UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2
2000/2001

MAS223/623

Graph theory

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FIVE questions in Section A and THREE questions in Section B.

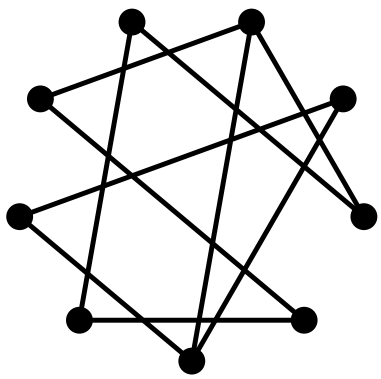
You may submit part of your answers to questions A1, A3, A5, B6 and B8 on the page of diagrams provided.

SECTION A

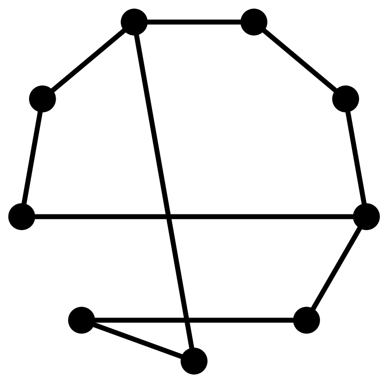
A1. (a) Write down the degree sequence of each of the graphs below.

(b) State which of the graphs are isomorphic to each other. Show these isomorphisms by labelling vertices. (You may give your answers to this part of the question on the sheet of diagrams provided.)

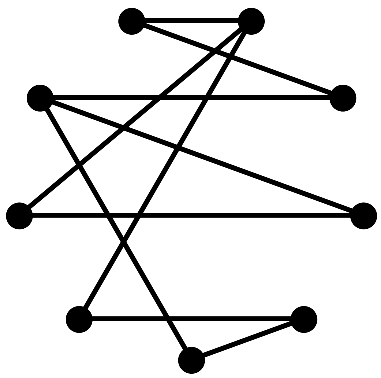
(c) Is it true that graphs with the same degree sequence are always isomorphic? Justify your claim, referring to the graphs below if appropriate.



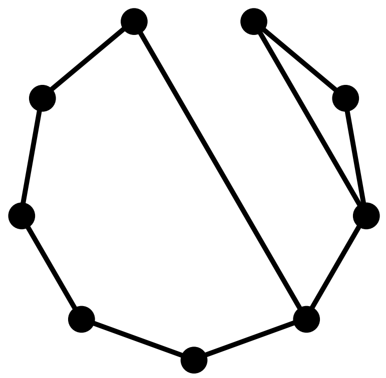
(A)



(B)



(C)



(D)

[9 marks]

A2. (a) State the Handshaking Lemma which relates the sum of degrees of vertices of a graph to the number of edges.

(b) Draw the complete bipartite graph $K_{3,5}$. How many vertices of degree 3 does $K_{3,5}$ have? How many vertices of degree 5 does $K_{3,5}$ have? Verify the Handshaking Lemma for $K_{3,5}$.

(c) How many vertices of degree 6 does $K_{6,11}$ have? (Do not draw $K_{6,11}$.) How many vertices of degree 11 does $K_{6,11}$ have? Use the Handshaking Lemma to compute the number of edges of $K_{6,11}$.

(d) Compute the number of edges of $K_{r,s}$, where $r, s \geq 1$.

[9 marks]

A3. In the graph G_3 below find the following and write down their defining sequences. (It suffices to write down the vertices of the sequences).

(a) A walk of length 6 which is not closed and not a trail.

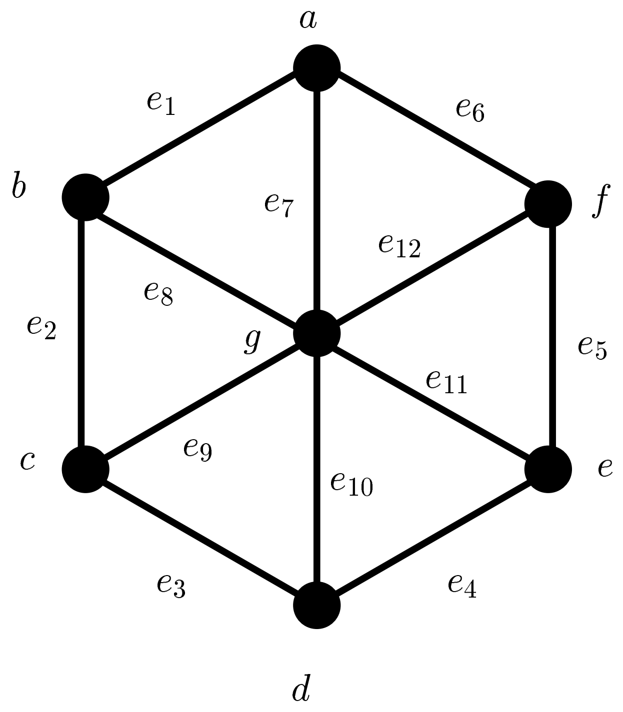
(b) An open path of length 4.

(c) A circuit of length 6 which is not a cycle.

(d) A cycle of length 3.

(e) A Hamiltonian cycle.

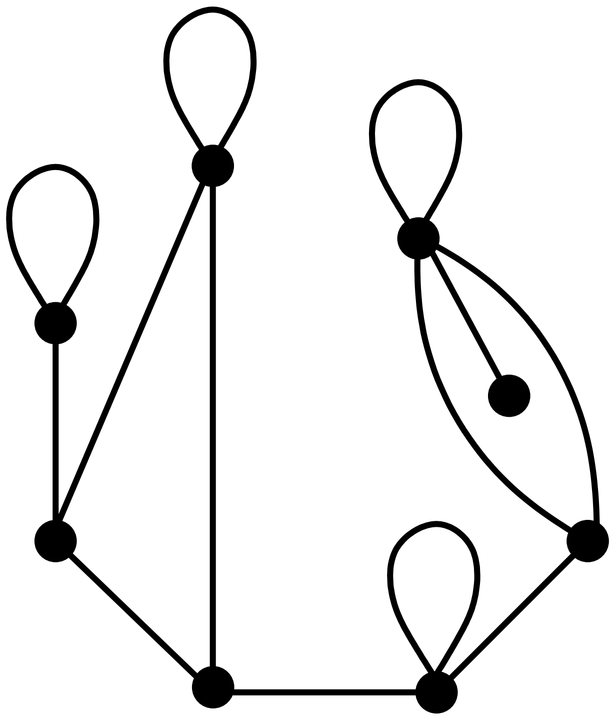
Construct an Eulerian graph by adding 3 new edges to G_3. (You may give your answer on the sheet of diagrams.)



G_3

[8 marks]

A4. Draw all 6 spanning trees for the graph below.



[6 marks]

A5. (a) Write down the definition of **proper edge–colouring** and of **edge–chromatic number**.

(b) Write down the edge–chromatic number of the complete graphs K_{2d} and K_{2d-1}.

(c) Find an edge–colouring of K_7 using as few colours as possible. (You may use the drawing of K_7 on the sheet of diagrams.)

[10 marks]

SECTION B

B6. (a) State Euler's Formula relating the number of vertices, edges and faces of a plane drawing of a connected graph. Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges.

(i) Show that if G has no faces of degree 3, 4 or 5 then

$$m \leq 3(n-2)/2.$$

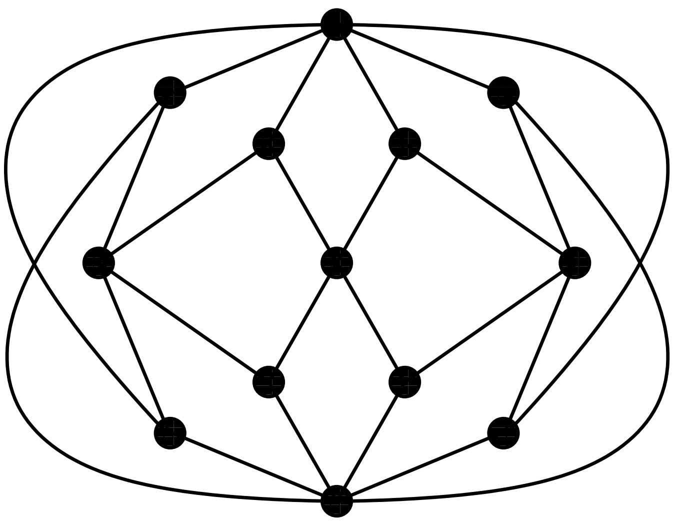
(ii) On the other hand show (using the previous part of the question) that if G is regular of degree 3 then G must have at least one face of degree 3, 4 or 5.

(iii) Now show that a simple connected planar graph which is regular of degree 3 and bipartite must have a face of degree 4. (You may use the fact that a bipartite graph contains no closed walk of odd length.)

(b) (i) Show that a bipartite graph contains no closed walk of odd length.

(ii) Show that the graph below is bipartite. (You may use the sheet of diagrams.)

(iii) Use the two previous parts of the question to show that the graph below is non–Hamiltonian.



[30 marks]

B7. (a) Let G be a simple graph with n vertices such that $\deg(v) \geq (n-1)/2$, for all vertices v of G.

(i) By considering the number of vertices incident to a fixed vertex of G show that each connected component of G contains $(n+1)/2$ vertices.

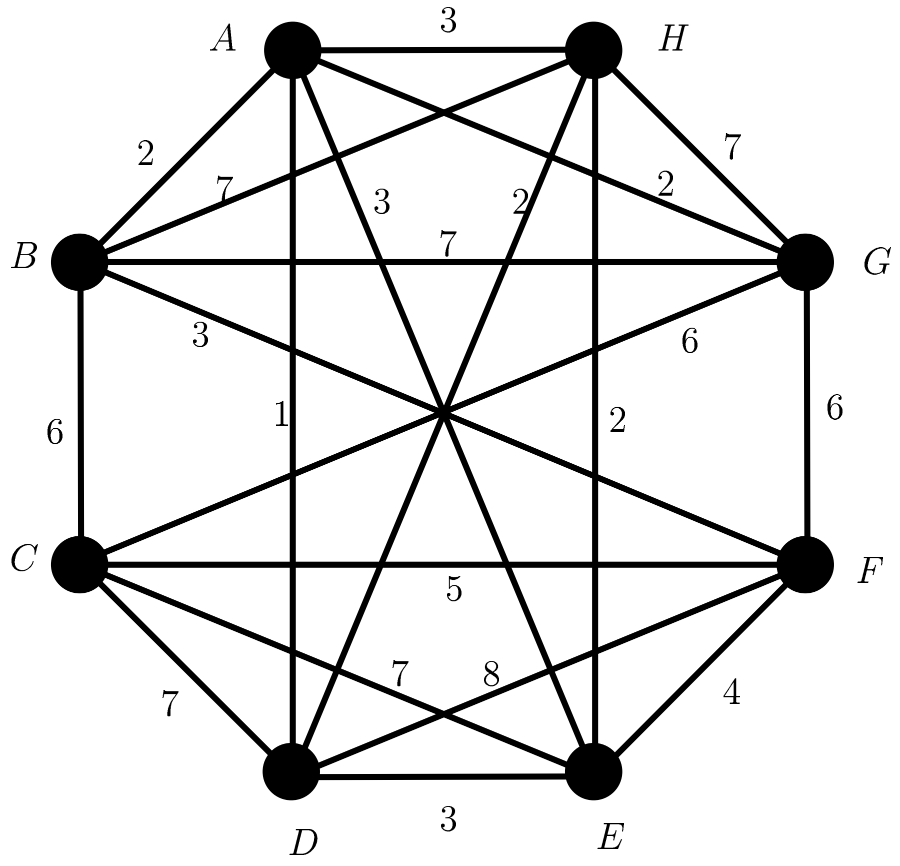
(ii) Show that G is connected.

(b) (i) Find all minimum weight spanning trees for the graph $G_7 - A$ obtained from the weighted graph G_7 below by removing vertex A.

(ii) Use the previous part of the question to find a lower bound for the Travelling Salesman Problem corresponding to the weighted graph G_7.

(iii) Explain why the solution to the Travelling Salesman Problem in G_7 must have weight greater that the lower bound found above.

(iv) Find a solution to the Travelling Salesman Problem in G_7.



G_7

[30 marks]

B8. (a) Let $P = u_0, \dots, u_k$ and $Q = v_0, \dots, v_k$ be open paths of length k in a connected graph G.

(i) Show that if P and Q have no vertex in common then there is a path R from a vertex u_r of P to a vertex v_s of Q, such that no vertex of R, except u_r and v_s, belongs to P or Q. Draw a diagram to illustrate this situation. What are the lengths (in terms of r and k) of the subpaths of P from u_0 to u_r and from u_r to u_k? Can both these subpaths have length less than $k/2$? Show that there is an open path in G of length greater than k.

(ii) Now suppose that P and Q are open paths of maximal length in G (i.e. G contains no path of length more than k). Show that P and Q have a vertex in common.

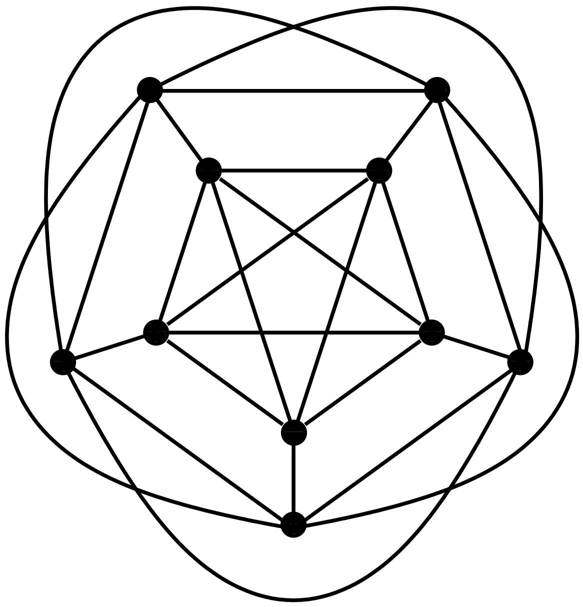
(b) (i) Define a k–**colouring** (of vertices) and the **chromatic number** $\chi(G)$ of a graph G without loops. State Brooke's Theorem for a simple connected graph G and a non–negative integer d such that $\deg(v) \leq d$, for all vertices v of G.

(ii) Using Brooke's Theorem give upper bound for the chromatic number $\chi(G_8)$ of the graph G_8 below. (Show your reasoning.)

(iii) Write down $\chi(G_8)$ and justify your claim.

(iv) Find a vertex colouring for G_8 using $\chi(G_8)$ colours. (You may use the sheet of diagrams.)

G_8:



[30 marks]