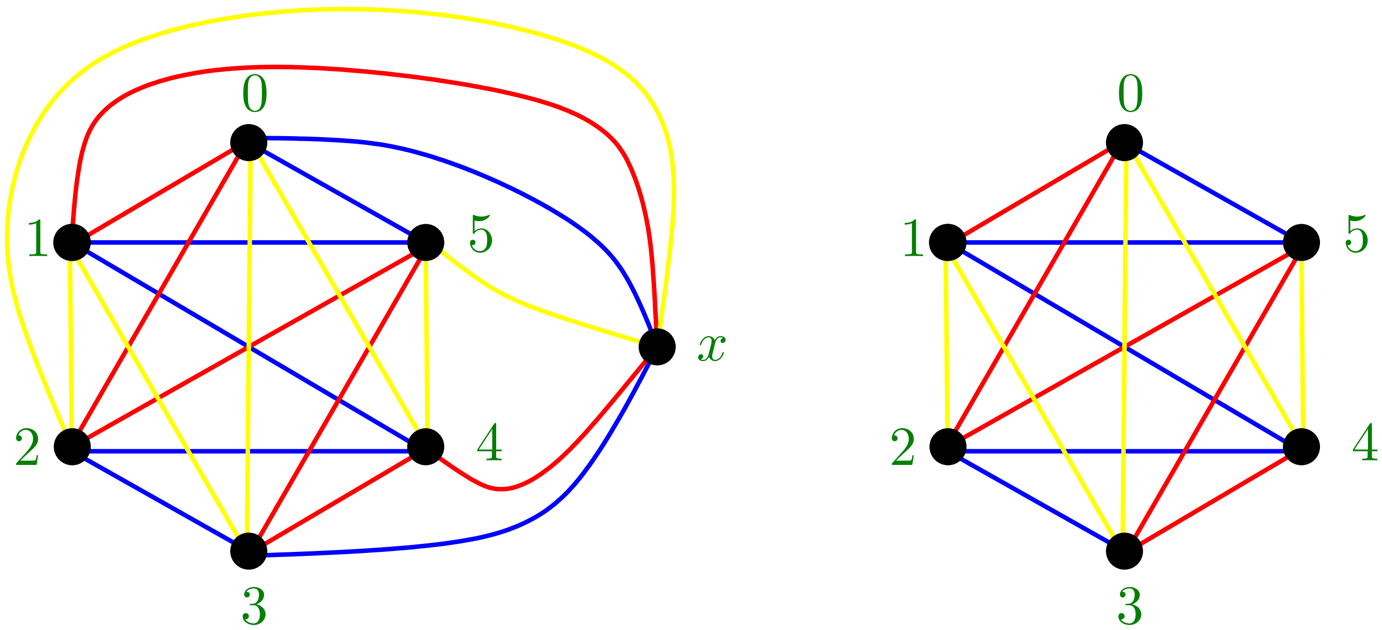
$$
\begin{array}{lcccccccccc}
C_1: & 0, & 2d-1, & 1, & 2d-2, & 2 & \ldots & d-1, & d, & x, & 0 \\
C_2: & 1, & 0, & 2, & 2d-1, & 3 & \ldots & d, & d+1, & x, & 1 \\
C_3: & 2, & 1, & 3, & 0, & 4 & \ldots & d+1, & d+2, & x, & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
C_{d-1}: & d-2, & d-3, & d-1, & d-4, & d & \ldots & 2d-3, & 2d-2, & x, & d-2 \\
C_d: & d-1, & d-2, & d, & d-3, & d+1 & \ldots & 2d-2, & 2d-1, & x, & d-1
\end{array}
$$

This results in the required decomposition into d Hamiltonian closed paths.

Decomposition into paths

Corollary 9.5. *The complete graph K_{2d} has a decomposition into d Hamiltonian paths.*

Example 9.6. The decomposition of K_7 into 3 Hamiltonian closed paths, given by Theorem 9.4, and the corresponding decomposition of K_6 into 3 Hamiltonian paths.



Example 9.7.

Theorem 9.8. *The complete graph* K_{2d} *has a decomposition into* $2d-1$ *paths of lengths* $1, 2, \ldots, 2d-1$*.*

Theorem 9.9. *Let* G *be a simple graph with* $n \geq 3$ *vertices.*

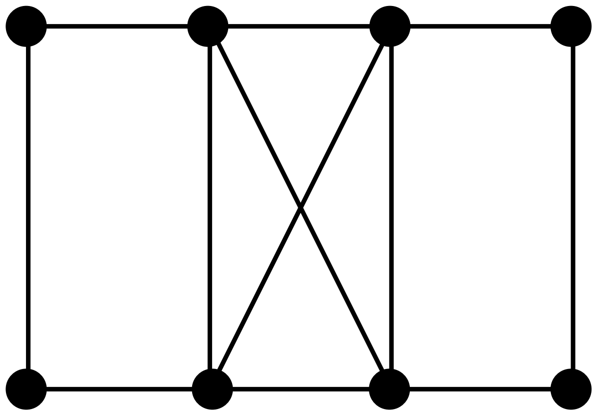
Suppose

$$\deg(u) + \deg(v) \geq n,$$

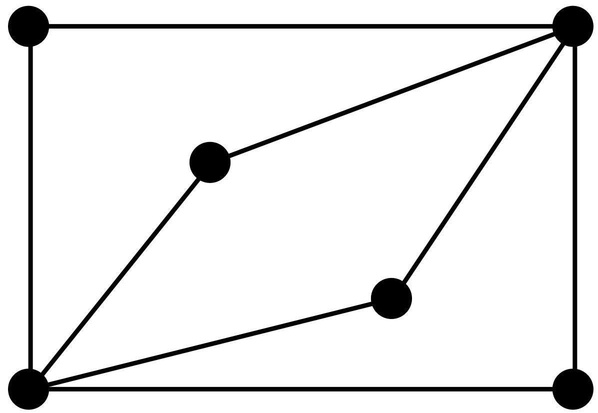
whenever u *and* v *are vertices of* G *which are not adjacent.*

Then G *is Hamiltonian.*

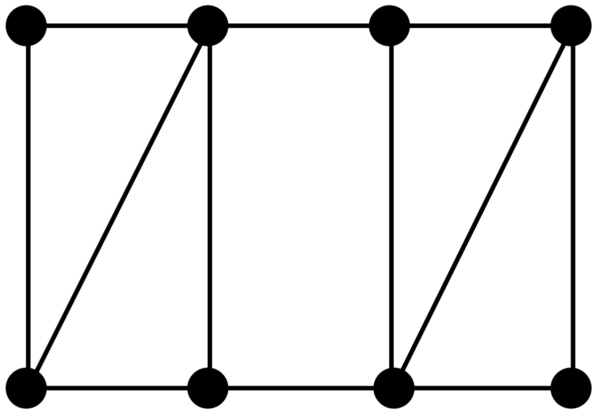
Example 9.10.



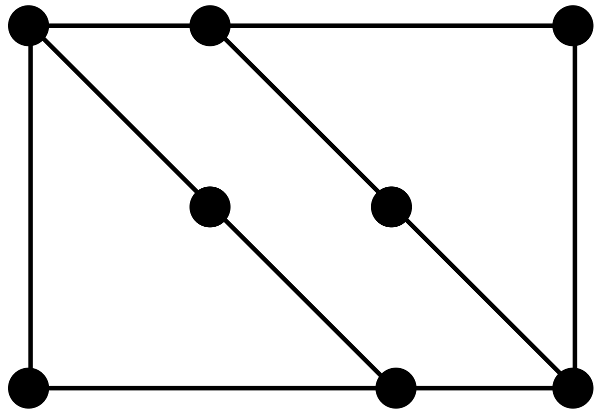
A graph which is Hamiltonian and Eulerian



A graph which is Eulerian and non–Hamiltonian



A graph which is Hamiltonian and non–Eulerian



A graph which is non–Eulerian and non–Hamiltonian

Trees

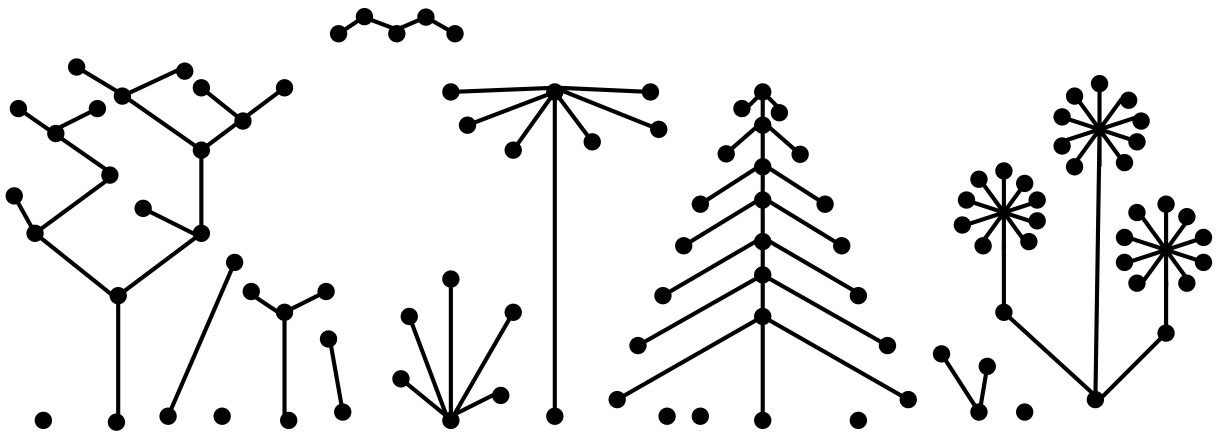
Definition 10.1.

1. A forest is a graph with no cycle.

2. A tree is a connected graph with no cycle.

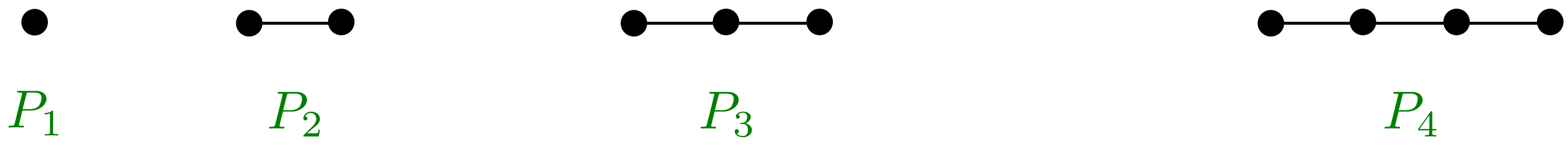
Example 10.2.

1. A forest:



2. The graphs of Example 4.6.2 are all trees.

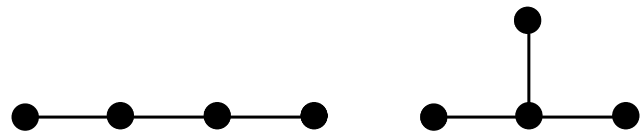
3. The path graph P_n, for $n \geq 1$, is a tree.



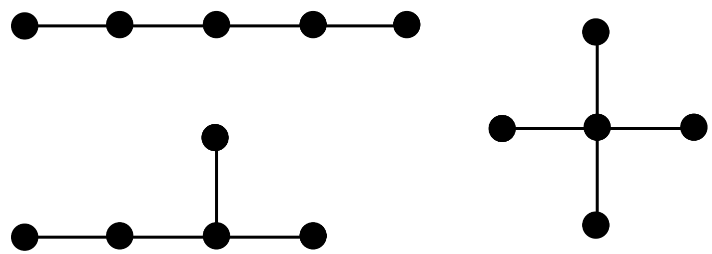
Example 10.3.

1. There is only one tree with one vertex, $N_1 = P_1$. There is only one tree with 2 vertices, $K_2 = P_2$. There is only one tree with 3 vertices, namely P_3.

2. There are 2 trees with 4 vertices:

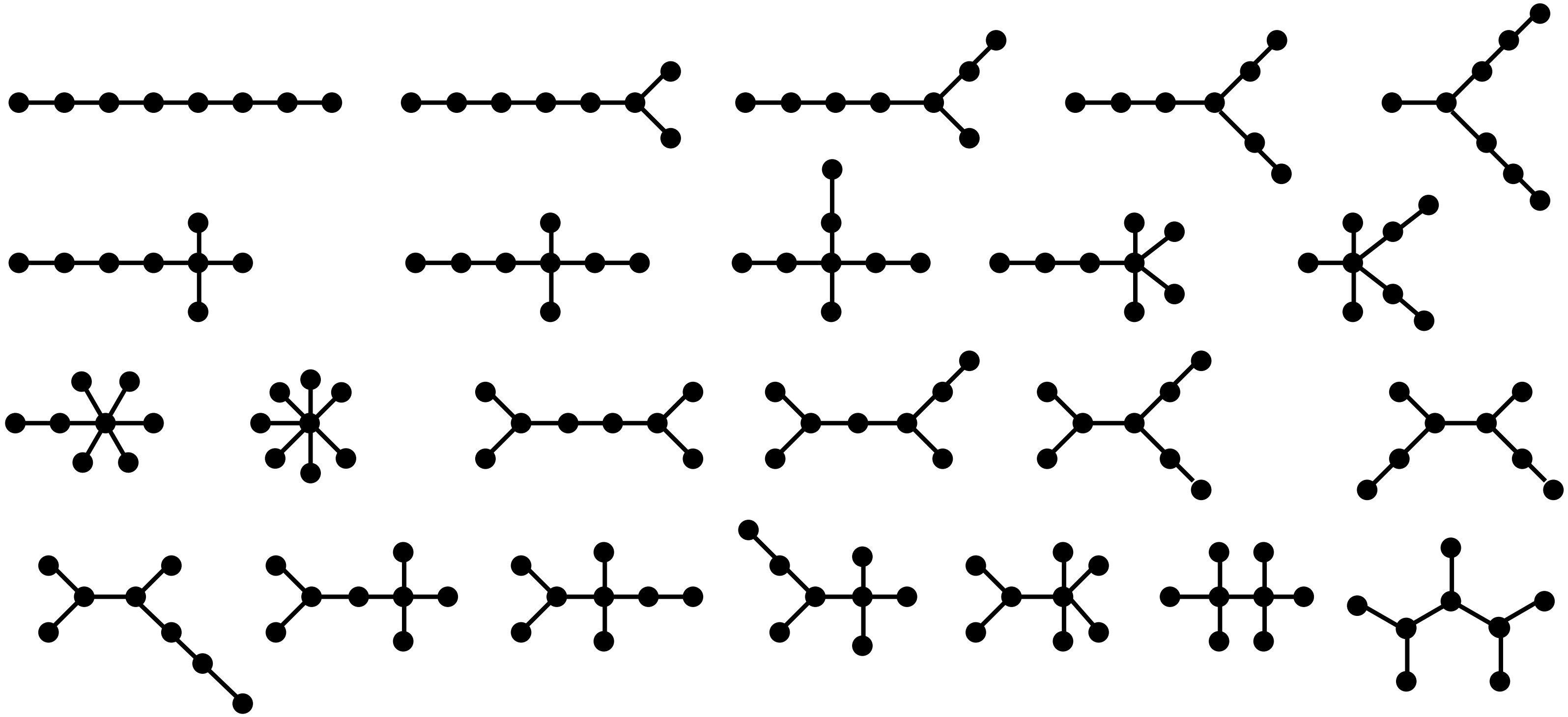


3. There are 3 trees with 5 vertices.



4. There are 6 trees with 6 vertices and 11 trees with 7 vertices (see the Exercises).

5. There are 23 trees with 8 vertices:



Characterising trees

Lemma 10.4. *If a graph G contains two distinct paths from vertices u to v then G contains a cycle.*

Proof. Amongst all pairs of distinct paths with the same initial and terminal vertices choose a pair p, q such that the sum of their lengths is minimal:

say

$$p = u_0, \ldots, u_m \quad \text{and} \quad q = v_0, \ldots v_n,$$

where $u_0 = v_0$ and $u_m = v_n$.

Suppose that $u_i = v_j$ for some i, j, with $0 < i < m$ and $0 \leq j \leq n$.

Then either there is a pair of distinct paths from u_0 to u_i or from u_i to u_m which have smaller lengths than p and q.

This contradicts our choice of p and q, so cannot occur.

It follows that $u_0, u_1, \ldots, u_m = v_n, v_{n-1}, \ldots, v_1, v_0 = u_0$ is a cycle.

Theorem 10.5. *A graph G is a tree if and only if*

(i) G has no loops and

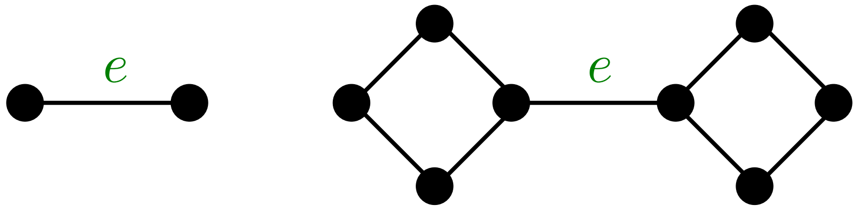
(ii) there is exactly one open path from u *to* v*, for all pairs* u, v *of vertices of* G*.*

Bridges

Definition 10.6. An edge e of a graph G is a bridge if $G - e$ has more connected components than G.

Example 10.7.

1. In these diagrams the edge e is a bridge:



2. If e is an edge of a cycle then e is not a bridge (Lemma 7.8).

Theorem 10.8. *A graph G is a tree if and only if G is connected and every edge of G is a bridge.*

Lemma 10.9. *Let G be a connected graph with m edges and n vertices. Then $n \leq m + 1$. Furthermore G contains a cycle if and only if $n < m + 1$.*

Theorem 10.10. *Let G be a connected graph with n vertices and m edges. Then G is a tree if and only if $n = m + 1$.*

This follows immediately from Lemma 10.9

Spanning Trees

Definition 10.11.
Let G be a graph. A spanning tree for G is a subgraph of G which

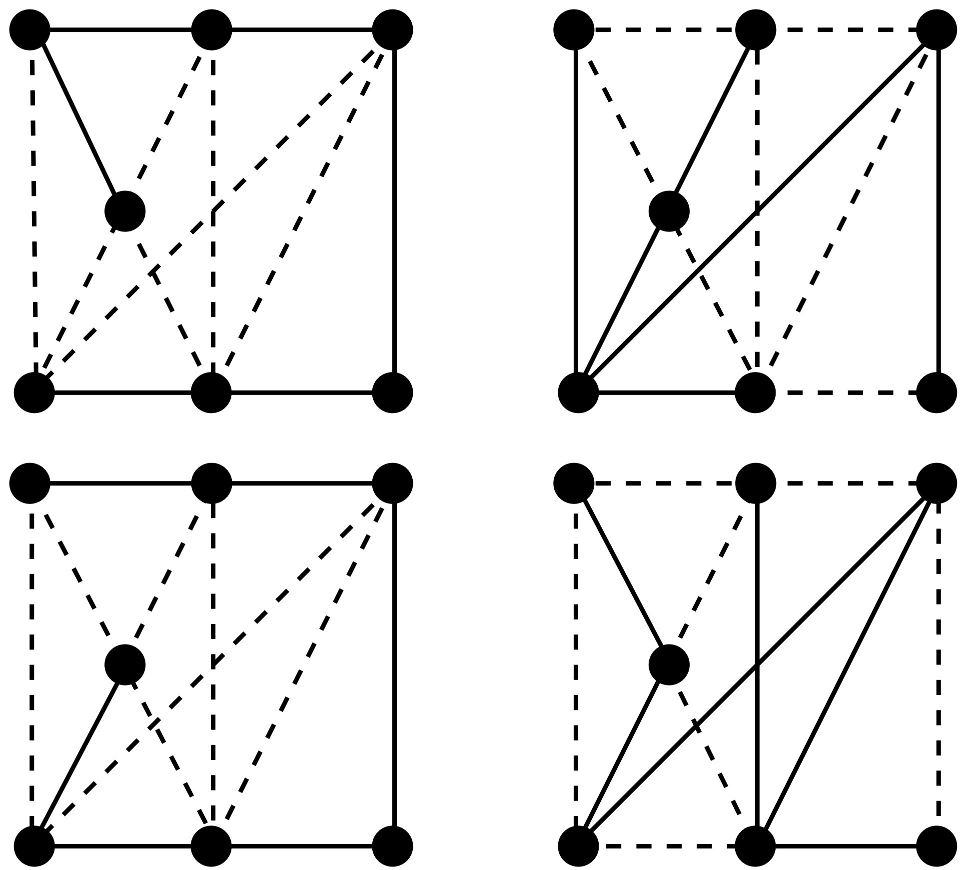
1. is a tree and

2. contains every vertex of G.

- A graph which has a spanning tree must be connected.

- A graph may have many different spanning trees.

Example 10.12. In the diagrams below the solid lines indicate some of the spanning trees of the graph shown: there are many more.



Theorem 10.13. *Every connected graph has a spanning tree.*

The above proof suggests an algorithm for construction of a spanning tree of a graph.

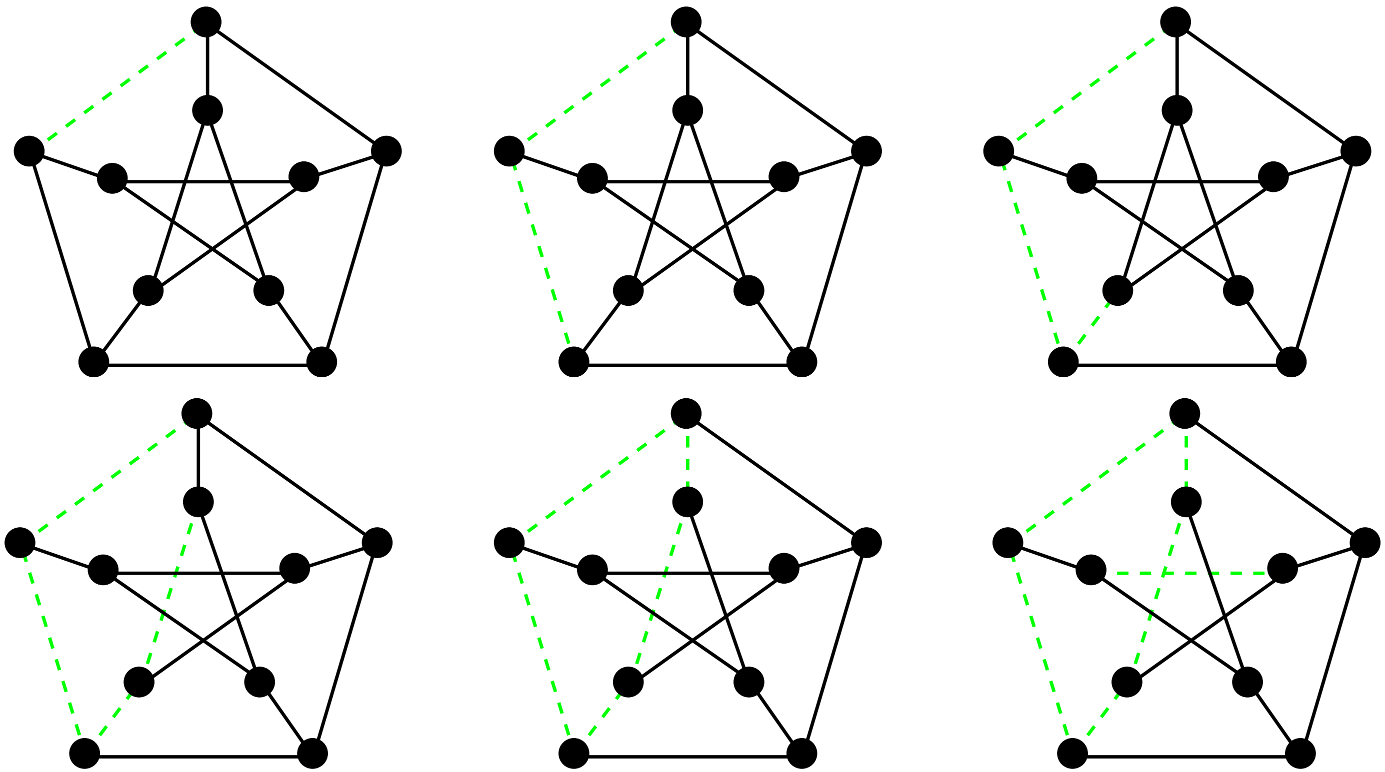
The cut–down algorithm
Given a connected graph G to construct a spanning tree:

1. If G is a tree stop.

2. Choose an edge e from a cycle and replace G with $G - e$. Repeat from 1.

Proof that this process results in a spanning tree is contained in the proof of Theorem 10.13.

The cut–down algorithm on the Petersen graph



The build–up algorithm

Given a connected graph G to construct a spanning tree:

1. Start with a graph T consisting of the vertices of G and no edges.

2. If T is connected stop.

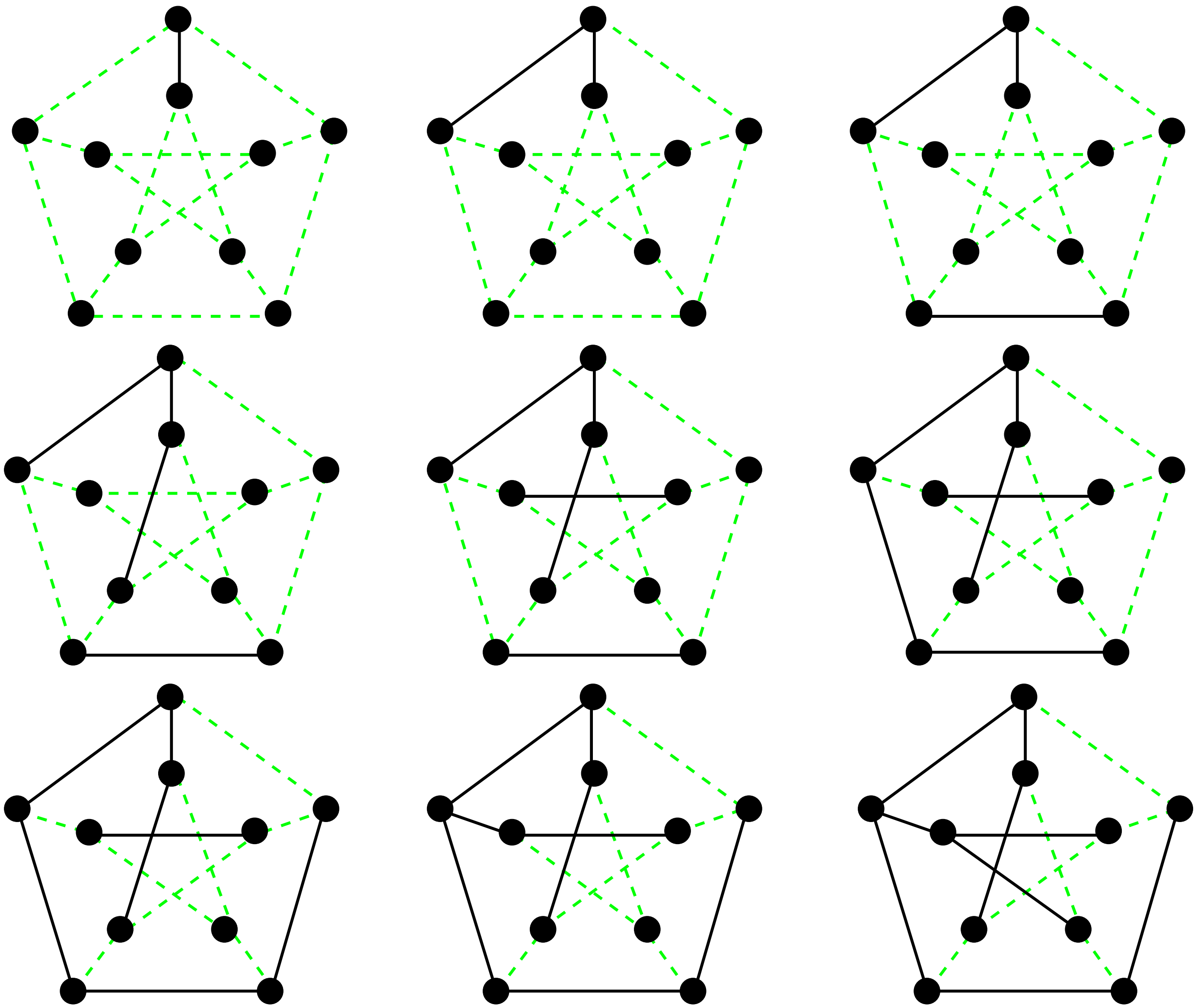
3. Add an edge e of G to T which does not form a cycle in T. Repeat from 2.

Proof that the build–up algorithm stops when T is a spanning tree for G is straightforward.

Strictly speaking neither of these is an algorithm as we have described no way of testing whether a graph contains a cycle or whether a graph is connected.

We shall see how to remedy this defect in the next section.

The build–up algorithm on the Petersen graph



Applications: A programmable spanning tree algorithm

$G = (V, E)$ a graph with at least 2 vertices and 1 edge.

Step 1 Choose an element $e \in E$, say $e = \{a, b\}$, with $a \neq b$.
Set $v_1 = a$, $v_2 = b$ and $t_1 = e$.
Start building T with vertices $V(T) = \{v_1, v_2\}$ and edges $E(T) = \{t_1\}$.
Set $i = 1$ and $j = 2$. (i is the number of the "base vertex", j is the number of the last vertex added.)

Step 2 If there is a vertex u of G which is adjacent to v_i and not in the subgraph T then

add 1 to j;
set $v_j = u$ and $t_{j-1} = \{v_i, v_j\}$ (an edge of G joining v_i to u);
continue to build up T by adding v_j to $V(T)$ and t_{j-1} to $E(T)$.

Step 3 If j is equal to the number of vertices of G then output the tree T and stop.

Step 4 If all vertices of G which are adjacent to v_i are in T then add 1 to i.

Step 5 If $i > j$ then output the message "G is not connected" and stop.

Otherwise repeat from Step 2.

Example 11.1.

Weighted graphs

Definition 11.2. Let G be a connected graph with edge set E. To each edge $e \in E$ assign a non–negative real number $w(e)$.

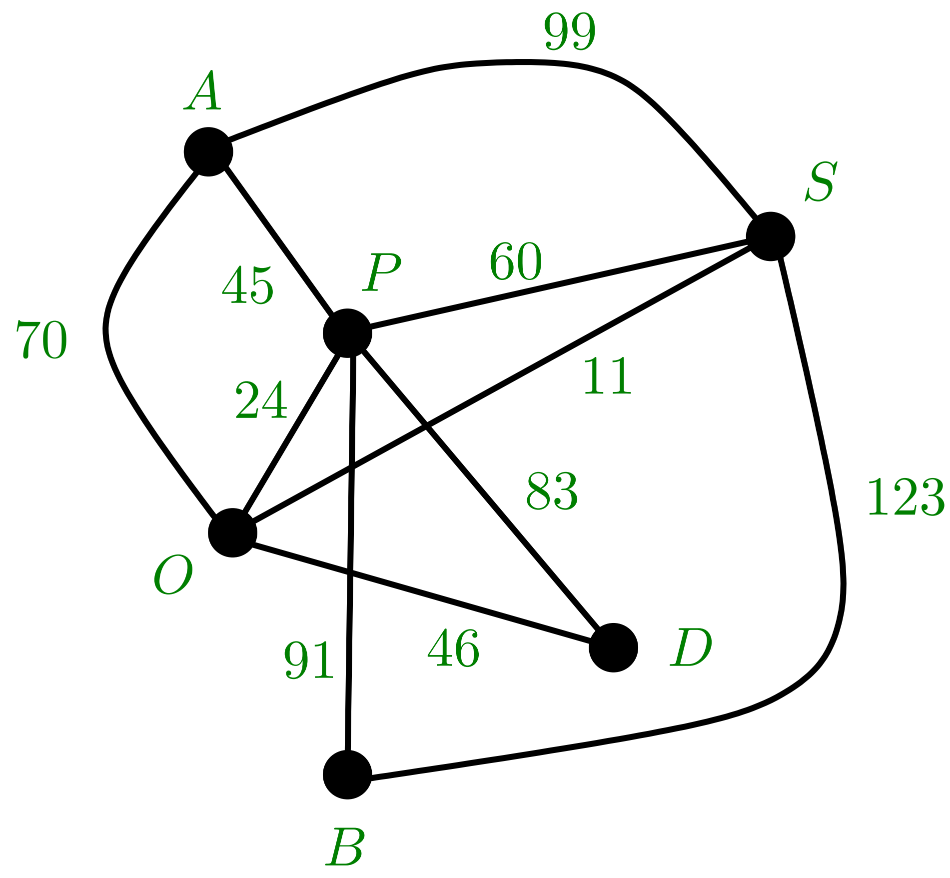
Then G is called a weighted graph and the number $w(e)$ is called the weight of e.

The sum

$$W(G) = \sum_{e \in E} w(e)$$

is called the weight of G.

Example 11.3.



The weight of edge $\{A, S\}$ is $w(\{A, S\}) = 99$

and the weight of edge $\{O, P\}$ is $w(\{O, P\}) = 24$.

The graph has weight $W(G) = 652$.

The Minimum Connector Problem

A subgraph of a connected graph G which contains all the vertices of G is called a spanning subgraph.

Every spanning graph must contain a spanning tree.

In a connected, weighted graph the problem of finding a spanning subgraph of minimal weight is called the minimal connector problem.

A spanning subgraph of minimal weight is always a spanning tree,

so the problem is to find a spanning tree of minimal weight.

The following algorithm does so. Again we leave aside the problem of testing for a cycle.

The Greedy Algorithm (also known as Kruskal's Algorithm)

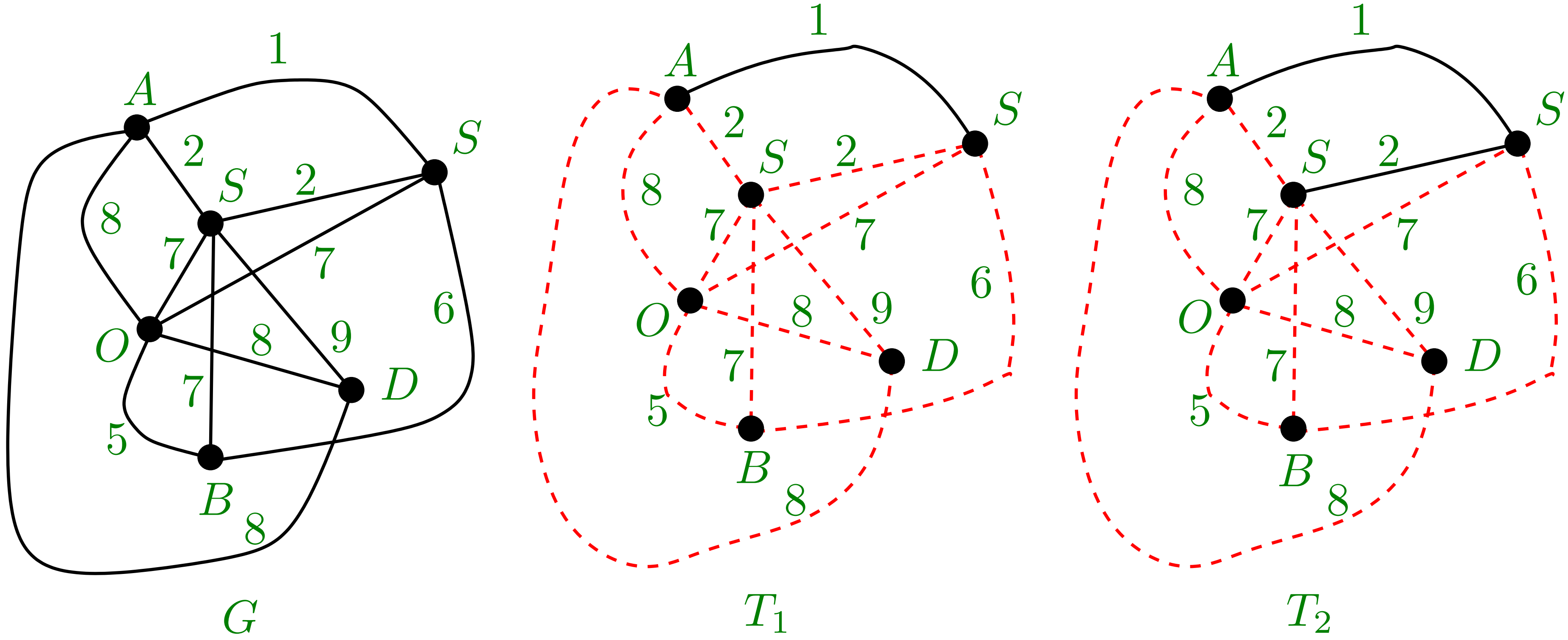
Let G be a connected weighted graph. To find a spanning tree T for G of minimal weight:

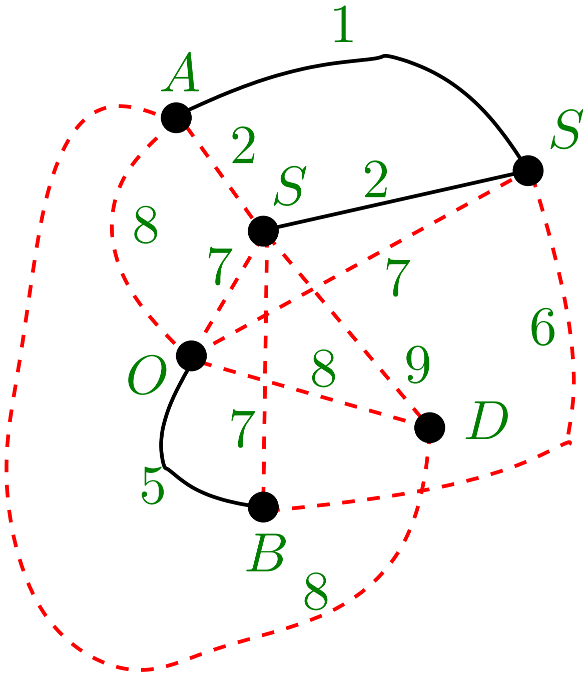
Step 1

Step 2

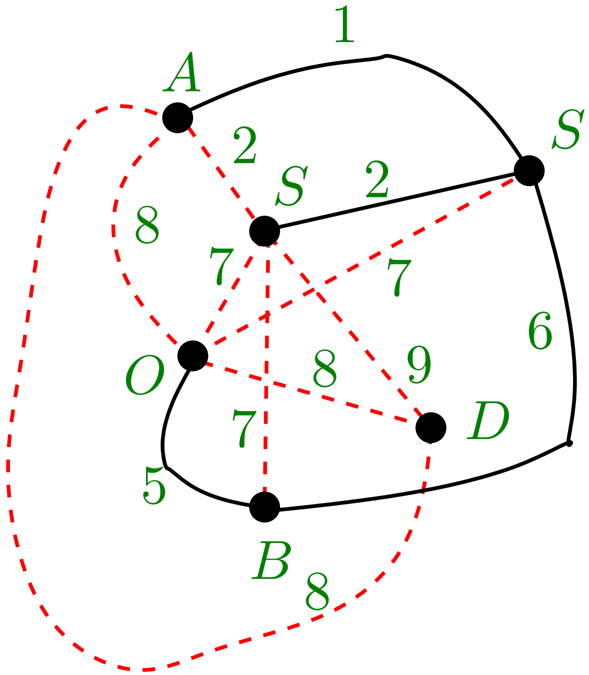
Step 3

Example 11.4.

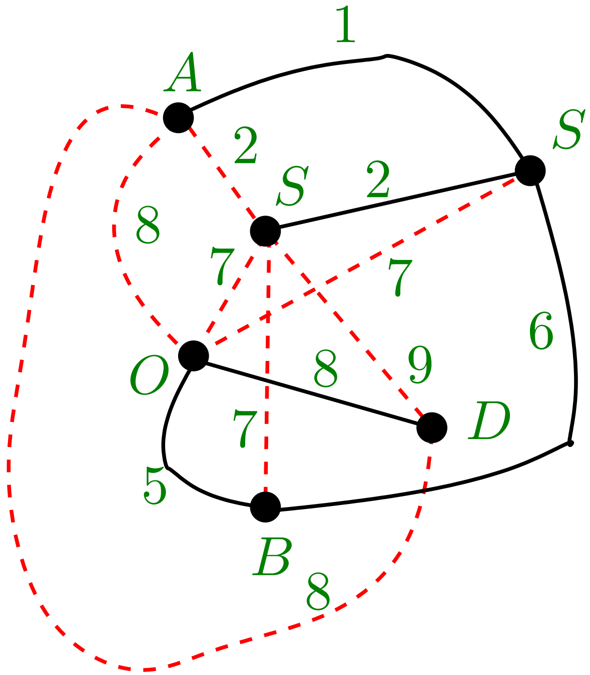




T_3



T_4



T_5

Some choices that have to be made in the running of the algorithm.

For instance, either of the edges of weight 2 could have been included in T.

A different choice results in a different minimal weight spanning tree, of which there may be many.

The Travelling Salesman Problem

A problem:

"Given a connected weighted graph G, find a closed walk in G containing all vertices of G and of minimal weight amongst all such closed walks."

This problem is very difficult to solve in general.

An easier problem: the Travelling Salesman problem:

"Given a connected weighted graph G, find a minimal weight Hamiltonian closed path in G."

easier — fewer possible solutions,

but still very difficult to solve.

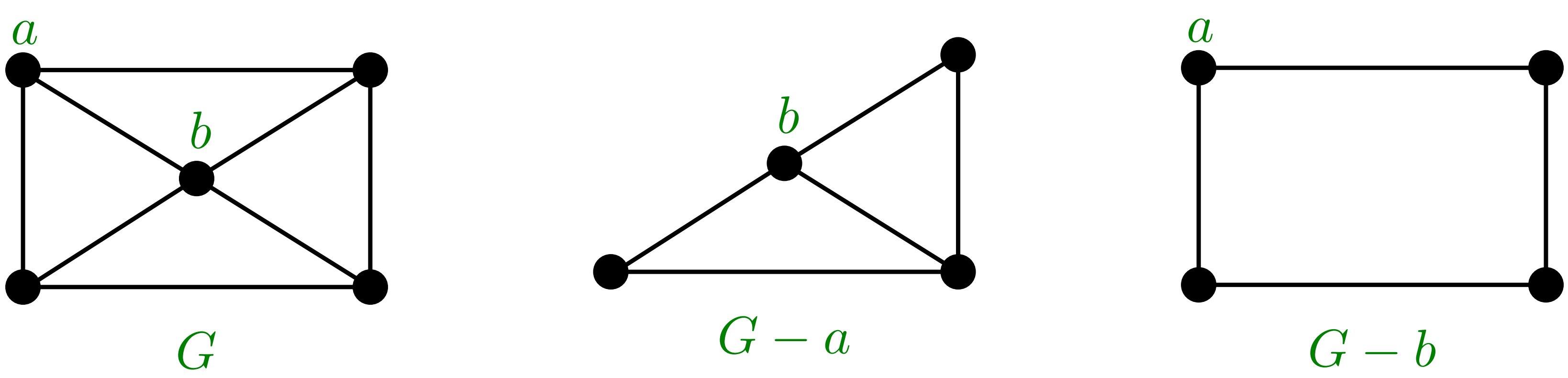
The algorithm for the Minimum Connector problem can be used to find a lower bound for the Travelling Salesman problem.

Deleting vertices

Definition 11.5. Let G be a graph and let v be a vertex of G.

The graph $G - v$ obtained from G by deleting v is defined to be the graph formed by removing v and all its incident edges from G.

Example 11.6.



A lower bound for the Travelling Salesman

Theorem 11.7. *If G is a weighted graph, C is a minimal weight Hamiltonian closed path in G and v is a vertex of G then*

$$w(C) \geq M + m_1 + m_2,$$

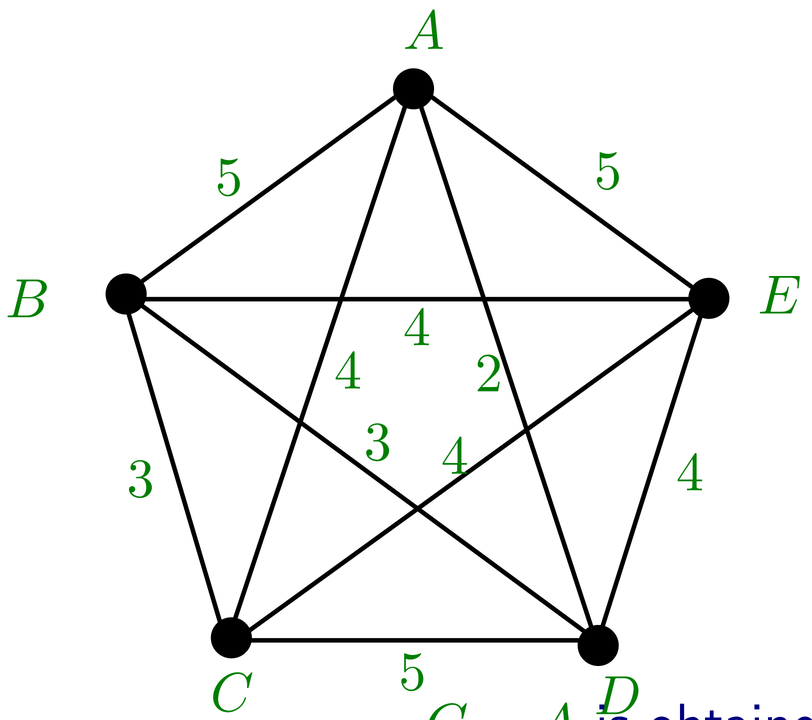
where M is the weight of a minimal weight spanning tree for $G - v$ and m_1 and m_2 are the weights of two edges of least weight incident to v.

As pointed out above the inequality in this Theorem may be strict.

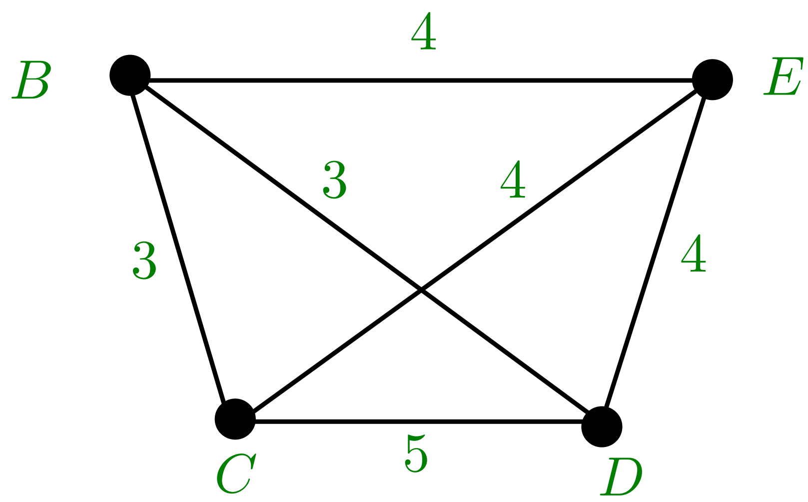
We obtain a lower bound for the Travelling Salesman problem, which in some cases may be smaller than the weight of minimal weight Hamiltonian closed path.

Example 11.8.

Find a lower bound for the Travelling salesman problem in the weighted graph G below by removing vertex A.

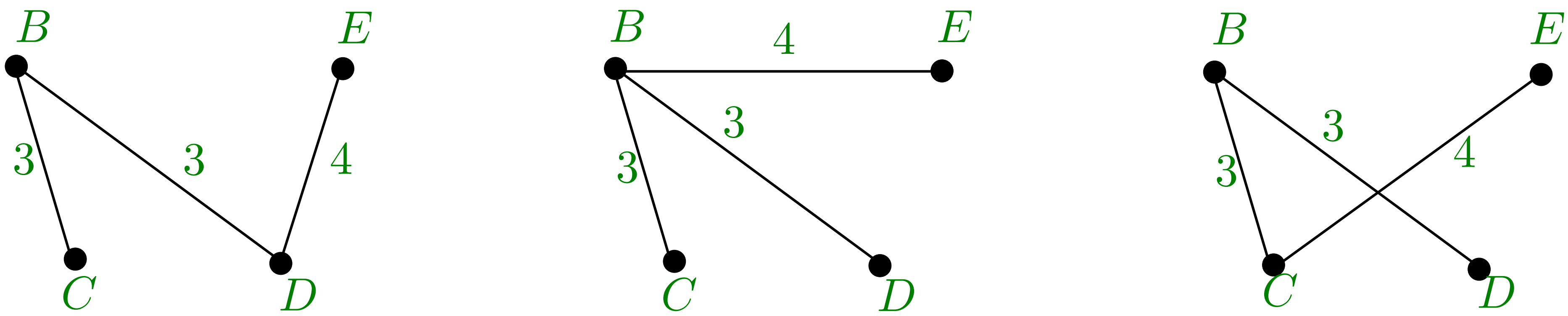


$G - A$ is obtained by removing vertex A.



The Greedy Algorithm on $G - A$ is used to find a minimal weight spanning tree.

The output is one of the 3 shown below: which all have weight 10, so $M = 10$.

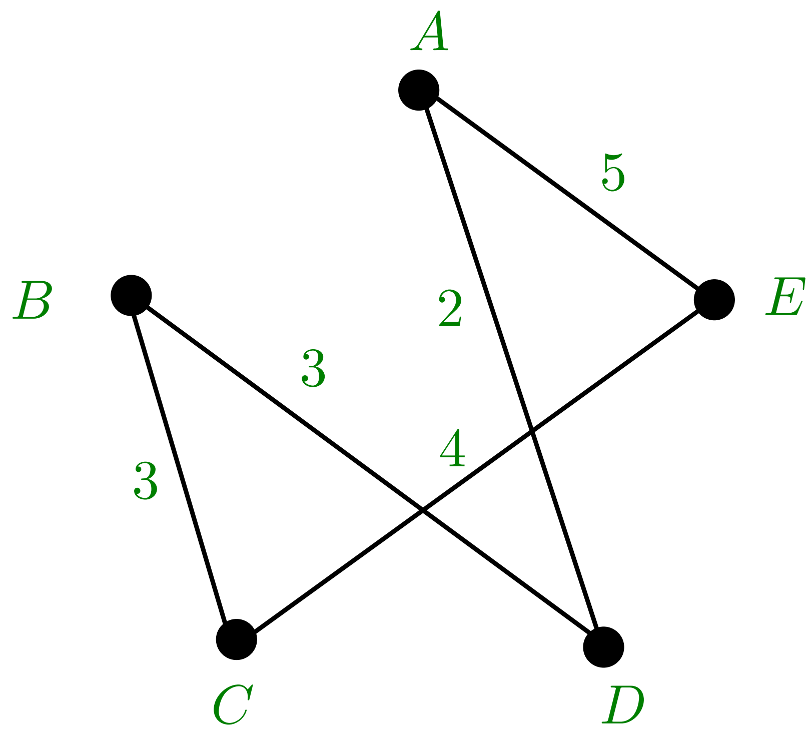


Spanning Tree 1 Spanning Tree 2 Spanning Tree 3

The edges of minimal weight incident to A are $\{A, C\}$ and $\{A, D\}$ which have weights $m_1 = 2$ and $m_2 = 4$.

Combining this information we have a lower bound of $10 + 2 + 4 = 16$.

A minimal weight Hamiltonian closed path



Planar Graphs

Definition 12.1. A graph is planar if it can be drawn in the plane without edges crossing. A plane drawing of a graph is a drawing of a graph in the plane which has no edge crossings.

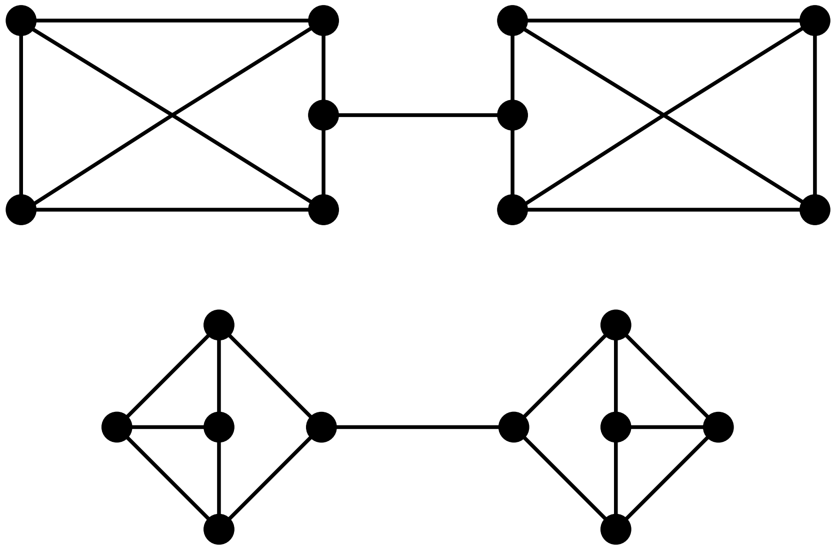
We shall refer to a plane drawing of a graph as a plane graph.

Note however that a drawing is merely a representation of a graph:

as always a graph consists merely of a pair of sets.

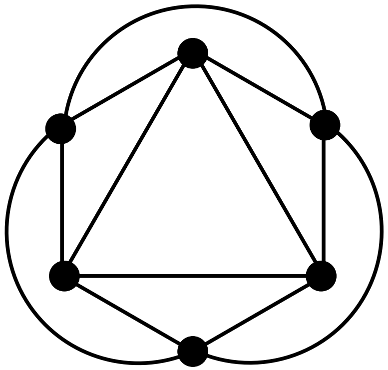
Example 12.2.

1.



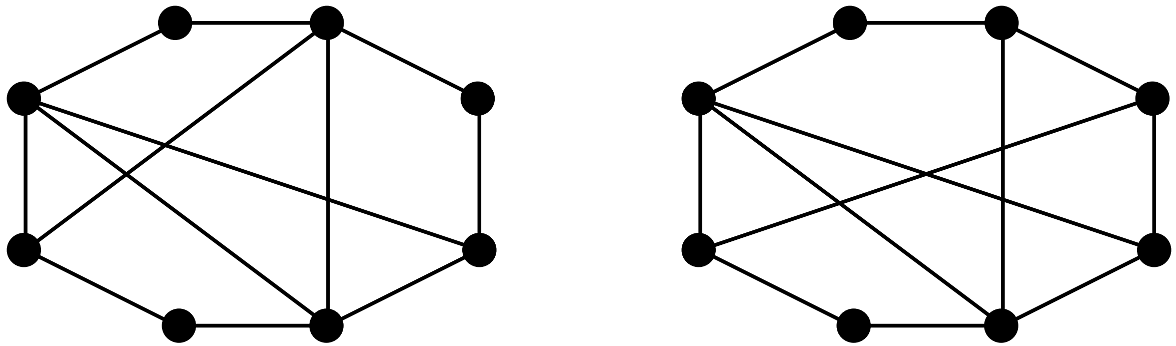
Example 12.2 cont.

2.



Example 12.2 cont.

3.



Faces

A plane drawing of a graph divides the plane up into polygonal regions which we call faces.

In the Example 12.2 the first plane graph divides the plane into 7 regions

and the Octahedron divides the plane into 8 regions.

(Note that in each case one of the regions is unbounded.)

Faces of a plane drawing

Definition 12.3. Let D be a plane drawing of a graph. If x is a point of the plane not lying on D then the set of all points of the plane that can be reached from x without crossing D is called a face of D. One face is always unbounded and is called the exterior face.

(To make a rigourous definition of face requires the Jordan Curve theorem, which says that:

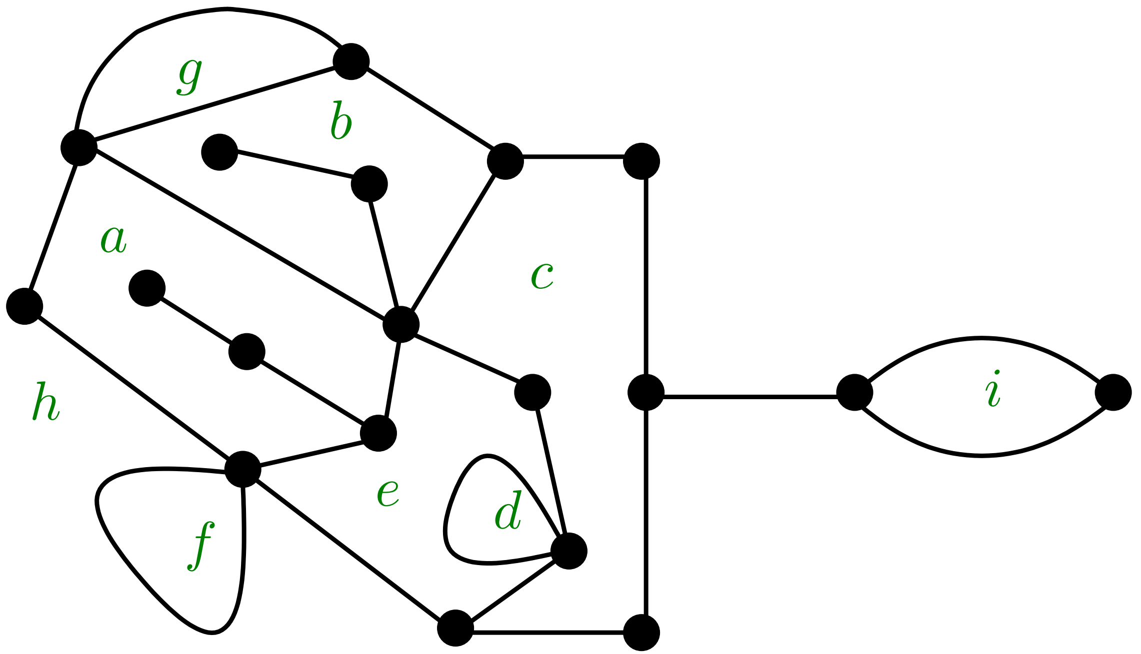
a simple closed curve in the plane divides the plane into two parts, one inside and one outside the curve.

This theorem is beyond the scope of this course.)

Example 12.4.

1. A plane drawing of a tree has one face (which is exterior).

2. The graph below has 9 faces labelled $a \ldots i$. Face h is the exterior face.



Euler's Formula

Theorem 12.5. *[Euler's Formula] Let G be a connected plane graph (i.e. a plane drawing of a connected graph) with n vertices, m edges and r faces. Then $n - m + r = 2$.*

Definition 12.6. Let F be a face of a plane graph. The degree of F, denoted $\deg(F)$ is the number of edges in the boundary of F, where edges lying in no face except F count twice.

(To compute $\deg(F)$ walk once round the boundary of F, counting each edge on the way.)

In Example 12.4.2 above we have

face	a	b	c	d	e	f	g	h	i
degree	9	8	8	1	7	1	2	14	2

Degrees of faces vs degrees of vertices

Compare the following to Lemma 3.1.

Lemma 12.7. *If G is a plane graph with m edges and r faces $F_1, \ldots, F_r$ then*

$$\sum_{i=1}^{r} \deg(F_i) = 2m.$$

Proof. Every edge meets either one or two faces.

Edges meeting only one face contribute 2 to the degree of their face.

Edges meeting two faces contribute 1 to the degree of each of their faces.

The result follows.

Finding non–planar graphs

Corollary 12.8. *If G is a simple connected planar graph with $n \geq 3$ vertices and m edges then $m \leq 3n - 6$.*

Corollary 12.9. *If G is a connected simple planar graph with $n \geq 3$ vertices, m edges and no cycle of length 3 then $m \leq 2n - 4$.*

Proof. A plane drawing of G can have no face of degree less than 4.

The proof proceeds as that of Corollary 12.8, except that this time $2m \geq 4r$.

Theorem 12.10. *The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are both non–planar.*

Subdivision

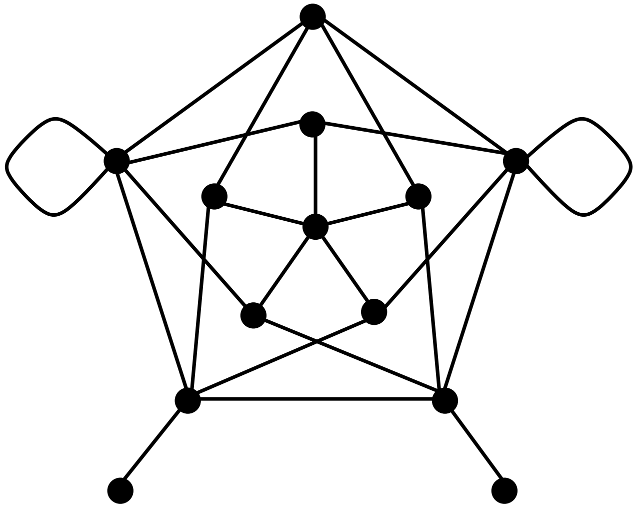
If a graph G is non-planar then any graph which contains G as a subgraph is also non–planar.

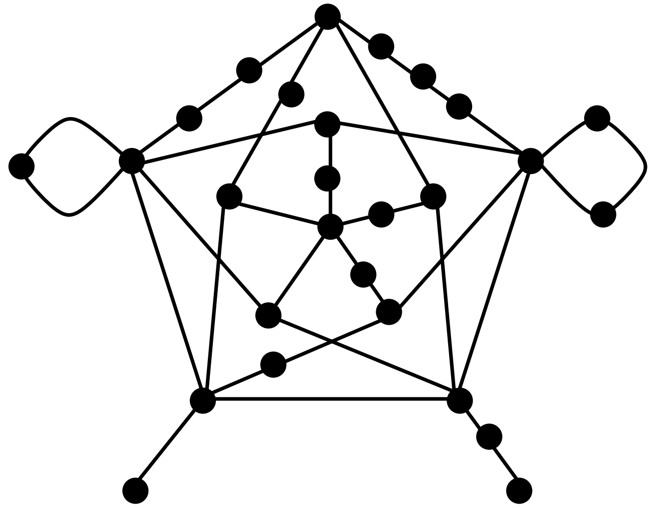
It follows that if a graph contains K_5 or $K_{3,3}$ as a subgraph it must be non-planar.

Definition 12.11. A graph H is a subdivision of a graph G if H is obtained from G by the addition of a finite number of vertices of degree 2 to edges of G.

It is possible to add no vertices and so a graph is a subdivision of itself.

Example 12.12. H below is a subdivision of G.





Planarity and subdivisions

The following theorem is an easy consequence of Theorem 12.10.

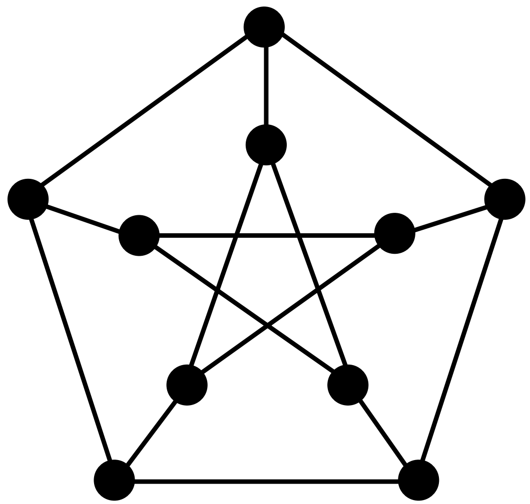
Theorem 12.13. *If G is a graph containing a subgraph which is a subdivision of K_5 or $K_{3,3}$ then G is non–planar.*

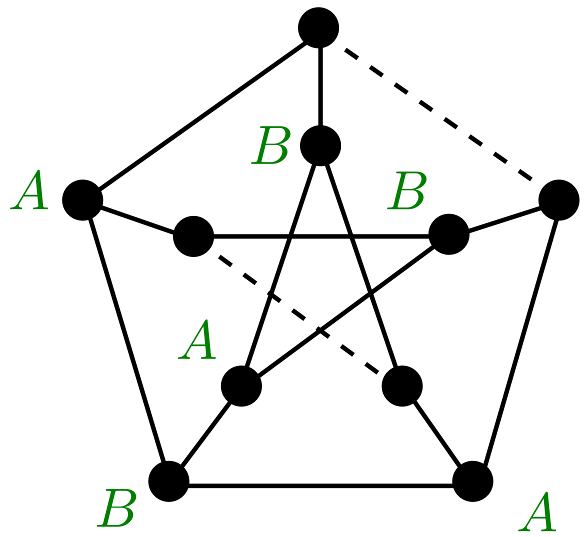
Example 12.14.

Neither Corollary 12.8 nor Corollary 12.9 are sufficient to show that the graphs of this example are non–planar.

1. The Petersen graph has 10 vertices and 15 edges.

The diagram on the right shows a subgraph which is a subdivision of $K_{3,3}$. Therefore the graph is non-planar. (Vertices which are not labelled A or B are those added in the subdivision.

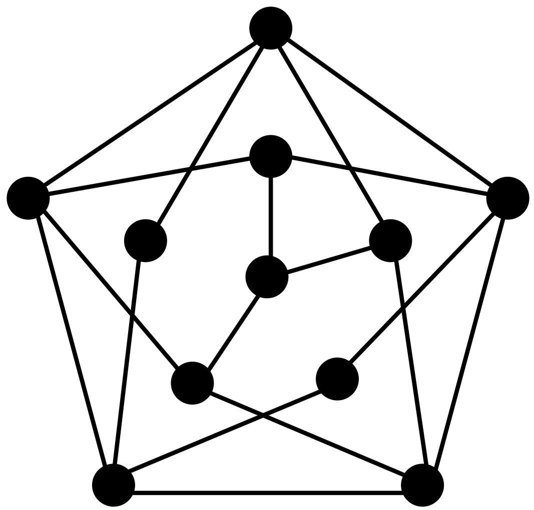


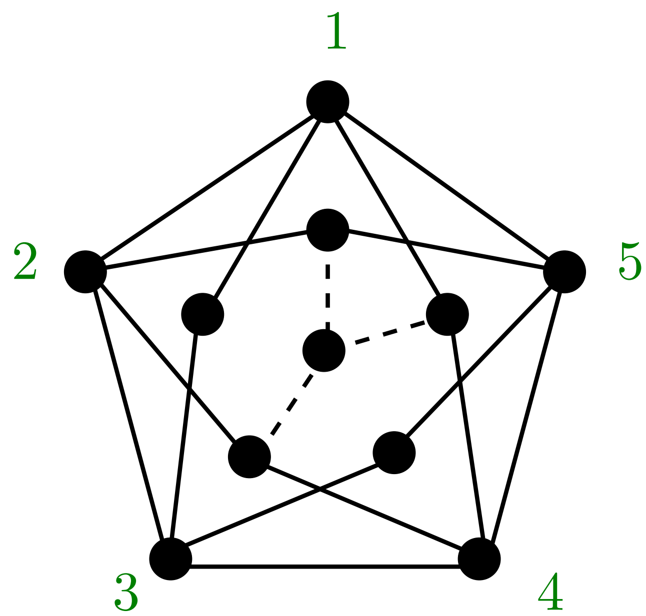


Example 12.14 cont.

2. The graph shown below has 11 vertices and 18 edges.

The right hand diagram shows a subgraph which is a subdivision of K_5. Therefore the graph is non-planar.





Kuratowski's Theorem

A more surprising theorem:

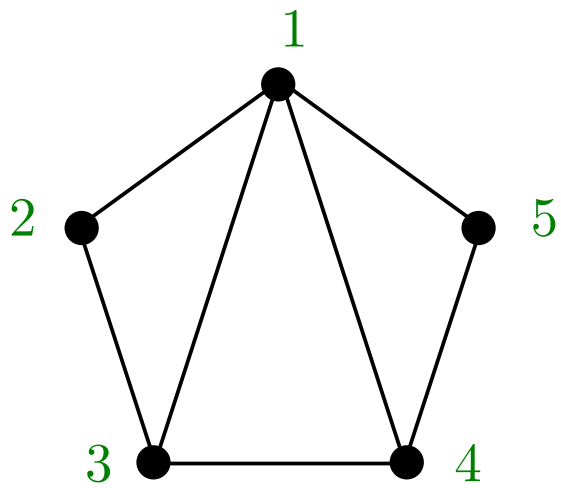
Theorem 12.15. *[Kuratowski] If* G *is a non–planar graph then* G *contains a subgraph which is a subdivision of* K_5 *or* $K_{3,3}$*.*

Vertex Colouring

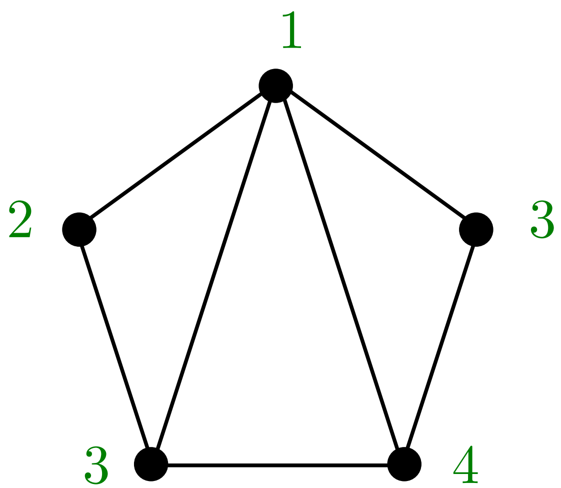
Definition 13.1. Let G be a graph without loops. A k–colouring of G is an assignment of k colours to the vertices of G such that no two adjacent vertices are assigned the same colour. If G has a k–colouring it is said to be k–colourable.

Example 13.2. We use colours $1, 2, 3, 4$ and 5.

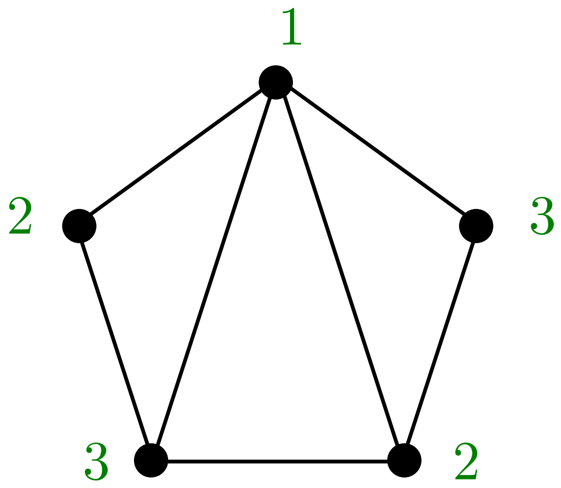
1. a 5–colouring:



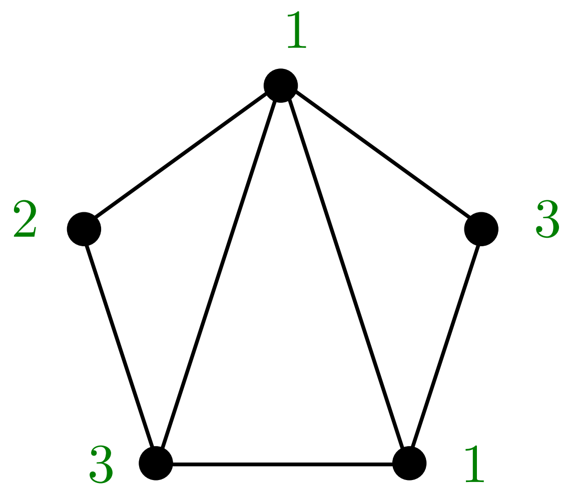
2. a 4–colouring:



3. a 3–colouring:

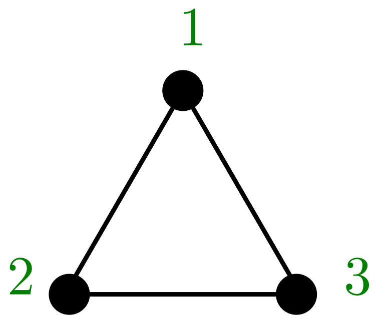


4. not a colouring:



Example 13.3.

1.



2.

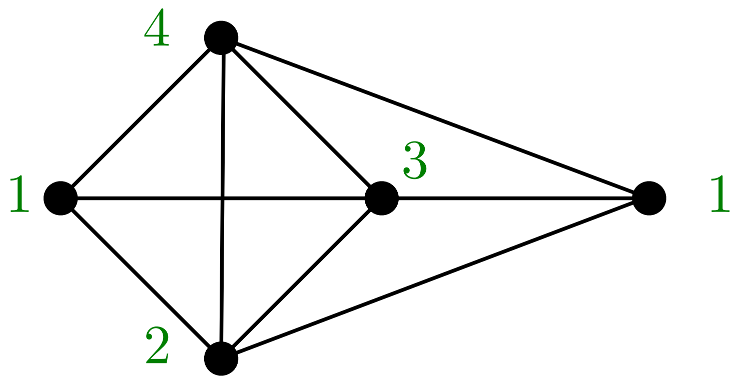
Definition 13.4. The chromatic number $\chi(G)$ of a graph G is the least positive integer k such that G has a k–colouring.

Example 13.5.

1. From Example 13.3 it follows that $\chi(K_d) = d$, for all $d \geq 2$.

2.

3.



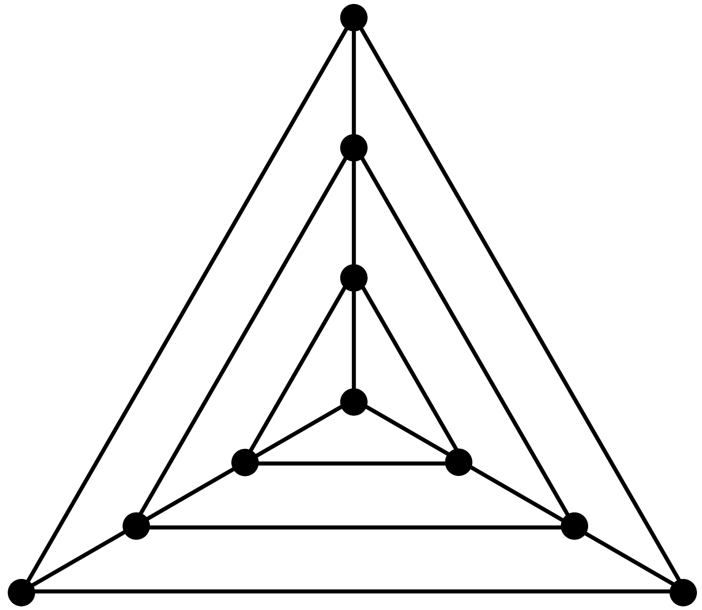
Theorem 13.6. *[Brooks] Let* G *be a connected simple graph and* d *a non–negative number such that* $\deg(v) \leq d$*, for all vertices* v *of* G*. If*

1. G *is not a cycle graph* C_n *with* n *odd and*

2. G *is not a complete graph* K_n

then $\chi(G) \leq d$.

Example 13.7. The graph G below has a subgraph isomorphic to K_4, so $\chi(G) \geq 4$.



Using Brooks' theorem $\chi(G) \leq 4$.

Hence $\chi(G) = 4$.

We have calculated $\chi(G)$ without colouring G

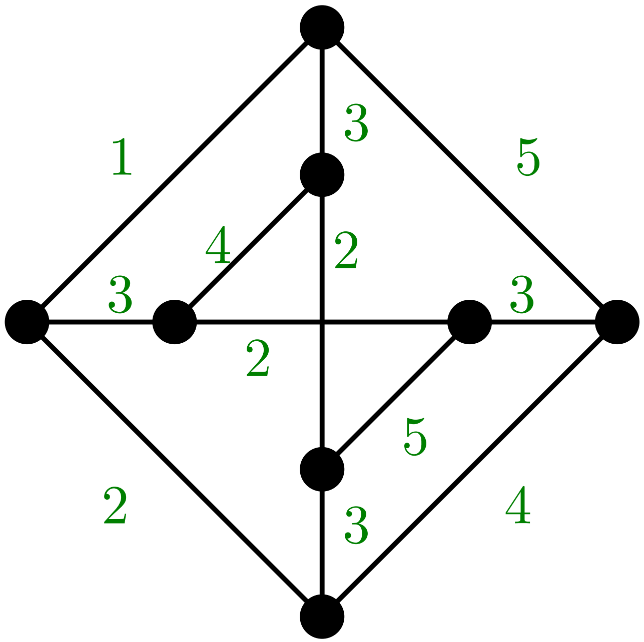
Edge colouring

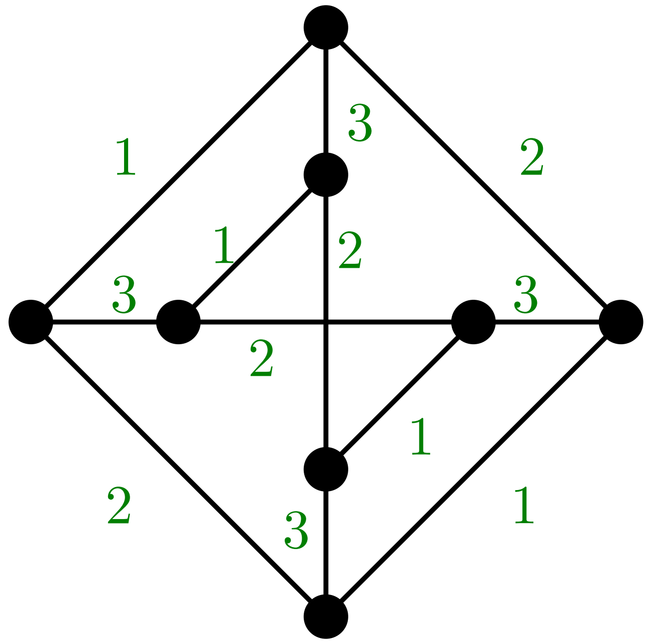
Definition 13.8. An edge–colouring using **k** colours of a graph G is an assignment of one of k colours to each edge of G.

A proper edge–colouring is one with the additional property that no two adjacent edges are assigned the same colour.

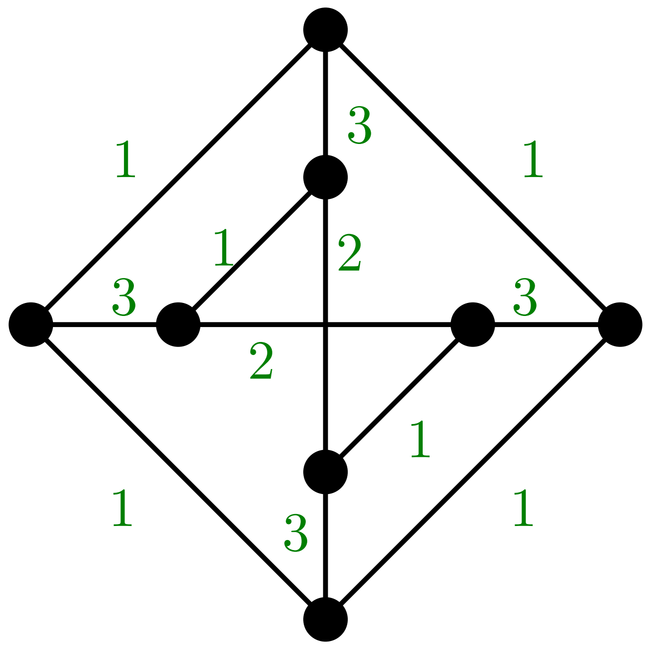
The edge–chromatic number of G is the smallest integer k such that G has a proper edge–colouring using k colours.

Example 13.9. 1. Proper edge–colourings using 5 and 3 colours:





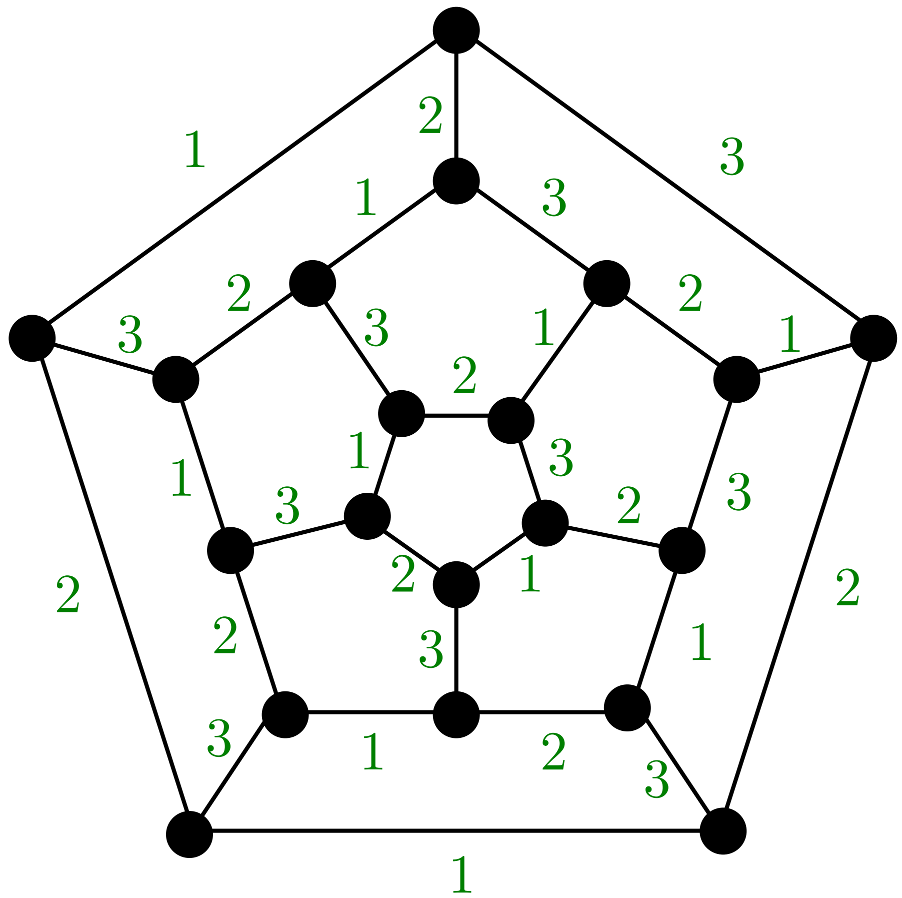
3. an edge–colouring using 3 colours which is not proper:



Lemma 13.10. *Let G be a graph and let d be the largest degree of a vertex of G. Then any proper edge–colouring of G uses at least d colours.*

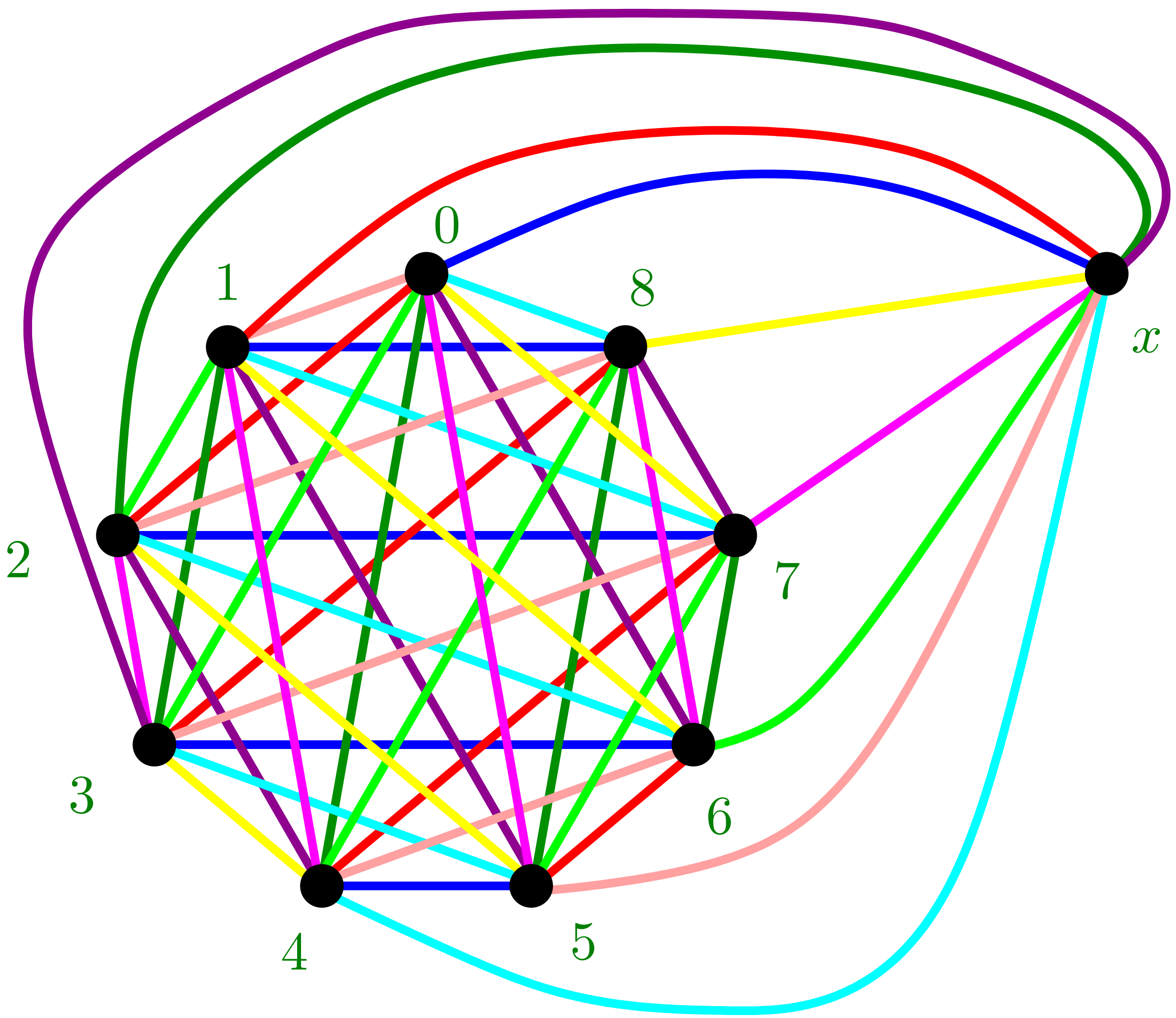
Example 13.11. 1.

2.



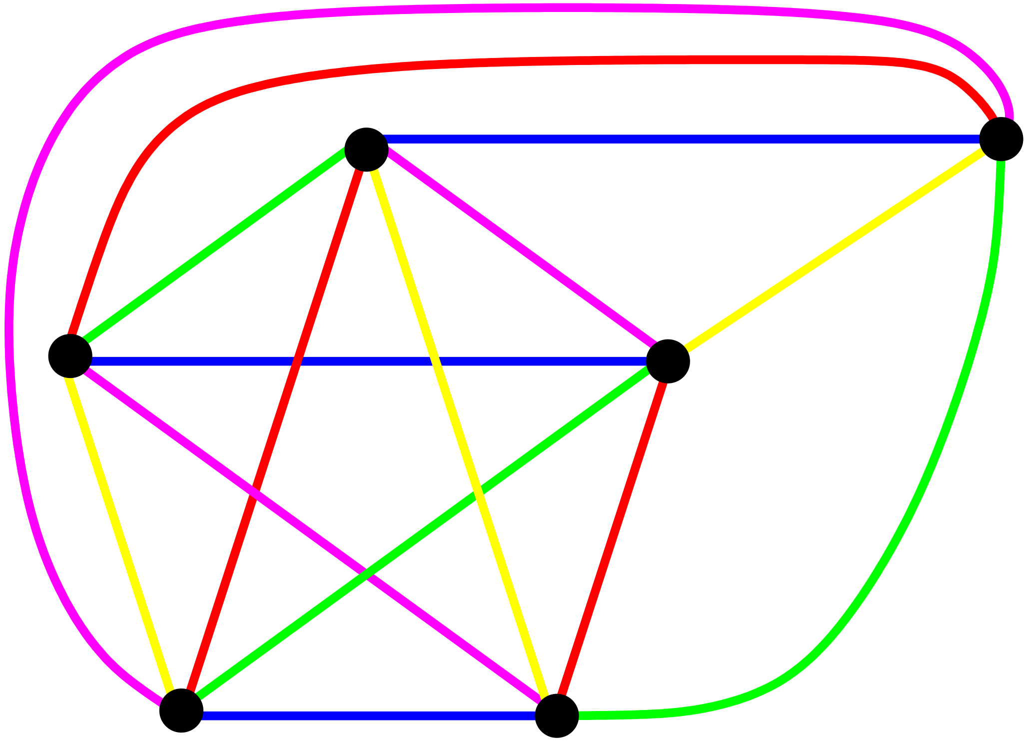
The Edge–Chromatic Number of K_n

Theorem 13.12. *The edge–chromatic number of* K_{2d} *is* $2d - 1$, *for all* $d \geq 1$.

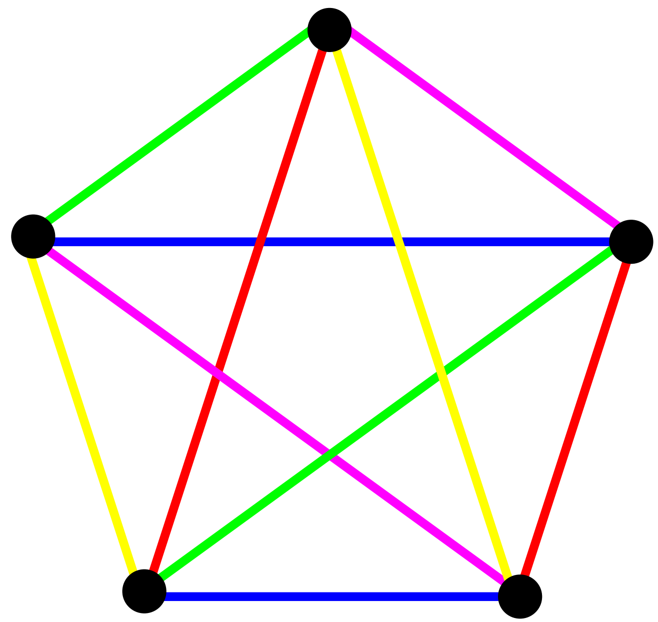


Corollary 13.13. *The edge–chromatic number of* K_{2d-1} *is* $2d-1$*, for all* $d \geq 1$.

Example 13.14. Using Theorem 13.13 and Corollary 13.14.



Proper edge–colouring of K_6 using 5 colours



Proper edge–colouring of K_5 using 5 colours

The Four–colour Problem

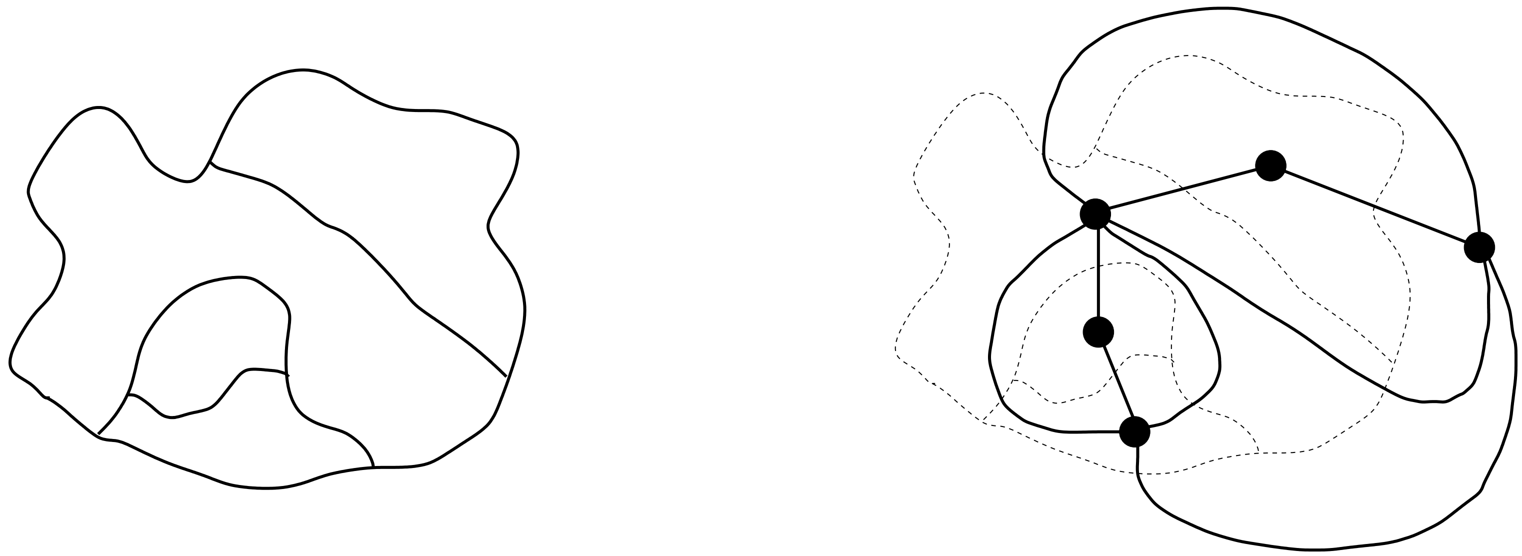
De Morgan's conjecture (1852): any map of countries can be coloured using only 4 colours, in such a way that countries with a common border have different colours.

Given a map of countries construct a plane drawing of a graph as follows.

Place one vertex in each country.

Join two vertices with an edge whenever their countries have a common border.

Example 13.15.



Reformulated in terms of graph theory:

If it can be shown that any planar graph without loops is 4–colourable then it follows that every map of countries can be coloured as specified.

Conjecture 13.16. *[The 4–colour conjecture] Every simple planar graph is 4–colourable.*

History

1852 De Morgan proposes the 4–colour conjecture.

1873 Cayley presents a proof to the London Mathematical Society. The proof is fatally flawed.

1879 Kempe publishes a proof; which collapses.

1880 Tait gives a proof which turns out to be incomplete.

1976 Appel & Haken at the University of Illinois prove the 4–colour conjecture: using thousands of hours of CPU time on a Cray computer.

A problem with Appel & Haken's proof is that the program runs for so long that it is impossible to verify manually. We cannot even to be sure that the hardware performed well enough, over such an extended period, to give a reliable result.

The 5 and 6-colour Theorems

Theorem 13.17.
Every simple planar graph G is 6–colourable.

A proof of a 5–colour theorem can be found in most introductory texts on graph theory.

In 1880 Tait made the following connection between 4–colouring of faces and edge–colouring.

Theorem 13.18. *Let G be a plane drawing of a graph which is connected, regular of degree three and has no bridges or loops. Then the faces of G can be coloured using 4 colours if and only if G has a proper edge–colouring using 3 colours.*