UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1
Mock Exam

MAS121

Number Systems and the Foundations of Analysis

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FIVE questions in Section A and THREE questions in Section B.

SECTION A

A1. (a) Find the greatest common divisor of 1400 and 37730.

(b) Find integers x and y such that

$$1400x + 37730y = \gcd(1400, 37730).$$

(c) Which of the following equations have integer solutions? In each case either find integer solutions u and v or explain (briefly) why no solution exists.

(i) $1400u + 37730v = 210$;

(ii) $1400u + 37730v = 102$.

(d) Find the general solution for those equations in part (c) above which have a solution.

[20 marks]

A2. (a) Let a, b, c and d be integers such that $a|b$ and $c|d$. Prove that $ac|bd$.

(b) Show that

$$5n^2|(5n^2 + 3)^2 - 9,$$

for all $n \in \mathbb{Z}$.

[5 marks]

A3. Show that n^2 has the form $5k$, $5k + 1$ or $5k + 4$, with $k \in \mathbb{Z}$, for all integers n.

[10 marks]

A4. (a) Complete the table below for multiplication modulo 9 using only the integers $0, 1, 2, \ldots, 8$.

$\times$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1			
3	0								
4	0								
5									
6									
7									
8									

(b) Which integers have inverses modulo 9?

(c) Find all solutions to the congruence $3x \equiv 6 \pmod 9$.

[10 marks]

A5. Use Fermat's method of factorisation to find factors of 253, showing all your working. (You may use the fact that $15 < \sqrt{253} < 16$.)

[5 marks]

SECTION B

B6. (a) State the prime divisor property.

(b) Let p and q be prime numbers and let n be an integer $n \geq 2$. Suppose that d is a positive integer such that $q \nmid d$. For $i = 0, \ldots, q-1$, write $p + id = qa_i + r_i$, for some integers a_i and r_i, where $0 \leq r_i < q$.

(i) Show that $q \nmid (j - i)$, for all integers i, j with $0 \leq i < j \leq q - 1$.

(ii) Show that if $0 \leq i < j \leq q - 1$ then $r_i \neq r_j$.

(iii) Use part B6(b)(ii) of the question to show that $r_t = 0$, for some t with $0 \leq t \leq q - 1$. Use this to show that $q|p + td$.

(c) Let p be a prime number, let c be a positive integer and let n be an integer $n \geq 2$. Assume that all the numbers

$$p, p + c, p + 2c, \ldots, p + nc$$

are prime. Show that $n < p$. Now show that if q is a prime number and $q \leq n$ then $q \nmid p + ic$, for $i = 1, \ldots, n$. Combine this with part B6(b) of the question to show that if q is a prime number and $q \leq n$ then $q|c$.

[25 marks]

B7. The Fibonacci numbers are generated by the rules $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$, for $n \geq 2$. The first 9 Fibonacci numbers are therefore $1, 1, 2, 3, 5, 8, 13, 21, 34$.

(a) Show that if r and s are integers, not both zero, then $\gcd(r+s, s) = \gcd(r, s)$.

(b) Prove by induction that $\gcd(f_n, f_{n+1}) = 1$, for all $n \geq 1$.

(c) Show that, for all $n \geq 2$,

$$nf_{n+2} - f_{n+3} = (n+1)f_{n+2} - f_{n+4}.$$

(d) Show by induction that

$$f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 2,$$

for all $n \geq 1$.

[**25 marks**]

B8. (a) Let a be an integer with collected prime factorisation $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Write down the collected prime factorisation of a^2. Show that if b is an integer such that $a^2|b^2$ then $a|b$.

(b) Let a, b and c be natural numbers such that

$$a^2 + b^2 = c^2.$$

Show that if d is an integer such that $d|a$ and $d|b$ then $d|c$. Show that if $a \equiv 0 \pmod 3$ and $b \equiv 0 \pmod 3$ then $c \equiv 0 \pmod 3$.

(c) Show that if n is an integer then $n^2 \equiv 0$ or $1 \pmod 3$. Show also that if $n^2 \equiv 0 \pmod 3$ then $n \equiv 0 \pmod 3$.

(d) Let a, b and c be natural numbers such that a, b and c have no common divisor and

$$a^2 + b^2 = c^2.$$

Show that either

$$a \not\equiv 0 \pmod 3 \text{ and } b \equiv 0 \pmod 3$$

or

$$a \equiv 0 \pmod 3 \text{ and } b \not\equiv 0 \pmod 3.$$

[**25 marks**]