UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1
2006/7

MAS2203/3203

Graph theory

MOCK

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

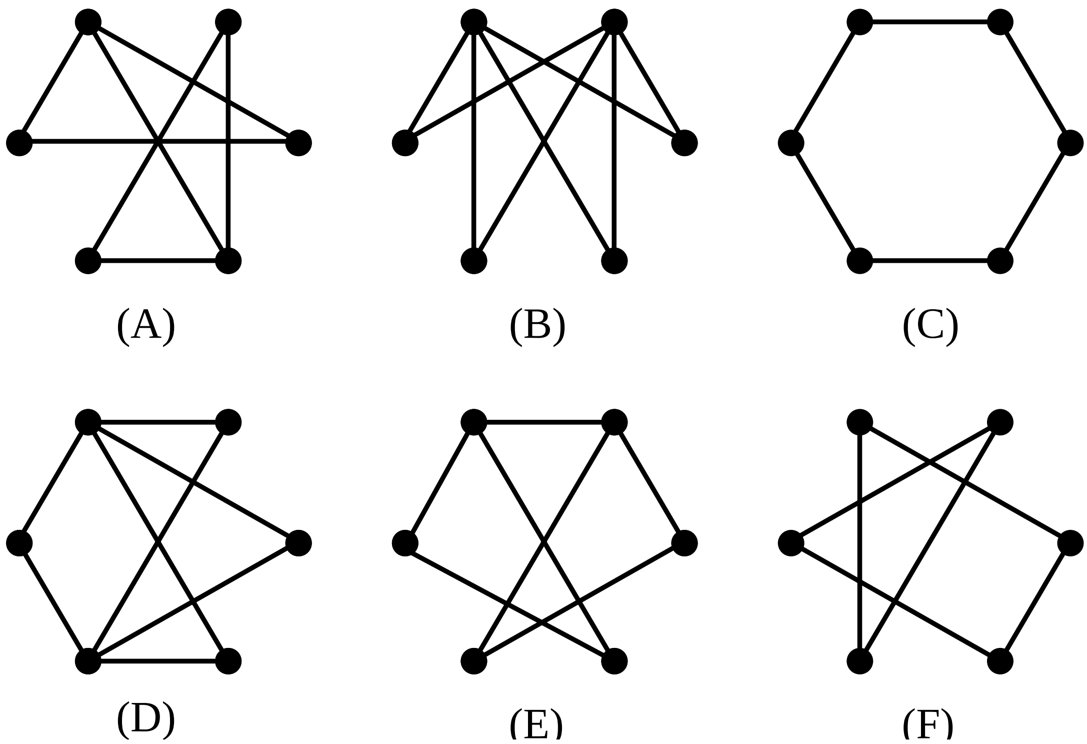
Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are SEVEN questions in Section A and THREE questions in Section B.

You may submit part of your answers to questions A1, A3 and A7 on the page of diagrams provided.

SECTION A

A1. State which of the following graphs are isomorphic to each other. Show these isomorphisms by labelling vertices. (You may give your answers to this part of the question on the sheet of diagrams provided.)



[6 marks]

A2. Define a *bipartite graph*. Draw the complete bipartite graph $K_{2,5}$. How many edges has the complete bipartite graph $K_{m,n}$, where m and n are positive integers?

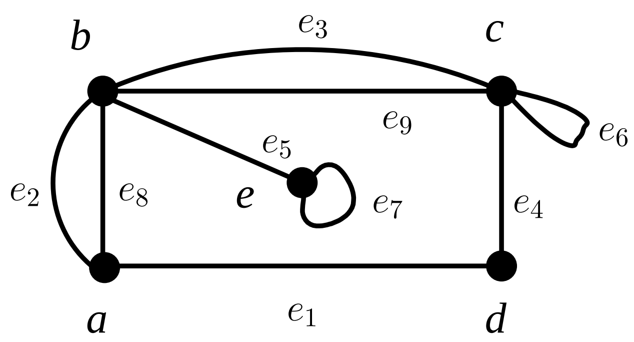
[6 marks]

A3. In the graph below, which has vertices a, b, c, d, e and edges $e_1, \cdots, e_9$, give an example of

(a) A walk which is not closed and is not a trail;

(b) A trail which is not closed and is not a path;

(c) A closed trail which has positive length and is not a cycle.

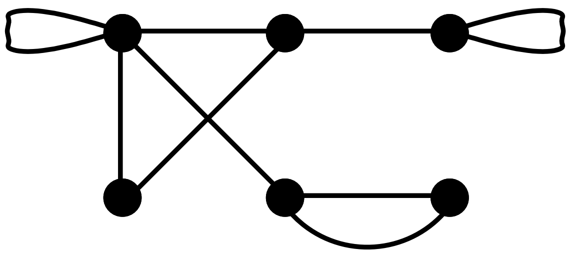


[6 marks]

A4. Define an *Eulerian circuit*. (You may assume the definition of a circuit.) State a result about connected graphs which characterizes Eulerian graphs in terms of degrees of vertices.

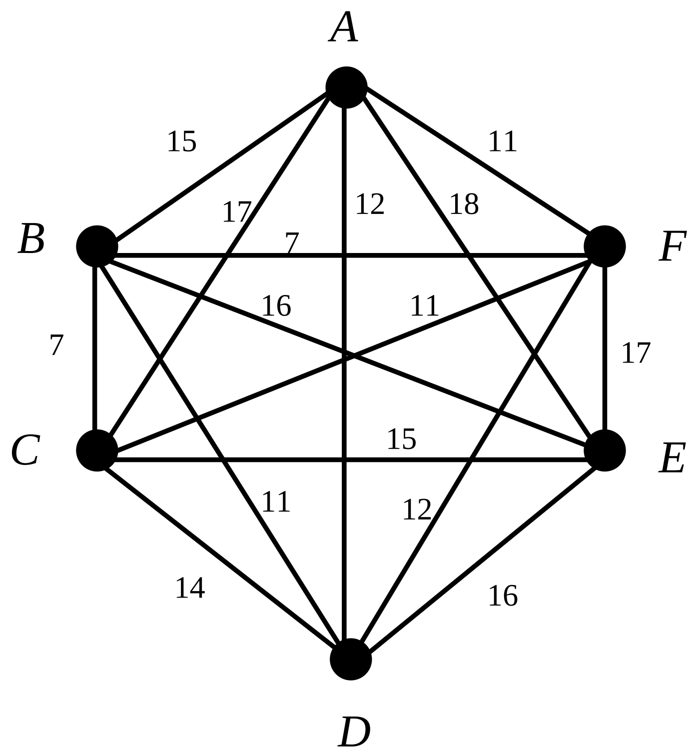
[4 marks]

A5. Draw all 6 spanning trees for the graph below.



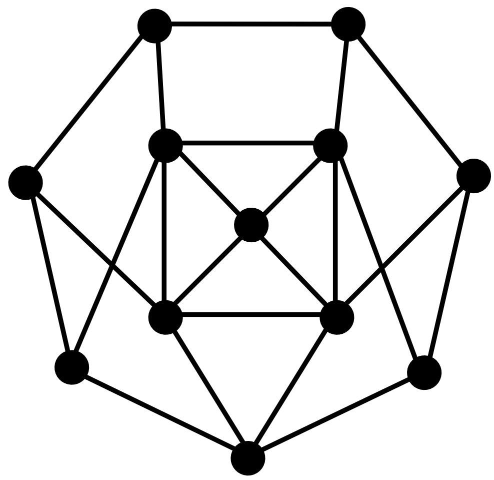
[6 marks]

A6. Find a lower bound for the travelling salesman problem corresponding to the weighted graph below, by removing vertex A.



[6 marks]

A7. Let G be a simple graph. Define a *k–colouring* of G and the *chromatic number* of G. Display a 3–colouring of the graph below. (You may give your answers to this part of the question on the sheet of diagrams provided.) Explain why this graph has chromatic number 3.



[6 marks]

SECTION B

B8. (a) Let n be an integer $n \geq 3$. Show that a simple graph G with $n-2$ vertices has at most $\frac{1}{2}(n^2-5n+6)$ edges.

(b) Let G be a simple connected graph with $n \geq 3$ vertices and let u and v be vertices of G which are not adjacent.

(i) Show that the graph G_1, formed from G by removing u and v and all their incident edges, has exactly

$$|E(G)| - (\deg(u) + \deg(v))$$

edges.

(ii) Use **??** and **??** to show that

$$\deg(u) + \deg(v) \geq |E(G)| - \frac{1}{2}(n^2 - 5n + 6).$$

(iii) Now suppose that

$$|E(G)| \geq \frac{1}{2}(n^2 - 3n + 6).$$

Show that $\deg(u)+ \deg(v) \geq n$.

(c) Conclude, after accurately stating an appropriate Theorem from the course, that if G is a simple connected graph with $n \geq 3$ vertices and $|E(G)| \geq \frac{1}{2}(n^2-3n+6)$, then G is Hamiltonian.

[**30 marks**]

B9. (a) (i) Let T be a tree. Show that addition of a new edge to T creates a cycle.

(ii) Let G be a graph and suppose that u and v are vertices of G lying in separate connected components. Show that there is no path from u to v in G.

(iii) Let G be a graph with no cycles such that the addition of any new edge to G creates a cycle. Show that G is a tree.

(b) In this part of the question you may use the fact that a tree with n vertices has $n-1$ edges.

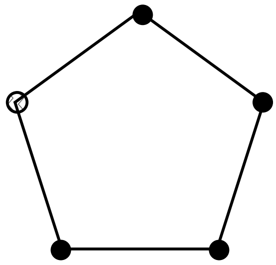
(i) Let F be a forest with k connected components and n vertices. Show that F has $n-k$ edges.

(ii) Let H be a graph with n vertices, $n-k$ edges and k connected components. Show that H is a forest.

[**30 marks**]

B10. A *plane triangulation* is a plane drawing of a graph D such that every face of D has degree 3.

(a) Draw a plane triangulation D which has the graph Q below as a subgraph. (Don't forget the exterior face.)



Q

(b) Let P be a plane graph.

(i) Let F be a face of P of degree $d > 3$. Show that by adding a new vertex inside F a new plane drawing can be made in which F is replaced by d faces of degree 3.

(ii) Explain how to construct a plane triangulation D which has P as a subgraph.

(iii) If P is a k–colourable prove that there exists a $k+1$–colourable plane triangulation D which has P as a subgraph.

(iv) Suppose P has a proper edge–colouring using K colours and that D is a plane triangulation which has P as a subgraph. Does D necessarily have a proper edge–colouring using K colours? If so prove it; if not give a counter–example.

(c) Let D be a plane triangulation with n vertices and m edges. Use Euler's Theorem to prove that $m = 3n - 6$.

[30 marks]