Preamble

- Introduction to Pure Mathematics.
- Main theme – the notion of proof.
- The booklet contains blanks which you fill in during the lectures:
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- The notes and other course information can be found on the web at
  `www.mas.ncl.ac.uk/~najd2/teaching/mas1202/`
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A puzzle

Move forward to a time after the collapse of the banking system when we have returned to bartering. In the university 1 loaf of bread can be exchanged for 11 apples and a chocolate cake can be exchanged for 15 apples. A professor has baked a batch of cakes and a student turns out to have a dozen loaves of bread and hundreds of apples. The professor wants just one apple, so would like to exchange some cakes for one apple and some loaves of bread. Can this be done, and if so how?

Solution

1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22
23	24	25	26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63	64	65	66

Is there more than one solution? We can describe the problem algebraically. Let a, b and c and stand for the value of an **a**pple, a **c**ake and a loaf of **b**read, respectively. Then $c = 15a$ and $b = 11a$.

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1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22
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34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55
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More barterning

Now suppose that at bottle of French wine is worth 30 apples and a bottle of English wine is worth 24 apples. A lecturer has a crate of french wine and some apples and the professor now wants 6 apples, but only has a crate of English wine. Can a fair transaction be made so that the prof ends up with 6 apples?

We can describe the problem algebraically again. Let f and e stand for the values of **F**rench and **E**nglish wine, respectively. Solve to find whole numbers x and y which are both positive.

Solution

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	39	30
31	32	33	34	35
36	37	38	39	40

Equations with integer solutions

The crucial feature of these problems are that we are only interested in solutions which are natural numbers (defined in Section A.6).Solutions would be very easy to find if we allowed ourselves to use rational numbers or real numbers (see Section A.6).

On the other hand integer solutions are no easier to find than natural number solutions (integers are also defined in Section A.6).

This chapter looks into some of the properties of natural numbers and integers that, among other things, prove useful in solving problems such as the bartering ones above.We'll look at a a step by step recipe which would give us a number, like 6 in the second problem above, which can be used to simplify the problem and in fact determines whether or not there is a solution. We shall investigate, in some detail, how and why this works.

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Greatest Common Divisors

What is the biggest positive number that divides both 24 and 30?

Make two lists.

Positive divisors of 24 : 1, 2, 3, 4, 6, 8, 12, 24

Positive divisors of 30 : 1, 2, 3, 5, 6, 10, 15, 30

Pick the largest number which appears on both of the lists, which is 6.

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Example 1.1

Find the biggest number which divides both 2028 and 2600.

Positive divisors of

2028 : 1, 2, 3, 4, 6, 12, 13, 26, 39, 52, 78, 156, 169, 338, 507, 676, 1014, 2028

2600 : 1, 2, 4, 5, 8, 10, 13, 20, 25, 26, 40, 50, 52, 65, 100, 104, 130, 200, 260,325, 520, 650, 1300, 2600

The biggest number dividing both 2028 and 2600 is 52.

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The Euclidean Algorithm

The biggest natural number which divides both natural numbers a and b is called the **greatest common divisor** of a and b.

Given natural numbers a and b we wish to find their greatest common divisor.

The recipe works as follows.

EA1. Input the pair (b, a), with $0 < a < b$.

EA2. Write $b = aq + r$, where q and r are integers with $0 \leq r < a$.

EA3. If $r = 0$ then **output** $\gcd(a, b) = a$ and **stop**.

EA4. Replace the ordered pair (b, a) with (a, r).
Repeat from EA2.

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Example 1.2

Find the greatest common divisor d of 12 and 63. Find $x, y \in \mathbb{Z}$ such that $12x + 63y = d$.

Example 1.3

Find the greatest common divisor d of 2600 and 2028. Find integers x and y such that $d = 2600x + 2028y$.

We write out the results of Step EA2 as the algorithm runs:

$$(2600, 2028) \qquad 2600 = 2028 \cdot 1 + 572 \tag{1.1}$$

(1.2)

(1.3)

(1.4)

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This gives $\gcd(2600, 2028) = 52$, as we found in Example 1.1.

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(572, 312) \quad & 572 = 312 \cdot 1 + 260 && (1.3)\\
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(312,260) \quad & 312 = 260 \cdot 1 + 52 && (1.4)\\
(260,52) \quad & 260 = 52 \cdot 5 + 0. && (1.5)
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To find the integers x, y we work back from (1.4) to (1.1).

$$\begin{aligned}
52 &= 312 - 260 \cdot 1 && \text{from (1.4)} \\
&= 312 - (572 - 312 \cdot 1) = 312 \cdot 2 - 572 && \text{from (1.3)} \\
&= (2028 - 572 \cdot 3) \cdot 2 - 572 = 2028 \cdot 2 - 572 \cdot 7 && \text{from (1.2)} \\
&= 2028 \cdot 2 - (2600 - 2028 \cdot 1) \cdot 7 = 2028 \cdot 9 - 2600 \cdot 7 && \text{from (1.1).}
\end{aligned}$$

Thus $52 = 2600 \cdot (-7) + 2028 \cdot 9$ so we may take $x = -7$ and $y = 9$.

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Example 1.4

Find the greatest common divisor d of 2028 and 626. Find $x, y \in \mathbb{Z}$ such that $2028x - 626y = d$.

$$(2028, 626) \qquad 2028 = 626 \cdot 3 + 150 \tag{1.6}$$

(1.7)

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(1.9)

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This gives $\gcd(2028, 626) = 2$.

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$$(626, 150) \qquad 626 = 150 \cdot 4 + 26 \tag{1.7}$$

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(626, 150) \quad & 626 = 150 \cdot 4 + 26 && (1.7)\\
(150, 26) \quad & 150 = 26 \cdot 5 + 20 && (1.8)\\
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This gives $\gcd(2028, 626) = 2$.

Example 1.4

Find the greatest common divisor d of 2028 and 626. Find $x, y \in \mathbb{Z}$ such that $2028x - 626y = d$.

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Thus $2 = 2028 \cdot 96 - 626 \cdot 311$ so we may take $x = 96$ and $y = 311$.

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Divisibility in the integers

Definition 1.5

Let a and b be integers. If there exists an integer q such that $b = qa$ then we say that a **divides** b, or $a|b$,.

A definition establishes once and for all the meaning of a word. From now on whenever we say "divides" we mean what it says above, nothing more, nothing less.

Other ways of saying $a|b$ are that a is a **factor** of b, a is a **divisor** of b or b is a **multiple** of a.

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From the definition we can easily check that $6|18$ because $18 = 6 \cdot 3$.

In the same way we see that 6 divides $24, 12, 6, 0$ and -6.

Also it's "obvious" $7 \nmid 16$ and $-11 \nmid 32$: but these are not immediate consequences of the definition of division.

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The **modulus** or **absolute value** of a real number x is denoted $|x|$ and is given by the formula

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

The definition above is what is known as a definition by cases.

For example

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Theorem 1.10 (**The Division Algorithm)**

Let a and b be integers with $a \neq 0$. Then there exist unique integers q and r such that $b = aq + r$ and $0 \leq r < |a|$.

(1) The condition that $a \neq 0$ is necessary. If it's left out then the statement becomes untrue.

(2) There are two parts to the conclusion of the Theorem. Firstly it says that such q and r do exist. Secondly it says that q and r are unique.

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Every integer n can be written as $n = 2q + r$, with $0 \leq r < 2$.

If $r = 0$ we say n is **even** and if $r = 1$ we say n is **odd**.

We've used the Division Algorithm (Theorem 1.10) to partition of integers into odd and even.

Example 1.12

Here we have partitioned the integers into three:
those that leave remainder 0,
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on applying the Division Algorithm with $a = 3$.

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Properties of Division

Example 1.15

Consider the equation $112 = 20 \cdot 5 + 12$.

Why are the gcd's are both the same?

Lemma 1.16

Let s, t and u be integers, which are not all zero, such that

$$s = tq + u,$$

for some $q \in \mathbb{Z}$. Then $\gcd(s,t) = \gcd(t,u)$.

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Show that any integer that divides both s and t must also divide u.

Then show that any integer that divides both t and u must also divide s.

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We can write $337 = 11 \cdot 30 + 7$ so ...

Lemma 1.18

1. $n|n$, *for all integers* n.
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Step(5) **Conclude that the result holds.** It cannot happen that $m > n$ because this forces $n > n$, which is impossible. The conclusion is $m \leq n$.

Why the Euclidean Algorithm works

Example 1.19

Consider the Equations (1.6)–(1.11).

Stringing all these facts together we have

$$2 = \gcd(6, 2)$$

that is $\gcd(2028, 626) = 2$.

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$$\begin{aligned} 2 &= \gcd(6,2) \\ &= \gcd(20,6) \\ &= \gcd(26,20) \\ &= \gcd(150,26) \\ &= \gcd(626,150) \\ &= \gcd(2028,626), \end{aligned}$$

that is $\gcd(2028,626) = 2$.

Why the Euclidean Algorithm works

Example 1.19

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Example 1.20

Consider the Equations (1.1)–(1.5).

$$\gcd(2600, 2028) = \gcd(2028, 572), \text{using Equation (1.1)}$$

that is $\gcd(2600, 2028) = 52$.

Example 1.20

Consider the Equations (1.1)–(1.5).

$$\gcd(2600, 2028) = \gcd(2028, 572), \text{using Equation (1.1)}$$
$$\gcd(2028, 572) = \gcd(572, 312), \text{using Equation (1.2)}$$

that is $\gcd(2600, 2028) = 52$.

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$$\gcd(2600, 2028) = \gcd(2028, 572), \text{using Equation (1.1)}$$
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And another thing

Given two integers a and b we can work back through the output of the Euclidean algorithm, as we did in Examples 1.2, 1.3 and 1.4, to find integers x and y such that $ax + by = \gcd(a, b)$.

Theorem 1.21

Let a and b be integers, not both zero, and let $d = \gcd(a, b)$. Then there exist integers u and v such that $d = au + bv$.

The input to the Euclidean algorithm is a pair of positive integers. What if $a < 0$?

$\gcd(a, b) = \gcd(-a, b) = \gcd(-a, -b) = \gcd(a, -b)$ and from this it follows that the Theorem holds in all cases.

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An application

Example 1.22

Find integers x and y such that $2600x + 2028y = 104$.

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In Example 1.3 we ran the Euclidean Algorithm and found $\gcd(2600, 2028) = 52$.

Once we'd done so we were able to use the equations generated to find integers x and y such that

$$2600 \cdot (-7) + 2028 \cdot 9 = 52. \tag{1.12}$$

Example 1.23

Find integers x and y such that $-72 = 12378x - 3054y$.

First we run the Euclidean Algorithm to find $\gcd(12378, 3054)$.

$$(12378, 3054) \qquad 12378 = 3054 \cdot 4 + 162 \tag{1.13}$$

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$$6 = 12378u + 3054v.$$

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Thus

$$6 = 12378 \cdot 132 + 3054 \cdot (-535) \qquad (1.19)$$

and we may take $u = 132$ and $v = -535$.

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Existence of solutions

Lemma 1.24

Let a, b and c be integers (a, b not both zero). The equation

$$ax + by = c \tag{1.20}$$

has integer solutions x, y if and only if $\gcd(a, b) | c$.

If and only if

The phrase “if and only if” in this Lemma is an important part of the conclusion.

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The second statement is the converse of the first.

(More generally, the converse of “If A is true then B is true” is “If B is true then A is true”.)

To prove the lemma we must prove both statements because it is possible for the converse of a true statement to be false.

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The converse is "If I can swim then I am a frog", and this is commonly regarded as false.

More precise mathematical examples are not hard to find.

Synonyms

There are several different ways of saying things like “if ... then ...” and “... if and only if ...”.

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if A then B	$A \Rightarrow B$	B if A
if B then A	$A \Leftarrow B$	A if B
A if and only if B	$A \Leftrightarrow B$	A iff B

Example 1.25

Are there integers x and y such that $2600x + 2028y = 130$?

Example 1.26

For which c does the equation $72x + 49y = c$ have a solution?

$\gcd(72, 49) = 1$
so the equation $72x + 49y = c$ has a solution for every choice of c.

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Concluding remarks

Fix integers a and b and let $d = \gcd(a,b)$.

Lemma 1.24 tells us that the equation $ax + by = c$ has a solution if and only if $d|c$. So

1. there is a solution if $d = c$ and
2. there is no solution if $0 < c < d$.

Conclusion: d is the smallest positive integer that can be written in the form $ax + by$, with $x, y \in \mathbb{Z}$.

Now suppose that for our choice of a, b there exist integers u and v such that $au + bv = 1$.

e.g. take $a = 25132$ and $b = 15079$, then $3a - 5b = 1$.

What can we say about $\gcd(a,b)$ in this case?

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What can we say about $\gcd(a,b)$ in this case?

Concluding remarks

Fix integers a and b and let $d = \gcd(a, b)$.

Lemma 1.24 tells us that the equation $ax + by = c$ has a solution if and only if $d|c$. So

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Objectives

After covering this chapter of the course you should be able to:

(i) use terms such as Definition, Lemma, and proof with confidence;

(ii) read and understand simple proofs;

(iii) remember Definition 1.5 of *a* divides *b*, for integers *a* and *b*;

(iv) apply this definition to prove simple divisibility properties;

(v) state the Division Algorithm and be able to use it to demonstrate properties of integers;

(vi) remember the definition of greatest common divisor of two integers;

(vii) apply this definition to prove results;

(viii) apply the Euclidean algorithm and explain why it works;

(ix) find solutions to equations of the kind given above.

More Apples and Wine

The professor has been awarded a pay increase and decides to throw a party. He wants French wine for this party. Unfortunately in this department the pay is in bottles of English wine. Lecturers in Classics are paid in French wine and apples; so the professor wishes to trade English wine for apples and French wine.

The prof still wants to eat six apples, as it happens.

Can the professor buy sufficient wine to make a really memorable party?

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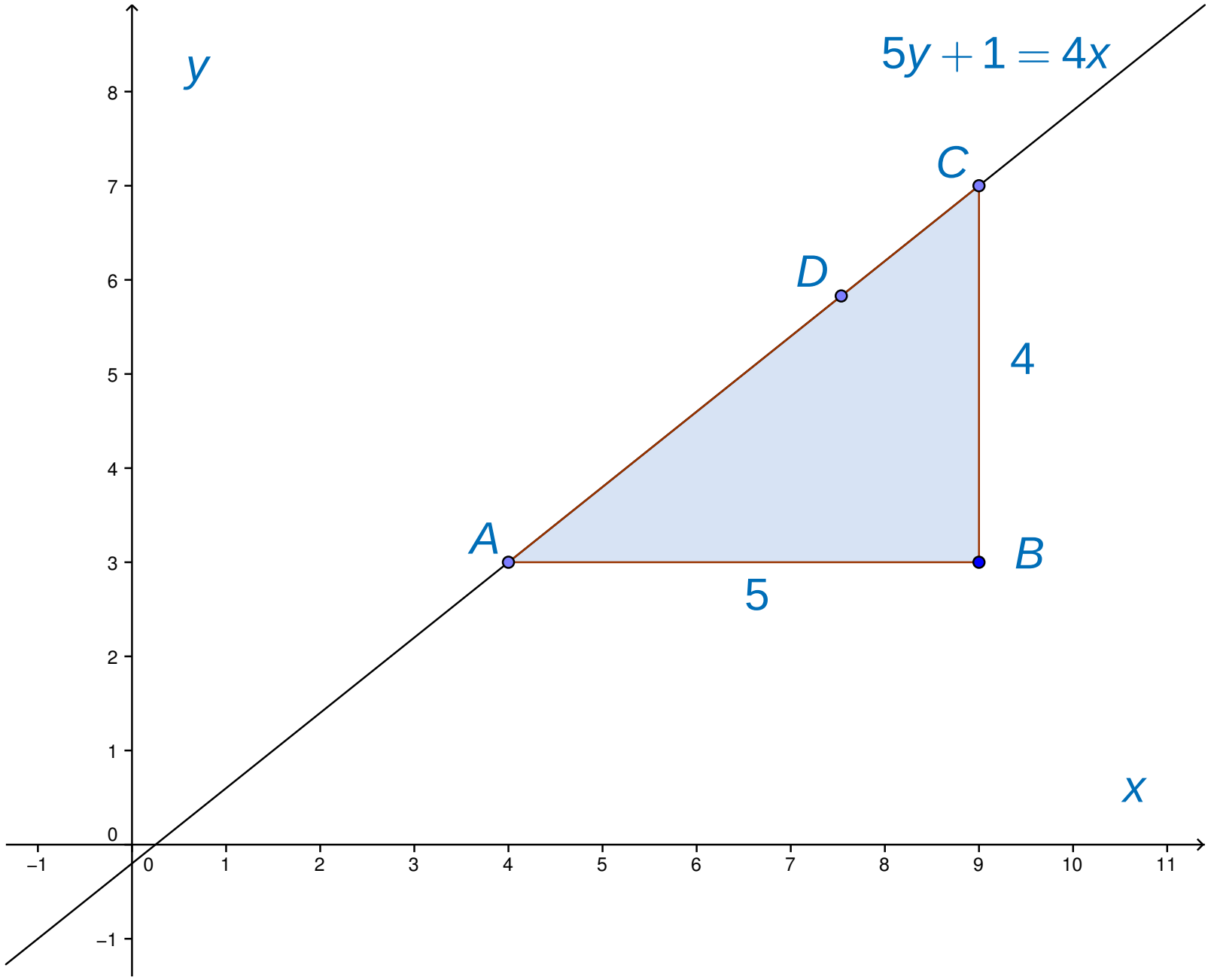
Solutions to a bartering problem



Section overview

In this section we'll develop enough of the theory of integers to enable us to write down a formula which tells us exactly which values of x and y are solutions to equations of this type for which we seek integer solutions (linear Diophantine equations). The main new idea we need is that of pairs of "coprime" numbers.

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Greatest common divisors again

The Euclidean Algorithm, run on natural numbers a and b, gives not only $\gcd(a,b)$ but also integers u and v such that

$$\gcd(a,b) = au + bv.$$

This gave us Theorem 1.21:

Let a and b be integers, not both zero, and let $d = \gcd(a,b)$. Then there exist integers u and v such that $d = au + bv$.

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Second proof of Theorem 1.21

Suppose that we have positive integers a and b.

Consider the set

$$S = \{ak + bl \in \mathbb{Z} : ak + bl > 0 \text{ and } k, l \in \mathbb{Z}\}.$$

This is a set of positive integers.

We shall prove the theorem by showing that it's smallest element is $\gcd(a, b)$.

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It is a fundamental property of numbers that every non-empty set of positive integers has a smallest element.

It's easy to see S is non-empty as it contains, for example $a + b$.

Therefore S has a smallest element, s say. Then

$$s = ak + bl, \text{ for some } k, l \in \mathbb{Z}. \tag{2.1}$$

Now, using the Division Algorithm, we can write

$$a = sq + r, \text{ where } 0 \leq r < s.$$

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$$\begin{aligned} a &= (ak + bl)q + r \\ &= a(kq) + b(lq) + r, \end{aligned}$$

so

$$r = a(1 - kq) + b(-lq), \text{ with } 0 \leq r < s.$$

If $r \neq 0$ then we have $r \in S$ and $r < s$, a contradiction.

Therefore $r = 0$ and $a = sq$. That is, $s|a$.

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Substitution in (2.1) gives

$$s = c(uk) + c(vl) = c(uk + vl).$$

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This completes the proof that $s = \gcd(a, b)$

and we've already found k, l such that $s = ak + bl$,

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Coprime integers

Definition 2.1

If a and b are integers with $\gcd(a,b) = 1$ then we say that a and b are **coprime**.

Example 2.2

6 and 35 are coprime and

$$6 \cdot 6 - 1 \cdot 35 = 1.$$

What about

$$11375 \cdot 3085622 - 7469451 \cdot 4699 = 1?$$

We have u and v such that $11375u + 7469451v = 1$.

Does this mean $\gcd(11375, 7469451) = 1$?

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> A corollary is something which follows easily from a previously proven fact.

Proof. This is an if and only if proof so has two halves.

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Assume that there are integers u and v such that $au + bv = 1$.

Let $d = \gcd(a, b)$.

We have $d = 1$, so a and b are coprime, as required.

Euclid's Lemma

Lemma 2.4

Let a, b and c be integers with $\gcd(a, b) = 1$. If $a|bc$ then $a|c$.

Application to solving equations

A Linear Diophantine Equation is one of the form

$$ax + by = c,$$

where a, b and c are integers and we seek integer solutions.

Lemma 1.24 states that such an equation has a solution if and only if $\gcd(a,b)|c$.

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Theorem 2.5

Let a, b, c be integers and let $d = \gcd(a, b)$. The equation

$$ax + by = c \tag{2.2}$$

has an integer solution if and only if $d \mid c$.

If $d \mid c$ then equation (2.2) has infinitely many solutions

and if $x = u_0$, $y = v_0$ is one solution then $x = u_1$, $y = v_1$ is a solution if and only if

$$u_1 = u_0 + (b/d)t$$

and

$$v_1 = v_0 - (a/d)t,$$

for some $t \in \mathbb{Z}$.

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Example 1.20 continued

Example 2.6

$\gcd(2600, 2028) = 52$ and the equation $2600x + 2028y = 104$ has a solution $x = -14, y = 18$.

As $2600/52 = 50$ and $2028/52 = 39$ the solutions to this equation are

$$x = -14 + 39t,\ y = 18 - 50t,\ \text{for } t \in \mathbb{Z}.$$

For each integer t we have a solution, some of which are shown below.

t	x	y
-2	-92	118
-1	-53	68
0	-14	18
1	25	-32
2	64	-82

Example 1.20 continued

Example 2.6

$\gcd(2600, 2028) = 52$ and the equation $2600x + 2028y = 104$ has a solution $x = -14, y = 18$.

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Example 2.7

Find all integer solutions to the equation $63x + 12y = 18$.

List all solutions with $x > -12$ and $y > 6$.

From Example 1.2 we have $\gcd(63, 12) = 3$ and as $3|18$ the equation has solutions.

In Example 1.2 we also found that $63 \cdot 1 + 12 \cdot (-5) = 3$.

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Example 2.8

Find the general form for integer solutions to the equation $12378x + 3054y = 42$.

Find all solutions x, y with $x > 0$ and $y > -2000$.

Find all solutions with $x > 0$ and $y > 0$.

In Example 1.23 we found that $\gcd(12378, 3054) = 6$ and since $6|42$ this equation has solutions.

In the given example we also found
$12378 \cdot 132 + 3054 \cdot (-535) = 6$.

Multiplying through by 7 gives
$12378 \cdot 132 \cdot 7 + 3054 \cdot (-535) \cdot 7 = 42$.

This gives a particular solution

$$x = 132 \cdot 7 = 924 \text{ and } y = (-535) \cdot 7 = -3745.$$

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For the general form of the solution, in this case we have $a/d = 12378/6 = 2063$ and $b/d = 3054/6 = 509$.

The general form of the solution is therefore

$$x = 924 + 509t \text{ and } y = -3745 - 2063t,$$

for $t \in \mathbb{Z}$.

(We can check this is correct: with $t = 1$ we verify that $12373 \cdot 1433 + 3054(-5808) = 42$.)

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For solutions with $x > 0$ we require $924 + 509t > 0$, that is $t > -924/509$.

As t is an integer we therefore require $t \geq -1$.

We have solutions with $y > -2000$ if and only if $-3745 - 2063t > -2000$

if and only if $3754 + 2063t < 2000$

if and only if $t < -1754/2063$

if and only if $t \leq -1$.

Therefore there is a unique solution with $x > 0$ and $y < -2000$, which we obtain by setting $t = -1$,

namely

$$x = 415, y = -1682.$$

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Objectives

After covering this chapter of the course you should be able to:

(i) recall Theorem 1.21 and understand its proof;

(ii) define a coprime pair of integers;

(iii) recall Corollary 2.3 and Euclid's Lemma and understand their proofs;

(iv) find the general form of the solution of a linear Diophantine equation in two variables.

A professor decides to reward his students by buying them drinks in the bar after lectures. Drinks are bought for those who answer questions correctly. The questions get harder throughout the lecture so the number of drinks per question increases. A correct answer to the 1st question merits 1 drink. The 2nd gets 3, the 3rd gets 5, the 4th gets 7 and so on. The prof needs a tray to carry all these drinks and discovers that a square tray is always the perfect shape to carry all the drinks. Why's that then?

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Listing the number of drinks bought:

Questions answered	Drinks bought	Tray size
1	1	1×1
2	$1+3=4$	2×2
3	$1+3+5=9$	3×3
4	$1+3+5+7=16$	4×4
5	$1+3+5+7+9=25$	5×5.

i.e. the sum of the first n positive odd numbers is n^2, at least for $n = 1, 2, 3, 4$ and 5.

Does this hold for all positive integers greater than n?

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i.e. the sum of the first n positive odd numbers is n^2, at least for $n = 1, 2, 3, 4$ and 5.
Does this hold for all positive integers greater than n?

To answer this question we can begin by finding the difference between: the sum of the first n positive odd numbers:

$$1+3+\cdots+(2n-1)$$

and the sum of the first $n+1$ positive odd numbers:

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which is clearly $2n+1$.

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Now find the difference between n^2 and $(n+1)^2$:

$$(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$$

again.
The difference between the nth and $(n+1)$th sums of odd integers is the same as the difference between n^2 and $(n+1)^2$.
This means that **if**

$$1 + 3 + \cdots + (2n-1) = n^2 \tag{3.1}$$

then the first $n+1$ sum to $(n+1)^2$ (because both sides of (3.1) are increased by $2n+1$): that is

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Therefore if (3.1) holds then (3.2) holds as well.

We know that (3.1) holds for $n = 5$so (3.2) holds for $n = 5$ as well.

This implies though that (3.1) holds for $n = 6$; so (3.2) holds for $n = 6$.

In turn this means (3.1) holds for $n = 7$; so (3.2) holds for $n = 7$ ... and so on. Continuing like this we can see that (3.1) holds for all positive integers n.

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This implies though that (3.1) holds for $n = 6$; so (3.2) holds for $n = 6$.
In turn this means (3.1) holds for $n = 7$; so (3.2) holds for $n = 7$ ... and so on. Continuing like this we can see that (3.1) holds for all positive integers n.

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We have used the following simple property of sets of positive numbers. Assume that $P(n)$ is a statement, for all $n \in \mathbb{N}$.
That is we have statements $P(1)$, $P(2)$, $P(3), \ldots$.
The **Principle of Induction** goes like this. Assume it can be shown

(1) that $P(1)$ is true and

(2) that if $P(k)$ is true then $P(k+1)$ is true, for $k \geq 1$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

A property, like the Principle of Induction, which we do not try to prove because we believe it is a law of nature is called an axiom.

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Example 3.1

Suppose that we wish to prove

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1},$$

for all $n\in\mathbb{N}$.

Here $P(n)$ is the statement

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

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Change of basis

It is possible to start induction at some point other than $n = 1$.
In this case we use the following version of the Principle of Induction.

Let $s \in \mathbb{Z}$. Assume that $P(n)$ is a statement, for all $n \geq s$.
Assume further

$(1')$ that $P(s)$ is true and

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Example 3.2 (Bernoulli's Inequality)

Prove that

$$(1+x)^n \geq 1+nx, \text{ for all } n \in \mathbb{Z}, n \geq 0, \text{ and for all } x \in \mathbb{R}, x > -1.$$

Example 3.3

Show that $2^n > n^3$, for all $n \geq 10$.

Note that $2^9 = 512 < 729 = 9^3$, so the result does not hold when $n = 9$.

In fact, for any positive integer t and sufficiently large n we have $2^n > n^t$. In our proof $t = 3$ and we show exactly what "sufficiently large" means in this case.

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Objectives

After covering this chapter of the course you should be able to:

(i) understand the principle of proof by induction;

(ii) carry out proof by induction, both starting with the integer 1 and starting with an integer other than 1;

Prime Numbers

It follows from the definition of division that every integer n is divisible by ± 1 and by $\pm n$.

Amongst the positive integers a special case is the integer 1 which has only one positive divisor, namely 1.

All other positive integers n have at least 2 positive divisors, 1 and n, and may have more.

Definition 4.1

A positive integer $p > 1$ is called a **prime** if the only positive divisors of p are 1 and p.

An integer greater than one which is not prime is called **composite**.

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For example $2, 5, 7, 11, 13, 17$ and 19 are prime

whilst the first few composite integers are:

4 which is divisible by 2

1 is neither prime nor composite.

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The prime divisor property

A fundamental property of prime numbers is the following.

Theorem 4.2

If p is a prime and $p|ab$ then $p|a$ or $p|b$.

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If $3|bc$ then either $3|b$ or $3|c$.

The same goes for 29: if $29|bc$ then $29|b$ or $29|c$.

This does not hold for all integers.
For instance $6|24$ and $24 = 8 \cdot 3$, so $6|8 \cdot 3$ but $6 \nmid 8$ and $6 \nmid 3$.

Once we've discussed prime factorisation it will be easy to see why this property doesn't hold for any composite integers.

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For example, if $3|abc$ then, from the Theorem either $3|ab$ or $3|c$.

If $3|ab$ then, from the Theorem again, $3|a$ or $3|b$.

Therefore, if $3|abc$ then $3|a$ or $3|b$ or $3|c$.

Corollary 4.4

If p is prime and $p|a_1 \cdots a_n$ then $p|a_i$, for some i.

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Inductive Step: Suppose that $p | a_1 \cdots a_{n+1}$.

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If $p|a$ the inductive hypothesis implies that $p|a_i$, for some i with $1 \leq i \leq n$.

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If $p|a$ the inductive hypothesis implies that $p|a_i$, for some i with $1 \leq i \leq n$.

If $p|b$ then $p|a_{n+1}$.

Hence $p|a_i$, for some i, as required.

Prime Factorisation

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We aim to show that every positive integer greater than one has a prime factorisation and that this prime factorisation is unique, up to the order in which the prime factors occur.
For instance

$$2 \cdot 5 \cdot 2 \cdot 7,$$
$$2 \cdot 7 \cdot 2 \cdot 5,$$
$$7 \cdot 2 \cdot 2 \cdot 5$$

are all prime factorisations of 140 but are regarded as the same because the number of 2's, 5's and 7's is the same in each.

Example 4.5

Write 7 and 21 as a products of primes:

$$7 \quad \text{and} \quad 3 \cdot 7$$

We consider these as products of primes of length one and two respectively.
We'll see that all positive numbers > 1 are products of primes – and no number has more than one factorisation as a product of primes (if we count correctly).

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The Fundamental Theorem of Arithmetic

Theorem 4.6

Every integer $n > 1$ is a product of one or more primes. This product is unique apart from the order in which the primes occur.

Proof.

Step(1) Prove that every $n > 1$ has a prime factorisation.

Step(2) Prove that prime factorisations are unique.

The Fundamental Theorem of Arithmetic

Theorem 4.6

Every integer $n > 1$ is a product of one or more primes. This product is unique apart from the order in which the primes occur.

Proof.

Step(1) Prove that every $n > 1$ has a prime factorisation.

Step(2) Prove that prime factorisations are unique.

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Rational numbers

Before continuing we shall pause to see that this theorem really did need proving: that it is not a universal truth that holds in all situations.

To begin with consider the rational numbers $\mathbb{Q}$.

We can factor 2 as

$$2 = 4 \cdot (1/2) = 8 \cdot (1/4) = \cdots = 2^n \cdot (1/2^{n-1}) = \cdots =$$

and in general as $2q \cdot (1/q)$, for any non-zero element $q \in \mathbb{Q}$.

Therefore there is no hope of anything like Theorem 4.6 holding in $\mathbb{Q}$.

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E-numbers

To see how the uniqueness part of the Theorem might fail: let E denote the set of all even integers:

$$E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}.$$

If we add two elements of E we obtain another element of E: if $2n$ and $2m$ are arbitrary elements of E then

$$2m + 2n = 2(m + n) \in E.$$

(The same is true of subtraction.)
If we multiply together two elements of E the result is an element of E:

$$2m \cdot 2n = 2(2mn) \in E.$$

We can therefore regard E as a number system, the E-number system, with operations of addition and multiplication.

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E-division

We shall can also define division.

Definition 4.7

If a and b are elements of E then we say that a E-divides b if $b = aq$, where q is an element of E. Write $a|_E b$ if a E-divides b.

With this definition $2|_E 8$ because $8 = 2 \cdot 4$ and $4 \in E$.
2 does not E-divide 6 because $6 = 2 \cdot 3$ and $3 \notin E$.
Also $2|_E 4$ and $4|_E 24$
but $2 \nmid_E 10$.
Also $2 \nmid_E 2$ and $4 \nmid_E 4$ as $2 = 2 \cdot 1$ and $4 = 4 \cdot 1$.
More generally $n \nmid_E n$, for all E-numbers n.

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E-primes

Now we can define E-prime numbers (but here we don't have to worry about 1 which is not an E-number, and no number divides itself).

Definition 4.8

A positive E-number n is called an E-prime if it has no E-divisor.

2 is E-prime, 4 is not, 6 is E-prime.
The first few E-primes are

$$2, 6, 10, 14, 18, 22, 26, 30.$$

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E-prime factorisation

The numbers 4, 8 and 12 have E-prime factorisations

$$4 = 2 \cdot 2, \quad 8 = 2 \cdot 2 \cdot 2 \text{ and } 12 = 2 \cdot 6.$$

In fact Theorem 4.6 can be adapted to show that every E-number has an E-prime factorisation.
However 60 has two prime factorisations

$$60 = 2 \cdot 30 \text{ and } 60 = 6 \cdot 10.$$

Therefore the uniqueness part of Theorem 4.6 does not extend to E-numbers.

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Collected prime factorisation

It is often convenient to write the prime factorisation of an integer with all like primes collected together, in ascending order, and with exponential notation.

For example we could write the prime factorisations of 140 and 2200 as

$$140 = 2^2 \cdot 5 \cdot 7 \text{ and}$$

We call this the **collected prime factorisation** of an integer n or say that we've written n in **standard form**.

From the Fundamental Theorem of Arithmetic it follows that collected prime factorisations are unique.

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Collected prime factorisation

Corollary 4.9

Let $n > 1$ be an integer. Then n may be written uniquely as

$$n = p_1^{a_1} \cdots p_k^{a_k},$$

where $k \geq 1$, $p_1 < \cdots < p_k$, p_i is prime and $a_i \geq 1$.

The square root of 2

If n is a positive integer and has collected prime factorisation $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

then $n^2 = (p_1^{\alpha_1} \cdots p_k^{\alpha_k})(p_1^{\alpha_1} \cdots p_k^{\alpha_k})$

An integer m is of the form n^2, for some integer n, if and only if every prime in the prime factorisation of m has even exponent.

Corollary 4.10

There is no rational number r such that $r^2 = 2$. That is $\sqrt{2} \notin \mathbb{Q}$.

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Primality testing

Is n prime?

Test it for divisibility by all prime numbers p such that $1 < p < n$.

Better to use the following lemma.

Lemma 4.11
An integer $n > 1$ is composite if and only if it has a prime divisor p such that $p \leq \sqrt{n}$.

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Example 4.12

To find all primes in the range 1 to 45:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43$$

is now a complete list of primes between 1 and 45.

This method of constructing lists of primes is known as the Sieve of Eratosthenes.

In fact it is still too inefficient to use in practice to determine if a large number is prime.

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A Theorem of Euclid

The following theorem appears in Book IX of the Elements, a mathematical textbook written by Euclid: a Greek mathematician who lived around 300 bc.

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Objectives

After covering this chapter of the course you should be able to:

(i) define prime and composite numbers;

(ii) recall the prime divisor property, Theorem 4.2, and understand its proof;

(iii) recall the Fundamental Theorem of Arithmetic, Theorem 4.6, and understand its proof;

(iv) recognise and write down the prime factorisation and standard form or collected prime factorisation of an integer;

(v) use the sieve of Eratosthenes;

(vi) recall the statement of Theorem 4.13 and understand its proof.

Casting Out Nines

This is a method of testing integers for divisibility by 9.

Procedure 5.1

Given a non–negative integer (written in base 10) repeat the following steps (in any order) until a number less than 9 is obtained.

1. *Cross out any digits that sum to 9 or a multiple of 9.*
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The result is the remainder of division of n by 9.

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Example 5.2

Cast out Nines from 215763401.

Example 5.3

Check the computation

$$215763401 \times 51422218 = 11095032642643428.$$

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The Telephone Number Trick

1. Write down your telephone number.
2. Write down your telephone number with digits reversed.
3. Subtract the smaller of these two numbers from the larger.
4. By casting out nines from the result decide whether or not it is divisible by 9.

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The “Odd & Even” Number System

Red, white and blue arithmetic

Congruence

In the Red, White and Blue number system we collected together all integers which left remainder 0, 1 or 2 after division by 3, and called them white, red or blue.

We saw that a and b are the same colour if and only if $3 | b - a$.

Generalising this from 3 to an arbitrary integer n leads us to the definition of congruence.

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Generalising this from 3 to an arbitrary integer n leads us to the definition of congruence.

Definition 5.4

Let n be a positive integer and let $a, b \in \mathbb{Z}$.

If $n|b-a$ then we say that a is **congruent** to b **modulo** n, and write

$$a \equiv b \pmod{n}.$$

For instance $17 \equiv 5 \pmod{12}$ and $216 \equiv 6 \pmod{7}$.

As in the case $n = 3$ above, $a \equiv b \pmod{n}$ if and only if a and b both leave the same remainder after division by n.

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If $n|b-a$ then we say that a is **congruent** to b **modulo** n, and write

$$a \equiv b \pmod{n}.$$

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As in the case $n = 3$ above, $a \equiv b \pmod{n}$ if and only if a and b both leave the same remainder after division by n.

In fact, if

$$a = nq + r \text{ and } b = np + r, \text{ where } 0 \leq r < n \qquad (5.1)$$

then

$$b - a = n(p - q),$$

so $n|b - a$: that is $a \equiv b \pmod{n}$.

On the other hand if $a \equiv b \pmod{n}$ then $n|b - a$ so $b - a = np$, for some p.

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Example 5.5

Congruence modulo 2 gives rise to the Odd and Even number system.

Example 5.6

Congruence modulo 3 gives rise to the Red, White and Blue number system.

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Example 5.7

Suppose $n = 10$.

Then $0 \equiv 10 \pmod{10}$, $10 \equiv 101090 \pmod{10}$, $11 \equiv 121 \pmod{10}$ and $27 \equiv 253427 \pmod{10}$.

Every positive integer is congruent to its last digit (written to base 10).

In particular integers congruent to 0 all end in the digit 0.

These are exactly the integers divisible by 10.

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Congruence is not the same as equality but it does share some of the properties of equality.

If we have any integers a, b and c and n is a positive integer

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Let's check them for congruence.

The first one is easy since $n|0 = a - a$, for all integers a.
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Modular arithmetic

Arithmetic with congruences is called modular arithmetic.

We've already seen a couple of examples: Odd & Even arithmetic and Red, White and Blue arithmetic.

The idea is to add and multiply integers in the usual way but to regard two numbers as the same if they are congruent.

There is a possible problem with this. Suppose we work modulo 10, that is $n = 10$.

Now take two integers which are congruent modulo 10, say 23 and 3. We are to regard these as the same.

This means that if we do something to one, say add 6, then we should get the same answer as if we add 6 to the other.

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Let's see:

$$23+6=29 \quad \text{and} \quad 3+6=9.$$

This is alright because $29 \equiv 9 \pmod{10}$ and so we regard 29 and 9 as the same.

Does this always work? The purpose of the next Lemma is to reassure us that it does.

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Modular arithmetic is consistent

Lemma 5.8

Let n be a positive integer. Suppose that a, b, u and v are integers such that

$$a \equiv u \pmod{n}$$

and

$$b \equiv v \pmod{n}.$$

Then

(i) $-a \equiv -u \pmod{n}$;

(ii) $a + b \equiv u + v \pmod{n}$ *and*

(iii) $ab \equiv uv \pmod{n}$.

We prove parts (i) and (iii) here, leaving part (ii) as an exercise.

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Lemma 5.9

Every integer is congruent modulo n to one and only one of the integers in the list $0, 1, \ldots, n-1$.

Proof.

This follows from the division algorithm because if $a \in \mathbb{Z}$ then we can write $a = nq + r$, with $0 \leq r < n$.

Then $n|a-r$ so $a \equiv r \pmod{n}$ and r is in the given list.

If $a \equiv r \pmod{n}$ and $a \equiv s \pmod{n}$ then, from the above, $r \equiv s$ with $0 \leq r < n$ and $0 \leq s < n$.

Assuming that $r > s$ then $n|r-s$ and $n > r \geq r-s$, contradicting Lemma 1.18.3.

Thus a is congruent to only one integer in the list. □

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Example 5.10

In modular arithmetic we can always avoid computation with large numbers.

For example working modulo 10 we have

$$74598987903520453 24 \equiv 4 \pmod{10}$$

and

$$9874558754423 \equiv 3 \pmod{10}.$$

Therefore

$$74598987903520453 24 \cdot 9874558754423 \equiv 4 \cdot 3 = 12 \equiv 2 \pmod{10}.$$

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Similarly, working modulo 7 we have

$$4543362 \equiv 5 \pmod 7.$$

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Divisibility by 9

When we write a number like 20195 to base 10 we are expressing the number

$$2 \times 10^4 + 0 \times 10^3 + 1 \times 10^2 + 9 \times 10^1 + 5$$

in shorthand (there's a 1 in the 100's column etc.).

Applying this argument in general we write

$$a_m a_{m-1} \cdots a_1 a_0$$

for the number

$$a_m \times 10^m + a_{m-1} \times 10^{m-1} + \cdots + a_1 \times 10 + a_0.$$

As $10^k \equiv 1 \pmod 9$, for $k = 1, \ldots, m$, we have

$$a_m a_{m-1} \cdots a_1 a_0 \equiv a_m + a_{m-1} + \cdots + a_1 + a_0 \pmod 9. \quad (5.2)$$

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As $10^k \equiv 1 \pmod 9$, for $k = 1, \ldots, m$, we have

$$a_m a_{m-1} \cdots a_1 a_0 \equiv a_m + a_{m-1} + \cdots + a_1 + a_0 \pmod 9. \quad (5.2)$$

Divisibility by 9

When we write a number like 20195 to base 10 we are expressing the number

$$2 \times 10^4 + 0 \times 10^3 + 1 \times 10^2 + 9 \times 10^1 + 5$$

in shorthand (there's a 1 in the 100's column etc.).

Applying this argument in general we write

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Casting out nines again

Suppose we cast out nines (Procedure 5.1) from an integer m.

In Step 1 we cross out any digits which sum to a multiple of 9.

The sum of these digits is congruent to zero modulo 9 so, from (5.2), the result is an integer congruent to m modulo 9.

In Step 2 we add the digits and again, from (5.2), the result is an integer congruent to m modulo 9.

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In Step 2 we add the digits and again, from (5.2), the result is an integer congruent to m modulo 9.

Thus the casting out nines procedure results at every stage in an integer congruent to m modulo 9.

The procedure ends with a number r such that $0 \leq r < 9$ and $r \equiv m \pmod 9$.

As $9|m-r$, from which it follows that $m = 9q + r$, for some $q \in \mathbb{Z}$ and $0 \leq r < 9$.

That is, the output from Casting out Nines is the unique remainder (guaranteed by the division algorithm) on attempting division of m by 9.

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Divisibility by 9

The following lemma follows from (5.2).

Lemma 5.11

An integer is divisible by 9 if and only if the sum of its digits is divisible by 9.

Example 5.12

Are either of 215763401 or 215743401 divisible by 9?

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Now $10^2 \equiv 0 \pmod 4$. Thus, for example,

$$1932526 = (19325 \times 100) + 26 \equiv 26 \pmod 4$$

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More generally, if $a_m \cdots a_1 a_0$ is an integer written to base 10

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$$a_m \cdots a_1 a_0 = (a_m \cdots a_2 \times 100) + a_1 a_0 \equiv a_1 a_0 \pmod 4.$$

Therefore

$$a_m \cdots a_1 a_0 \equiv 0 \pmod 4 \quad \text{if and only if} \quad a_1 a_0 \equiv 0 \pmod 4.$$

That is

$$4 \mid a_m \cdots a_1 a_0 \Leftrightarrow 4 \mid a_1 a_0.$$

Example 5.13

Does 4 divide 937475900345 or 80345003732?

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Inverses in modular arithmetic

If we work in the rational numbers $\mathbb{Q}$ we can find a multiplicative inverse for any non-zero element.

For example the inverse of $11/201$ is $201/11$.

The same is true in $\mathbb{R}$ where the inverse of $x \neq 0$ is $1/x$.

In general if x is a number and y has the property that $xy = 1$ then we say that x has **inverse** y.

Most elements of $\mathbb{Z}$ don't have inverses in $\mathbb{Z}$. For example 2 has no inverse.

In fact ± 1 are the only elements of $\mathbb{Z}$ which have inverses. What about arithmetic modulo n.

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Inverses modulo n

Example 5.14

Try to find the inverse of 2 modulo 6.

Example 5.15

Do either 3 or 7 have inverses modulo 10?

Example 5.16

Which numbers have inverses modulo 8?

Inverses modulo n

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Lemma 5.17

An integer a has an inverse modulo n if and only if

$$\gcd(a,n) = 1.$$

What happens if we do arithmetic modulo a prime number p?

In this case, for every integer a either

1. $p \nmid a$ in which case $\gcd(a,p) = 1$ or
2. $p|a$ in which case $a \equiv 0 \pmod{p}$.

Thus every integer which is not congruent to zero modulo p has an inverse.

In this way arithmetic modulo p resembles arithmetic in $\mathbb{Q}$ more closely that arithmetic in $\mathbb{Z}$.

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Example 5.18

Write out the multiplication table for arithmetic modulo 5 with the integers $0, 1, 2, 3$ and 4.

Hence find the inverse of every integer which is not congruent to zero modulo 5.

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Solving Congruences

Example 5.19

Find all integers x such that

$$6x \equiv 4 \pmod{8}. \tag{5.3}$$

We call such equations **congruences** and this is an example of a **linear** congruence.

If $x = a$ is a solution and $a \equiv b$ then $x = b$ is also a solution: so if there's one solution there are infinitely many.

Every integer is congruent to one of

$$0, 1, \ldots, n-1 \text{ modulo } n$$

so we seek solutions to congruences in this range.

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Exhaustive search

x	0	1	2	3	4	5	6	7
$6x \pmod 8$								

From the table we see that the only solutions are $x = 2$ and $x = 6$.

Cancellation does not always work when solving congruences.

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Cancellation does not always work when solving congruences.

A method of solution

$$ax \equiv b \pmod{n} \tag{5.4}$$

x is a solution to (5.4) if and only if $n|(ax-b)$

if and only if $ax - b = ny$, for some integer y

if and only if $ax - ny = b$, for some $y \in \mathbb{Z}$.

From Theorem 2.5 this has a solution if and only if $\gcd(a,n)|b$.

Therefore, if $d = \gcd(a,n)$ then the congruence $ax \equiv b \pmod{n}$ has solutions if and only if $d|b$.

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Finding all solutions to $ax \equiv b \pmod{n}$

If $d = \gcd(a,n) | b$ then we can use the Euclidean algorithm to find a particular solution to the equation

$$ax - ny = b. \qquad (*)$$

If $d|b$ and $x = u$, $y = v$ is a solution to the equation $(*)$

then the general solution is

$$x = u - (n/d)t$$

and

$$y = v - (a/d)t,$$

for $t \in \mathbb{Z}$.

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where t runs through the integers $\mathbb{Z}$.

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If $d = \gcd(a,n) | b$ then we can use the Euclidean algorithm to find a particular solution to the equation

$$ax - ny = b. \qquad (*)$$

If $d|b$ and $x = u$, $y = v$ is a solution to the equation $(*)$

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How many of the solutions to congruence (5.4) which we have found are congruent?

If $d|b$ and $x = u$ is one solution to the congruence (5.4)

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Summary

Theorem 5.20

Let a, b and n be integers with $n > 0$ and let $d = \gcd(a, n)$.

Then the congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d \mid b$.

If $d \mid b$ then there are exactly d pairwise incongruent solutions.

Example 5.21

Find all solutions to the congruence

$$2x \equiv 3 \pmod{6}.$$

Example 5.22

Find all solutions to the congruence $6x \equiv 9 \pmod{15}$.

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Cancellation again

Example 5.23

Compare the solutions to the congruences

$$2x \equiv 4 \pmod 6 \text{ and } x \equiv 2 \pmod 6.$$

Random numbers: an application

In situations where we require random numbers we often wish to give a machine the task of generating these numbers.

In many cases we'd also like the machine to be able to reproduce the sequence of random numbers that it outputs so that we can verify our results.

Such sequences cannot be truly random and are called **pseudo-random**.

Pseudo-random numbers are often generated by computer but this means that we need to find good algorithms to produce them.

The art and science of random number generation is highly developed and very sophisticated. You can see this by looking at the web page Random number generators – The pLab Project Home Page at http://random.mat.sbg.ac.at/.

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D.H. Lehmer's method (1949)

To generate a sequence of "random looking" integers

$$a_0, a_1, a_2, \ldots$$

use the following process.

1. Fix a positive number n and two integers m and c, with $2 \leq m < n$ and $0 \leq c < n$.
2. Choose a start value a_0, such that $0 \leq a_0 \leq n$.
3. Generate elements of the sequence successively using the formula

$$a_{k+1} = ma_k + c \pmod{n}, \quad \text{where} \quad 0 \leq a_{k+1} < n.$$

If a large value of n is chosen the sequence appears random, at least to start with.

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Example 5.24

With $n = 800$, $m = 71$, $c = 57$, and $a_0 = 2$ the first ten elements of the sequence are

$$2, 199, 586, 63, 530, 87, 634, 271, 98, 615.$$

Now altering a_0 to 551 the sequence produced is

$$551, 778, 95, 402, 599, 186, 463, 130, 487, 234.$$

Keeping everything fixed except $n = 8000$ we obtain

$$551, 7178, 5695, 4402, 599, 2586, 7663, 130, 1287, 3434.$$

With $n = 40$, $m = 22$, $c = 20$ and $a_0 = 13$ we obtain

$$13, 26, 32, 4, 28, 36, 12, 4, 28, 36, 12.$$

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Of course such sequences are not random (by definition) and we have a formula for the terms.

Theorem 5.25

The kth term of the sequence generated by the process above is

$$a_k = \left(m^k a_0 + \frac{c(m^k - 1)}{(m-1)} \right) \pmod{n},$$

with $0 \leq a_k < n$.

Analysis of "how random" a pseudo-random sequence is involves applying statistical tests to the sequence.
For instance the frequency of occurence of a particular integers in the sequence can be tested;
as can the frequency of occurence of pairs of integers.

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Objectives

After covering this chapter of the course you should be able to:

(i) recall the definition of congruence;

(ii) recall the statement of Lemma 5.8 and understand its proof;

(iii) do arithmetic modulo n;

(iv) understand how various divisibility tests work and be able to apply them;

(v) decide whether or not an integer has an inverse modulo n;

(vi) generate a sequence of random looking numbers.