4 Recursion and induction

We have already mentioned that some sets can be defined inductively.

- $\mathbb{Z}^+$ is defined as follows:-

  (1) $1 \in \mathbb{Z}^+$.

  (2) If $x \in \mathbb{Z}^+$ then $x + 1 \in \mathbb{Z}^+$.

  Nothing is in $\mathbb{N}$ which is not forced by either (1) or (2) to be in $\mathbb{Z}^+$.

- The set P of polynomials in x with real number coefficients is defined as follows:-

  (1) For any real number c, c is in P

  (2) If f is in P then so is xf, and so if $f + c$, for any real number c.

  Nothing is in P which is not forced by (1) or (2) to be in P.

- The set of $\mathcal{T}$ of binary trees can be defined as follows:-

  (1) The graph consisting of a single vertex is in $\mathcal{T}$.

  (2) If T is a binary tree and v is a leaf of T, then the graph formed from v by attaching two edges and two new vertices below v is in $\mathcal{T}$.

  Nothing is in $\mathcal{T}$ which is not forced by either (1) or (2) to be in $\mathcal{T}$.

What are the advantages of recognising that a set can be defined inductively?

- We can prove things about the set using the 'principle of mathematical induction', e.g. we can prove

  The sum of the first n positive integers (i.e. $1 + 2 + 3 + \ldots + n$) is

  $$\frac{n(n+1)}{2}.$$

  If T is a binary tree with n leaves then T has $2(n-1)$ edges.

- We can give recursive definitions for functions which have that set as their domain. Such definitions often give rise to very straightforward programming, e.g. look at the way we could program the factorial function on the positive integers.

How do we prove something using the principle of mathematical induction?

We split the argument into two pieces, the <u>base of the induction</u> and the <u>inductive step</u>. The base of the induction deals with the 'first' element of the set, defined by the first rule, and the inductive step deals with the second rule.

Example 4.1 *To prove that the sum of the first n positive integers (i.e.* $1 + 2 + 3 + \ldots + n$*) is*

$$\frac{n(n+1)}{2}$$

PROOF:

Base of the induction We prove the statement for $n = 1$. The sum of the first 1 positive integers is just the first positive integer, i.e. 1 itself. The formula $n(n+1)/2$ takes the value $1(1+1)/2 = 1$ when we put $n = 1$ into it. So the formula gives the correct answer when $n = 1$.

Hence the base of the induction is proved.

The inductive step Suppose that the sum of the first n positive integers is given by the formula

$$\frac{n(n+1)}{2}$$

To prove the inductive step we need to deduce from this that the sum of the first $n+1$ positive integers is given by the formula

$$\frac{(n+1)((n+1)+1)}{2}$$

or rather by

$$\frac{(n+1)(n+2)}{2}$$

Now the sum of the first $n+1$ positive integers is

$$1 + 2 + \ldots + n + (n+1) = (1 + 2 + \ldots + n) + (n+1)$$

and since we know the formula for the first n we can write this as

$$\frac{n(n+1)}{2} + (n+1)$$

When we tidy this up we get

$$(n+1)\left(\frac{n}{2} + 1\right) = (n+1)\frac{n+2}{2}$$

i.e. exactly the formula we were looking for.

Hence the inductive step is proved.

The principle of mathematical induction says that if we check just these two things we have proved that the formula holds for every positive integer. □

Example 4.2 *If T is a binary tree with n leaves then T has $2(n-1)$ edges.*

PROOF:

Base of the induction We need to prove the result for the binary tree with just one leaf, i.e. for the tree which is a single vertex. This tree has no edges. And this fits with the formula, since $2(1-1) = 0$.

So the base of the induction is proved.

The inductive step Now suppose that we know that the result holds for binary trees with n leaves. We want to deduce from this that the result holds for binary trees with $n+1$ leaves.

Now suppose that T' is a binary tree with $n + 1$ leaves. It was formed by attaching two edges and two leaves below a leaf v of a tree T with n leaves. v was a leaf of T but is not a leaf in T'. Every other leaf of T is also a leaf of T'. T' has two more edges than T. So, since T has $2(n - 1)$ edges, the number of edges in T' is

$$2(n-1) + 2 = 2n = 2((n+1) - 1).$$

So the inductive step is proved.

The principle of mathematical induction says that by checking just these two things we have proved that the formula holds for every binary tree. $\square$

More examples.

e.g. a set of size n has 2^n subsets.