MAS164/CSC163 homework exercises. Discrete Mathematics for computer science

1. List the elements of each of the following sets. If the set is empty, say so.
   (a) $\{x \in \mathbb{Z} : -3 < x \leq 6\}$
   (b) $\{n \in \mathbb{N} : n < 10\}$
   (c) $\{x \in \mathbb{R} : x^2 = 4\}$
   (d) $\{x \in \mathbb{R} : x^2 = -5\}$

2. Where $A = \{a, b, c, d, e, f, g, h\}$, $B = \{a, d, e, g, i, j\}$, $C = \{a, b, g, h, i, k, l\}$, and $\mathcal{U} = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$ is the universal set, calculate
   (a) $A \cap B$
   (b) $A \cup C$
   (c) $A \setminus B$
   (d) $B \setminus A$
   (e) $A \cup (B \cap C)$
   (f) $(A \cup B) \cap (A \cup C)$
   (g) A^c.

3. For sets A,B and C, prove the following, using Venn diagrams:-
   (a) $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$
   (b) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

4. Where A={a,b,c,d,e},B={1,2,3},C={x,y}, $D = \{0\}$, write down
   (a) $A \times B$
   (b) $B \times A$
   (c) $B \times B$
   (d) $A \times B \times C$
   (e) $B \times C \times D$
   (f) $|A \times C|$, i.e. the number of elements in $A \times C$.
   (g) $|A \times B \times C \times D|$.

5. A university database stores information in records, each of which consists of a student number, a module number, and a percentage mark (the students mark in that course). Given that there are 500 students, 30 possible modules, and 101 marks (varying from 0 to 100 %) how many possible different records are there?

6. Write down the subset of $\mathbb{Z}$ for which the unary relation $SmallSquare$ holds, where $SmallSquare(x)$ is defined to hold whenever $x^2 < 100$.

7. Draw the directed graph of the following binary relations on $\{1, 2, 3, 4, 5\}$:-

   (a) $>$, (b) $\geq$, (c) $=$,
   (d) $SquareOf$ defined by $SquareOf(x, y)$ when $y = x^2$.

8. Write down all related triples for the ternary relation $PlusEquals$ on $\{1, 2, 3, 4\}$ defined by the rule that $PlusEquals(x, y, z)$ when $x+y = z$.

9. Sketch the directed graph of each of the following functions and describe its range. In each case, is the function injective? Is it surjective? Is it bijective? Where it is invertible, sketch the directed graph of its inverse.

   (a) $f_1 : \{1, 2, 3\} \to \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined by $x \mapsto x^2$.
   (b) $f_2 : \{-1, 0, 1, 2, 3\} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined by $x \mapsto x^2$.
   (c) $f_3 : \{-1, 0, 1, 2, 3\} \to \{0, 2, 8, 18\}$ defined by $x \mapsto 2x^2$.
   (d) $f_4 : \{0, 1, 2, 3\} \to \{0, 2, 8, 18\}$ defined by $x \mapsto 2x^2$.

10. Sketch the cartesian graph of each of the following functions and describe its range. In each case, is the function $1-1$? Does it map onto its codomain?

    (a) $f_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, defined by the rule $f_1(x) = 1/|x|$
    (b) $f_2 : \mathbb{R} \to \mathbb{R}$ defined by the rule $f_2(x) = x|x|$
    (c) $f_3 : \mathbb{R} \to \mathbb{R}$ defined by the rule $f_3(x) = x - [x]$. (where $[x]$ is defined to be the largest integer less than or equal to x.

11. Functions $f : \mathbb{R} \to [0, \infty)$, $g : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to [0, \infty)$ are defined by the rules

$$f(x) = x^4, \quad g(x) = \sqrt{1 + x^3}, \quad h(x) = \frac{x^2 + 1}{2x + 1}$$

For each of the following composite functions, write down its domain, its codomain, and the rule which defines it.

$$(a) \quad f \circ g \quad (b) \quad f \circ g \circ h \quad (c) \quad g \circ f \quad (d) \quad g \circ f \circ h$$

12. The function $max : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined by the rule:-

$$\begin{aligned} max(x, y) &= x \quad \text{if} \quad x > y \\ &= y \quad \text{otherwise} \end{aligned}$$

Explain how to find the maximum of (a) 4 integers and (b) 6 integers by applying max more than once.

13. Write down the domain for each of the following partial functions.

(a)

$$f : \mathbb{R} \not\to \mathbb{R}, \quad f(x) = \frac{1}{x^2 - 3x + 2},$$

where this rule makes sense.

(b)

$$f : \mathbb{R} \not\to \mathbb{R} \quad f(x) = \frac{1}{\sqrt{x}},$$

where this rule makes sense.

14. Write down the rule and the domain of the partial function from $\mathbb{R}$ to $\mathbb{R}$ which is the left inverse of the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by the rule $x \mapsto 3x + 1$.

15. Write down the 5 values $f(2), \ldots f(6)$ of the functions $f : \mathbb{N} \to \mathbb{N}$ defined recursively as follows:-

(a) $f(1) = 1, \quad f(n) = n + f(n-1),$
(b) $f(1) = 1, \quad f(n) = n(2n-1)f(n-1),$
(c) $f(1) = 1, \quad f(n) = 2(2n-1)f(n-1).$

16. Write down recursive definitions for the functions

(a) $f : \mathbb{N} \to \mathbb{N}$ defined by the rule $f(n) = 2^n n!$
(b) $f : \mathbb{N} \to \mathbb{N}$ defined by the rule $f(n) = (2n)!$

that is, for each specify $f(1)$ and define $f(n)$ in terms of the images under f of integers less than n.

17. Draw examples of the following

    (a) A tree with 10 vertices and 7 leaves.

    (b) A forest with 2 components and a total of 10 vertices and 7 leaves.

    (c) A connected bipartite graph on 6 vertices.

18. (a) Draw a complete graph K_4 on 4 vertices, labelling the vertices $1, 2, 3, 4$. Draw also 2 different spanning trees, one with 2 leaves and one with 3 leaves. Mark the leaves.

    (b) Draw a complete graph K_5 on 5 vertices, labelling the vertices $1, 2, 3, 4, 5$. Draw also 3 different spanning trees, one with 2 leaves, one with 3 leaves and one with 4 leaves. Mark the leaves.

    (c) How many vertices and how many edges are there in a spanning tree for the complete graph K_n on n vertices? Draw an example of a spanning tree for K_{10} with 4 leaves.

19. Sketch rooted binary trees with each of the following numbers of vertices:- (a) 3 (b) 7 (c) 15 (d) 17.

    In each case mark the root.

20. Draw parsing trees for each of the following arithmetic expressions:-

    (a)
    $$(x + y)^5 * (x - y)^6 + (x - y)$$

    (b)
    $$((x + y * z)^6 - (x * y - z))^{12}$$

    (c)
    $$(5 * x + y^{10}) - (x * y + y * z)^{11}$$

21. Prove by induction that

    $$1^2 + 2^2 + 3^2 + \dots n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

    for any positive integer n.

22. Prove by induction that

$$1^3 + 2^3 + 3^3 + \dots n^3 = \frac{n^2(n+1)^2}{4}$$

for any positive integer n.

23. Prove by induction that

$$1 + 3 + 3^2 + \dots 3^n = \frac{3^{n+1} - 1}{2}$$

for any positive integer n.

24. Prove by induction that

$$n^4 + 2n^3 - n^2 - 2n$$

is divisible by 4 for any positive integer n.

25. Prove by induction that

$$2^n \geq n^2$$

for all $n \geq 4$

26. Prove by induction that if T is a binary tree with n non-leaves, for $n > 1$, then T has $n + 1$ leaves.

27. Which of the following is the negation of the statement 'Some people like applied mathematics.'?

    (a) Some people dislike applied mathematics.
    (b) Everybody dislikes applied mathematics.
    (c) Some people like pure mathematics.
    (d) Everybody likes applied mathematics.

28. Simplify each of the following to equivalent expressions which use only $\vee$ and $\neg$ and are as simple as possible.

    (a) $(p \to (q \vee \neg r)) \wedge (\neg p \wedge q)$
    (b) $(\neg p \wedge \neg q) \wedge (\neg r \to p)$

29. Which of the following logical expressions is equivalent to $(p \vee q) \wedge (p \vee \neg q)$?

(a) $p \wedge q$

(b) $p \vee q$

(c) p

(d) q

30. Show that $((p \to q) \wedge (q \to r)) \leftrightarrow (p \to r)$ is not a tautology, but that $((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (p \leftrightarrow r)$ is.

31. A woman who was captured by a crowd of football enthusiasts was promised her freedom if she could determine with a single 'yes' or 'no' the colour of their team's shirts. She knew the colour was either black or white. Unfortunately the crowd contained only two kinds of individuals: liars who invariably gave the wrong answer to any question they were asked, and truth-tellers, who invariably gave the right answer. How did the 'victim' save herself?

32. For each of the following statements about the positive integers, supply an equivalent statement in English without using variables. Which of these statements are true and which are false? Justify your answers briefly.

(a) $\forall x \exists y (x = y - 1)$.

(b) $\forall x \forall y [\exists u (x * y = 2 * u - 1) \to \exists v (x = 2 * v - 1) \wedge \exists w (y = 2 * w - 1)]$.

(c) $\exists x (x > 10000000 \wedge \exists u (x = u * u))$.

33. Which of the following are Boolean algebras? For those which are identify a unity element, a zero element, and explain how to construct a complement for each element of the Boolean algebra. For those which are not, identify one of the Boolean algebra axioms which fails.

(a) The set $\{0, 1\}$ with 'addition', $+$, defined by

$$0 + 0 = 0,\ 1 + 0 = 1,\ 0 + 1 = 1,\ ,1 + 1 = 0.$$

and with 'multiplication', $*$, defined by

$$0 * 0 = 0,\ 1 * 0 = 0,\ 0 * 1 = 0,\ ,1 * 1 = 1.$$

(b) The set of subsets of $\{1, 2, 3\}$, with 'addition' defined by $\cup$, and 'multiplication' by $\cap$.

(c) The set of subsets of $\{1, 2, 3\}$, with 'addition' defined by $\cap$, and 'multiplication' by $\cup$.

(d) The integers, with 'multiplication', $\otimes$, and 'addition', $\oplus$, defined by

$$x \otimes y = xy + x$$

and

$$x \oplus y = xy - x$$

(where xy is the usual product of two integers, and $+$ and $-$ are the usual operations of addition and subtraction).

(e) The integers with the usual operations of multiplication and addition.

(f) The set of positive divisors of 12, with 'addition' defined by

$$x + y = lcm(x, y),$$

and 'multiplication' defined by

$$x * y = gcd(x, y)$$

(g) The rational numbers with 'multiplication', $\otimes$, and 'addition', $\oplus$, defined by

$$x \otimes y = x \div y$$

and

$$x \oplus y = x - y$$

(where $\div$ and $-$ are the usual operations of division and subtraction).

(h) The set of subsets of $\{1, 2, 3\}$, with 'addition', $\oplus$, defined by

$$A \oplus B = A \setminus B,$$

and 'multiplication' by $\cap$.

(i) The set of subsets of $\{1, 2, 3\}$, with 'addition', $\oplus$, defined by

$$A \oplus B = (A \setminus B) \cup (B \setminus A),$$

and 'multiplication' by $\cap$.

(j) The set of positive divisors of 15, with 'addition' defined by

$$x + y = gcd(x, y),$$

and 'multiplication' defined by

$$x * y = lcm(x, y).$$

34. Using only the five Boolean algebra axioms, the fact that complements are unique and the principle of duality, deduce the following properties for a Boolean algebra (in any order you like - it may not be most sensible to start with (a)) :-

(a) $x + 1 = 1$

(b) $1' = 0$

(c) $x + (x * y) = x$

(d) $x * x = x$

(e) $(x * y)' = x' + y'$

(f) $x + x = x$

(g) $x * 0 = 0$

(h) $(x + y)' = x' * y'$

(i) $0' = 1$

(j) $x * (x + y) = x$

(k) $(x')' = x$

35. $f(x_1, x_2, x_3, x_4) = \max(x_1, x_2, x_3x_4)$.

36. Find the disjunctive normal form in 3 variables for the Boolean expression

$$(x_1 + x_1x_2x_3 + x_1'x_1x_2 + x_3x_2')x_3.$$

37. Simplify the Boolean expression (i.e. get rid of the brackets)

$$[((a'b + c')(ca + b'))' + a'b'c']'.$$

38. Which of the following statements are true for elements x, y and z of a Boolean algebra? Justify your answers **briefly** in each case.

(a) $x + yz = (x + y)(x + z)$;

(b) if $x + 1 = y + 1$ then $x = y$;

(c) if $x + y = x + z$ and $xy = xz$ then $y = z$.

(d) if $xy = 0$ then $x = 0$ or $y = 0$.