UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1
2002/2003

MAS221

Number Systems and the Foundations of Analysis

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FIVE questions in Section A and THREE questions in Section B.

SECTION A

A1. (a) Find the greatest common divisor of 1400 and 37730.

(b) Find integers x and y such that

$$1400x + 37730y = \gcd(1400, 37730).$$

(c) Find integers u and v such that

$$1400u + 37730v = 210.$$

[8 marks]

A2. (a) Let a, b, c and d be integers such that $a|b$ and $c|d$. Prove that $ac|bd$. Show further that $ac|(2ac + 5bd)$.

(b) Show that

$$5n^2|(5n^2 + 3)^2 - 9,$$

for all $n \in \mathbb{Z}$.

[8 marks]

A3. Show that n^2 has the form $5k$, $5k + 1$ or $5k + 4$, with $k \in \mathbb{Z}$, for all integers n.

[8 marks]

A4. (a) The prime factorisation of 107939 is $13 \cdot 19^2 \cdot 23$. Write down a list of positive divisors of 107939. You may give the prime factorisation of each divisor: i.e. write $13 \cdot 19$ instead of 247.

(b) Write down the collected prime factorisation of 3060 and 12285. Hence write down the collected prime factorisation of $\gcd(3060, 12285)$.

(c) Which of the following equations have integer solutions? Justify your claims but do not solve the equations.

(i) $3060x + 12285y = 95$;

(ii) $3060x + 12285y = 45$;

(iii) $3060x + 12285y = 135$.

[8 marks]

A5. (a) Complete the table below for multiplication modulo 9 using only the integers $0, 1, 2, \ldots, 8$.

$\times$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1			
3	0								
4	0								
5									
6									
7									
8									

(b) Which integers have inverses modulo 9?

(c) Find all solutions to the congruence $3x \equiv 6 \pmod 9$.

[8 marks]

SECTION B

B6. (a) State the prime divisor property.

(b) Let p and q be prime numbers and let n be an integer $n \geq 2$. Suppose that d is a positive integer such that $q \nmid d$. For $i = 0, \dots, q-1$, write $p + id = qa_i + r_i$, for some integers a_i and r_i, where $0 \leq r_i < q$.

(i) Show that $q \nmid (j - i)$, for all integers i, j with $0 \leq i < j \leq q - 1$.

(ii) Show that if $0 \leq i < j \leq q - 1$ then $r_i \neq r_j$.

(iii) Use part B6(b)(ii) of the question to show that $r_t = 0$, for some t with $0 \leq t \leq q - 1$. Use this to show that $q|p + td$.

(c) Let p be a prime number, let c be a positive integer and let n be an integer $n \geq 2$. Assume that all the numbers

$$p, p + c, p + 2c, \dots, p + nc$$

are prime. Show that $n < p$. Now show that if q is a prime number and $q \leq n$ then $q \nmid p + ic$, for $i = 1, \dots, n$. Combine this with part B6(b) of the question to show that if q is a prime number and $q \leq n$ then $q|c$.

(d) Let a and b be integers and assume that $a + sb = p$ is prime for some integer s. Set $t = s + kp$, where $k \in \mathbb{Z}$. Show that $a + tb$ is composite. Hence show that the sequence

$$a, a + b, a + 2b, a + 3b, \dots$$

contains infinitely many composite numbers.

[30 marks]

B7. The Fibonacci numbers are generated by the rules $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$, for $n \geq 2$. The first 9 Fibonacci numbers are therefore $1, 1, 2, 3, 5, 8, 13, 21, 34$.

(a) Show that if r and s are integers, not both zero, then $\gcd(r+s, s) = \gcd(r, s)$.

(b) Prove by induction that $\gcd(f_n, f_{n+1}) = 1$, for all $n \geq 1$.

(c) Show that, for all $n \geq 2$,

$$nf_{n+2} - f_{n+3} = (n+1)f_{n+2} - f_{n+4}.$$

(d) Show by induction that

$$f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 2,$$

for all $n \geq 1$.

(e) Show that $f_n^2 + f_n f_{n-1} = f_n f_{n+1}$, for all $n \geq 2$. Hence show that $f_n^2 = f_n f_{n+1} - f_n f_{n-1}$ and use this to show that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1},$$

for all $n \geq 1$.

[30 marks]

B8. (a) Let a be an integer with collected prime factorisation $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Write down the collected prime factorisation of a^2. Show that if b is an integer such that $a^2 | b^2$ then $a|b$.

(b) Let a, b and c be natural numbers such that

$$a^2 + b^2 = c^2.$$

Show that if d is an integer such that $d|a$ and $d|b$ then $d|c$. Show that if $a \equiv 0 \pmod 3$ and $b \equiv 0 \pmod 3$ then $c \equiv 0 \pmod 3$.

(c) Show that if n is an integer then $n^2 \equiv 0$ or $1 \pmod 3$. Show also that if $n^2 \equiv 0 \pmod 3$ then $n \equiv 0 \pmod 3$.

(d) Let a, b and c be natural numbers such that a, b and c have no common divisor and

$$a^2 + b^2 = c^2.$$

Show that either

$$a \not\equiv 0 \pmod 3 \text{ and } b \equiv 0 \pmod 3$$

or

$$a \equiv 0 \pmod 3 \text{ and } b \not\equiv 0 \pmod 3.$$

(e) Let s and t be natural numbers such that $\gcd(s,t) = 1$ and define

$$x = 2st,\ y = s^2 - t^2,\ z = s^2 + t^2.$$

Show that $x^2 + y^2 = z^2$. If p is an odd prime such that $p|x$ show that p does not divide y.

[30 marks]