6 Predicate logic

NB we Can use the IFAD tool here too!

We've already spent a little time studying propositional logic, manipulating formulae which consisted of variables representing propositions linked by the logical connectives $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftarrow$ and $\iff$. In propositional logic, we work from the principle that a proposition is either true or false, and we are interested in knowing which truth values of the propositional variables involved in a formula make the compound proposition it represents true, and which make it false.

But propositional logic can be a bit restrictive and doesn't allow very sophisticated arguments. We know that the truth value of a proposition may often depend on the value of a variable embedded in it. Sometimes we may want our logical arguments to take account of the variable, i.e. we may want to break the propositions in a formula down into their components, separating the variables involved from other parts of the proposition. Treating the proposition as a single entity doesn't allow very sophisticated arguments.

Consider for instance the following pair of statements. 'My cat is a siamese cat.' 'Every siamese cat has blue eyes.' Given that both these statements are true I can deduce a third statement 'My cat has blue eyes.' How? It's hard to explain. I have to look inside the statements and see how they are structured. The first statement is about one particular siamese cat (mine) and the second one about every siamese cat. In order to explain my argument I need the logical system I use to deal not with whole propositions, as propositional logic does, but also with some of the parts which make up the individual propositions, such as variables, constants, logical connectives and predicates. Predicate logic allows me to do this.

Remember a predicate is just a relation.

As examples of unary (one variabled) predicates we have $P =$'is a siamese cat', $Q =$'has blue eyes.' We can attach a variable x to define an open proposition $P(x)$, meaning 'x is a siamese cat', or $Q(x)$, meaning 'x has blue eyes. When x is set equal to the constant value my_cat, $P(x)$ becomes the proposition 'my_cat is a siamese cat', and $Q(x)$ the proposition my_cat has blue eyes. The fact that a constant value can be substituted for x in this waymeans that x is a <u>free variable</u> in each of $P(x), Q(x)$. (Later on we shall see examples of variables which are not free.)

The proposition 'My cat is a siamese cat' can also be reworded as 'My cat is in Siamese_cats' (where 'Siamese_cats' is defined to be the set of all siamese cats), and so can be expressed in terms of the binary predicate $\in$ as $my\_cat \in Siamese\_cats$.

So far we've managed to express the first and third statements in terms of the predicates $\in$ and Q, and constants my_cat and $Siamese\_cats$. The second statement can also be expressed in terms of the predicate Q. We could phrase it as 'For every siamese cat y, Q(y).' In fact we can write the whole statement using symbols, but we need to learn a new one first.

6.1 *The universal quantifier*

We use the symbol $\forall$ in predicate logic to mean 'for all', 'for every', 'for each'. $\forall x$ means 'for all x'. Often 'for all x' really means 'for all appropriate x', that is, for all x in some set of values which make sense in the current context. In that case we write $\forall x \in S$, where S is that appropriate set of values. the symbol $\forall$ is known as the universal quantifier.

We can write the statement 'For every siamese cat y, $Q(y)$', using the universal quantifier, as

$$\forall y \in Siamese\_cats\, Q(y)$$

Note that this statement is now a proposition, and not an open proposition. The variable y, which is free in $Q(y)$ is not free in this more complex statement, but is bound. It is no longer possible to substitute a constant value for y. We could change the name of the variable though. The statement

$$\forall x \in Siamese\_cats\, Q(x)$$

means exactly the same as the statment above.

In order to be able to manipulate formulae involving the universal quantifier, we need to know precisely when a formula involving it is true. Given a set of variables S and a unary predicate P, the proposition

$$\forall x \in S\, P(x)$$

is true precisely if the proposition $P(s)$ is true for every element s of S. Otherwise the proposition is false.

Similarly the proposition

$$\forall x\, P(x)$$

is true precisely if $P(x)$ is true for all possible values of x (whatever that means).

6.2 *The existential quantifier*

We use the symbol $\exists$ in predicate logic to mean 'there exists', 'there is', 'for some'. $\exists x$ means 'there exists an x'. We may use $\exists x$ or $\exists x \in S$, for some set S. Just like $\forall$, the existential quanitifier $\exists$ has the effect of turning a free variable into a bound variable.

Given a set of variables S and a unary predicate P, the proposition

$$\exists x \in S\, P(x)$$

is true precisely if the proposition $P(s)$ is true for at least one element s of S. The proposition is false if $P(s)$ is false for every element s of S.

Similarly the proposition

$$\exists x\, P(x)$$

is true precisely if $P(x)$ is true for at least one value of x.

It is perfectly possible to apply a (universal or existential) quantifier to a predicate involving several variables, or even to apply more than one quantifier. Each quantifier binds one variable (and of course two different quantifiers cannot simultaneously bind the same variable).

6.3 *Relationship between the universal and the existential quanitifiers.*

Notice that, for any unary predicate P, and for any set S, the following two identities hold.

$$\begin{aligned} \neg(\forall x \in S\, P(x)) &\equiv \exists x\, \neg P(x) \\ \neg(\exists x \in S\, P(x)) &\equiv \forall x\, \neg P(x) \end{aligned}$$

Examples 6.4 *of translation of statements from English into the predicate calculus.*

'The square of any integer is positive.' translates as

$$\forall x \in \mathbb{Z}\, x^2 > 0$$

'The sum of an odd integer and an even integer is always an odd integer'. translates as

$$\forall x \in OddInt\, \forall y \in EvenInt\, x + y \in OddInt$$

More examples.

The principle of mathematical induction uses the fact that following is a tautology:-

$$(P(1) \wedge (\forall n \in \mathbb{N} P(n) \to P(n+1))) \to (\forall n \in \mathbb{N} P(n)$$