MAS1002 Optimisation and Linear Methods: Problem Solutions

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1 Functions of One Variable

1.1 (a)

$$C(x) = \frac{1}{4}(x^2 - 3x) + 3.$$

The total amount of drug required is $D = 3$ so the amount x produced by process A must satisfy $0 \le x \le 3$.

$$C'(x) = \frac{x}{2} - \frac{3}{4},$$

so $C'(x) = 0$ if and only if $x = 3/2$. Therefore C has a unique stationary point at $x = 3/2$, which lies in the interval $[0, 3]$.

$$\begin{aligned} C(3/2) &= 2\frac{7}{16} \\ C(0) &= 3 \\ C(3) &= 3. \end{aligned}$$

Therefore C has global minimum at $x = 3/2$. That is $3/2$ units of zappo should be produced using process A and $3 - 3/2 = 3/2$ by process B.

(b) With $D = 1$ we have $C(x) = \frac{1}{4}(x^2 - 3x) + 1$ everything is the same as before except that now we require $0 \le x \le 1$. This means that there is no stationary point of C in the interval under consideration, that is $[0, 1]$. Therefore the global minimum must occur at an end point of $[0, 1]$. As $C(0) = 1$ and $C(1) = 1/2$ the global minimum is $x = 1$. Therefore only process A should be used.

(c) In this case the problem is to minimise

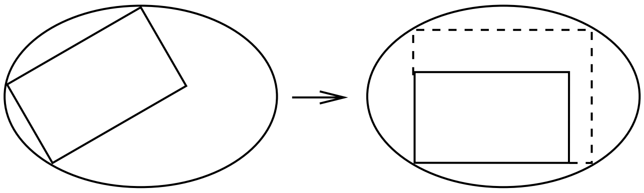
$$C(x) = 211x^5 + 9x^4 - 8x^3 + 32 + (10^4 - x)^5 - 5(10^4 - x)^2 + (10^4 - x) + \cos(10^4 - x) - 1,$$

where $0 \le x \le 10^4$. We have

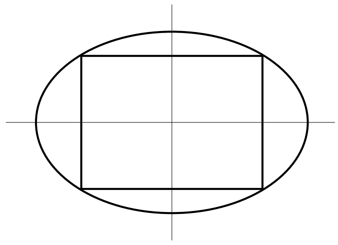
$$C'(x) = 1055x^4 + 36x^3 - 24x^2 - 5(10^4 - x)4 + 10(10^4 - x) + \sin(10^4 - x) - 1.$$

This does not look easy to solve. Maple finds a root at $x = 2078.44314$ and the value of $C(2078.44314)$ is less than $C(0)$ and $C(10^4)$. It therefore appears that the global minimum is (approximately) $x = 2078.44314$. However this result must be treated with caution as it is not really clear whether or not this is the only root of C' in the interval $[0, 10^4]$.

1.2 (a) If one of the corners of the rectangle does not touch the ellipse then the rectangle may be rotated inside the ellipse and enlarged.



Therefore we may assume that a rectangle of maximal area inscribed in the ellipse has vertical sides and horizontal top and bottom.



If one corner of this ellipse has coordinates (x, y) then the other corners are $(x, -y)$, $(-x, y)$ and $(-x, -y)$; because all corners lie on the ellipse. Assume the top right corner is (x, y), so x, y are non-negative. Each point on the ellipse in this top right quadrant is the corner of a unique rectangle with vertical sides and all four corners on the ellipse. The problem is therefore to find which of these points determines the rectangle of largest size.

Assume then that $x, y \geq 0$ and (x, y) lies on the ellipse. The area of the rectangle with corners (x, y), $(x, -y)$, $(-x, y)$ and $(-x, -y)$ is then $2x \times 2y = 4xy$. To reduce this to a problem of one variable we use the equation of the ellipse to find y in terms of x. Rearranging the equation and using the fact that $y \geq 0$ we obtain

$$y = \frac{3}{4}\sqrt{16 - x^2}.$$

As the points of the ellipse have real coordinates we must have $16 \geq x^2$ (to make $\sqrt{16 - x^2}$ real). This, and the fact that $x \geq 0$, means that $0 \leq x \leq 4$. Therefore the area of the rectangle is

$$A(x) = 3x\sqrt{16 - x^2}, \text{ where } 0 \leq x \leq 4.$$

The problem is to find the maximal value of A. As A is continuous it has a global maximum on $[0, 4]$. This must lie at one of the end-points of $[0, 4]$, at a stationary point or at a point where A is non-differentiable.

We have

$$A'(x) = \frac{3}{\sqrt{16 - x^2}}\left(16 - 2x^2\right).$$

Therefore $A'(x)$ is undefined at $x = 4$, which is the only point where A is non-differentiable in $[0, 4]$. Stationary points occur when $16 - 2x^2 = 0$, that is when $x = \pm\sqrt{8}$; so there is only one, at $x = \sqrt{8}$, in the interval $[0, 4]$.

The values of $A(x)$ at these points and at the end-points of $[0, 4]$ are

$$A(0) = 0,\ A(\sqrt{8}) = 24,\ A(4) = 0.$$

Therefore A has a global maximum at $x = \sqrt{8}$ where the value is $A(\sqrt{8}) = 24$. Hence the maximal area of a rectangle that can be inscribed in the given ellipse is 24. When $x = \sqrt{8}$ we have $y = (3\sqrt{8})/4$. The dimensions of this rectangle are therefore $2x$ wide by $2y$ high, that is $2\sqrt{8}$ wide by $(3\sqrt{8})/2$ high.

(b) The analysis unfolds in exactly the same way as for the previous part of the question, until the point where the equation of the ellipse is used to express y is terms of x. This time rearranging the equation and using the fact that $y \geq 0$ we obtain

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Note that we may assume that $a, b > 0$, as it makes no difference to the ellipse. As the points of the ellipse have real coordinates we must have $a^2 \geq x^2$ (to make $\sqrt{a^2 - x^2}$ real). This, and the fact that $x \geq 0$, means that $0 \leq x \leq a$. Therefore the area of the rectangle is

$$A(x) = 4xy = \frac{4bx}{a}\sqrt{a^2 - x^2},\ \text{where } 0 \leq x \leq a.$$

As A is continuous it has a global maximum on $[0, a]$. This must lie at one of the end-points of $[0, a]$, at a stationary point or at a point where A is non-differentiable.

We have

$$A'(x) = \frac{4b}{a\sqrt{a^2 - x^2}}\left(a^2 - 2x^2\right).$$

Therefore $A'(x)$ is undefined at $x = a$, which is the only point where A is non-differentiable in $[0, a]$. Stationary points occur when $a^2 - 2x^2 = 0$, that is when $x = \pm a/\sqrt{2}$; so there is only one, at $x = a/\sqrt{2}$, in the interval $[0, a]$.

The values of $A(x)$ at these points and at the end-points of $[0, a]$ are

$$A(0) = 0,\ A(a/\sqrt{2}) = 4ab,\ A(a) = 0.$$

Therefore A has a global maximum at $x = a/\sqrt{2}$ where the value is $A(a/\sqrt{2}) = 4ab$. Hence the maximal area of a rectangle that can be inscribed in the given ellipse is $4ab$. When $x = a/\sqrt{2}$ we have $y = b/\sqrt{2}$. The dimensions of this rectangle are therefore $2x$ wide by $2y$ high, that is $a\sqrt{2}$ wide by $b\sqrt{2}$ high.

1.3 (a)

$$\begin{aligned} f(W) &= \frac{1}{3}\ln(S) + \frac{2}{3}\ln(P) \\ &= \frac{1}{3}\ln(mW) + \frac{2}{3}\ln(24 - W). \end{aligned}$$

(b) The problem is now to find global maxima of $f(W)$, where $0 < W < 24$. If $W \to 0^+$ then

$$f(W) \to \frac{1}{3}\ln(mW) + \frac{2}{3}\ln(24) \to -\infty.$$

As $W \to 24^-$ so $24 - W \to 0^+$ and $f(W) \to -\infty$. Hence f has no global minimum on the interval between 0 and 24.

(c) As f is continuous at all points of $(0, 24)$ and tends to $-\infty$ as x approaches either end of this interval it has a global maximum (in the interval $(0, 24)$) and this occurs at a stationary point or a point of non-differentiability.

$$\begin{aligned} f'(W) &= \frac{1}{3} \cdot \frac{1}{mW} \cdot m + \frac{2}{3} \cdot \frac{1}{24 - W} \cdot (-1) \\ &= \frac{8 - W}{W(24 - W)}. \end{aligned}$$

f is differentiable at all points of $(0, 24)$, so the global maxima must occur at a stationary points. $f'(W) = 0$ if and only if $W = 8$. Hence there is a unique global maximum at $W = 8$. (At $W = 8$ we have $P = 16$ and $S = 8m$.)

(d) The global maximum occurs at $W = 8$ irrespective of the value of m. Therefore, assuming that you work to maximise $U(P, S) = f(W)$ the wage rate m will not affect the number of hours you work. If your employer raises m you will still choose to work for 8 hours.

2 Functions of Several Variables

2.1 To find the stationary points solve the equations $\partial f/\partial x = 4ye^{4xy} = 0$ and $\partial f/\partial y = 4xe^{4xy} = 0$. The unique solution is $(x, y) = (0, 0)$ and $f(0, 0) = 1$. As $x \to \infty$ we have $f(x, x) = e^{4x^2} \to \infty$, so f has no global maximum. As $x \to -\infty$ we have $f(x, 1) = e^{4x} \to 0^+$. As $f(x, y) > 0$, for all x and y it follows that f has no global minimum.

2.2 (a) Suppose there are N holidaymakers in total. As holidaymakers are uniformly distributed along the beach there are xN in $[0, x]$, $(y-x)N$ in $[x, y]$ and $(1-y)N$ in $[y, 1]$. The average distance walked is therefore

$$\frac{1}{N}\left(xN\frac{x}{2} + (y-x)N\frac{(y-x)}{4} + (1-y)N\frac{(1-y)}{2}\right) = \frac{x^2}{2} + \frac{(y-x)^2}{4} + \frac{(1-y)^2}{2}.$$

(b)

$$\frac{\partial f}{\partial x} = x - \frac{y-x}{2} = \frac{1}{2}(3x-y); \quad (2.1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(3y - x - 2). \quad (2.2)$$

These equations have unique solution $x = 1/4$ and $y = 3/4$. Therefore there is a unique stationary point at $(1/4, 3/4)$.

(c) Since each term in the sum is non-negative, for all x, y, it is clear that $f(x, y) \to \infty$ as $|(x, y)| \to \infty$. Therefore if we regard f as a function from $\mathbb{R}^2$ to $\mathbb{R}$ it has a global minimum (but no global maximum).

(d) From the previous part of the question the global minima occur at stationary points or points of non-differentiability. Since f is differentiable everywhere they occur only at the stationary point. Thus f has a global minimum at $(1/4, 3/4)$. The vendors should be placed at $x = 1/4$ and $y = 3/4$.

(e) If both vendors place themselves at the middle of the beach then the average distance walked to get an ice-cream will be $1/4$. On the other hand if you place the vendors as above the average distance will be $f(1/4, 3/4) = 3/32$. Therefore free market forces result in people having to walk much further (nearly 3 times as far) on average, to buy their ice-creams.

2.3 The region S is the inside of the triangle with vertices $(0, 0)$, $(10, 0)$ and $(0, 5)$.

We have $\partial g/\partial x = 3/x$ and $\partial g/\partial y = 2/y$. There are no stationary points in S. Therefore there are no local extrema in S. As the region S has no boundary it follows that g has no local maxima or minima on S.

3 Lagrange Multipliers

3.1 (a)

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = x_1^2 + 12x_1x_2 + 2x_2^2 - \lambda_1(4x_1^2 + x_2^2 - 25).$$

Therefore

$$\frac{\partial L}{\partial x_1} = 2x_1 + 12x_2 - 8\lambda_1 x_1 \quad \text{and} \qquad \text{(i)}$$

$$\frac{\partial L}{\partial x_2} = 12x_1 + 4x_2 - 2\lambda_1 x_2. \qquad \text{(ii)}$$

First eliminate λ_1:

$$\text{(i)} \times x_2 - \text{(ii)} \times 4x_1 = 12x_2^2 - 14x_1x_2 - 48x_1^2. \qquad \text{(iii)}$$

Now (iii)$= 0$ if and only if $6x_2^2 - 7x_1x_2 - 24x_1^2 = 0$ and we can regard this as a quadratic in x_1 and solve it in the usual way:

$$\begin{aligned} x_1 &= \frac{7x_2 \pm \sqrt{49x_2^2 + 4 \times 6x_2^2 \times 24}}{-48} \\ &= \frac{(7 \pm 25)x_2}{-48} \\ &= \frac{3x_2}{8} \text{ or } \frac{-2x_2}{3}. \end{aligned}$$

Now we use the condition that $f_1(x_1, x_2) = 0$. If $x_1 = 3x_2/8$ then

$$\begin{aligned} f_1(x_1, x_2) &= 4 \times \left(\frac{3x_2}{8}\right)^2 + x_2^2 - 25 \\ &= \frac{36x_2^2}{64} + x_2^2 - 25 \\ &= \frac{100x_2^2}{64} - 25, \end{aligned}$$

so $f_1(x_1, x_2) = 0$ if and only if $x_2^2 = 16$ if and only if $x_2 = \pm 4$. If $x_2 = \pm 4$ then $x_1 = 3x_2/8 = \pm 3/2$. This gives two admissable points which are also stationary points of L, namely $(3/2, 4)$ and $(-3/2, -4)$.
If $x_1 = -2x_2/3$ then

$$\begin{aligned} f_1(x_1, x_2) &= 4 \times \left(\frac{-2x_2}{3}\right)^2 + x_2^2 - 25 \\ &= \frac{16x_2^2}{9} + x_2^2 - 25 \\ &= \frac{25x_2^2}{9} - 25, \end{aligned}$$

so $f_1(x_1, x_2) = 0$ if and only if $x_2^2 = 9$ if and only if $x_2 = \pm 3$. If $x_2 = \pm 3$ then $x_1 = -2x_2/3 = \mp 2$; giving two more admissable points which are also stationary points of L, namely $(2, -3)$ and $(-2, 3)$.

L is differentiable everywhere. The solutions (x_1, x_2) to $f_1 = 0$ lie in a bounded region: say $|x_i| \le 5$, for $i = 1, 2$. Therefore the global maxima and minima of f_0 (with the given constraints) lie amongst the four points found above. We have $f_0(3/2, 4) = f_0(-3/2, -4) = 106\frac{1}{4}$ and $f_0(2, -3) = f_0(-2, 3) = -50$.

Therefore the global minima are $(2, -3)$ and $(-2, 3)$ and the global maxima are $(3/2, 4)$ and $(-3/2, -4)$.

(b)

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = x_1 x_2^2 x_3^3 - \lambda_1(x_1^2 + x_2^2 + x_3^2 - 1).$$

Therefore

$$\frac{\partial L}{\partial x_1} = x_2^2 x_3^3 - 2\lambda_1 x_1, \qquad \text{(i)}$$

$$\frac{\partial L}{\partial x_2} = 2x_1 x_2 x_3^3 - 2\lambda_1 x_2 \quad \text{and} \qquad \text{(ii)}$$

$$\frac{\partial L}{\partial x_3} = 3x_1 x_2^2 x_3^2 - 2\lambda_1 x_3. \qquad \text{(iii)}$$

First eliminate λ_1:

$$\text{(i)} \times x_2 - \text{(ii)} \times x_1 = x_2^3 x_3^3 - 2x_1^2 x_2 x_3^3 = x_2 x_3^3(x_2^2 - 2x_1^2) \qquad \text{(iv)}$$

and

$$\text{(i)} \times x_3 - \text{(iii)} \times x_1 = x_2^2 x_3^4 - 3x_1^2 x_2^2 x_3^2 = x_2^2 x_3^2(x_3^2 - 3x_1^2). \qquad \text{(v)}$$

Note that if either $x_1 = 0$ or $x_2 = 0$ or $x_3 = 0$ then $f_0 = 0$; so we shall assume for the meantime that $x_i \ne 0$, for $i = 1, 2$ and 3. Then (iv)=(v)= 0 implies $x_2^2 = 2x_1^2$ and $x_3^2 = 3x_1^2$. Now we use the condition that $f_1(x_1, x_2, x_3) = 0$. This gives

$$x_1^2 + 2x_1^2 + 3x_1^2 - 1 = 6x_1^2 - 1 = 0,$$

so $x_1 = \pm 1/\sqrt{6}$. Then $x_2^2 = 1/3$ and $x_3^2 = 1/2$; so $x_2 = \pm 1/\sqrt{3}$ and $x_3 = \pm 1/\sqrt{2}$. This gives a total of 8 admissable points which are stationary points of L, namely $(\gamma/\sqrt{6}, \delta/\sqrt{3}, \varepsilon/\sqrt{2})$, where $\gamma, \delta, \varepsilon \in \{-1, 1\}$.

L is differentiable everywhere. The solutions (x_1, x_2, x_3) to $f_1 = 0$ lie in a bounded region: say $|x_i| \le 1$, for $i = 1, 2, 3$. Therefore the global maxima and minima of f_0 (with the given constraints) lie amongst the eight points found above, or at a point(x_1, x_2, x_3) where x_1, x_2 or $x_3 = 0$. We have

$$f_0(\gamma/\sqrt{6}, \delta/\sqrt{3}, \varepsilon/\sqrt{2}) = \frac{\gamma\varepsilon}{12\sqrt{3}},$$

so the global mimima are the four points $(\gamma/\sqrt{6}, \delta/\sqrt{3}, \varepsilon/\sqrt{2})$ with $\gamma = -\varepsilon$ and the global maxima are the four of these points with $\gamma = \varepsilon$. (Points with $x_i = 0$ are neither global maxima or minima.)

3.2 There are £5 available to spend on x ice-creams, at 50p a time, and y rides, at £1 a time. Assuming that all the money is spent this means we should have $.5x + y = 5$, where $x, y \geq 0$, $x \leq 10$ and $y \leq 5$. Therefore we set

$$f_1 = \frac{x}{2} + y - 5 \text{ with } 0 \leq x \leq 10, 0 \leq y \leq 5.$$

Subject to this constraint the problem is to maximise the Utility Function $U(x, y)$, so we set $f_0(x, y) = U(x, y)$. As the Utility function is undefined for $x = 0$ or $y = 0$ we further assume that $x > 0$ and $y > 0$. Then

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = 3\ln(x) + 2\ln(y) - \lambda_1(\frac{x}{2} + y - 5).$$

Therefore

$$\frac{\partial L}{\partial x} = \frac{3}{x} - \frac{\lambda_1}{2} \quad \text{and} \qquad \text{(i)}$$

$$\frac{\partial L}{\partial y} = \frac{2}{y} - \lambda_1. \qquad \text{(ii)}$$

Now we must find points where (i)=(ii)= 0. First we eliminate λ_1:

$$2 \times \text{(i)} - \text{(ii)} = \frac{6}{x} - \frac{2}{y} \qquad \text{(iii)}$$

and (iii)= 0 if and only if $y = x/3$ (since $x, y > 0$).

Using the constraint f_1, with $y = x/3$, we obtain

$$f_1(x, y) = \frac{x}{2} + \frac{x}{3} - 5 = \frac{5x}{6} - 5 = 0$$

if and only if $x = 6$, in which case $y = x/3 = 2$.

As L is differentiable at all points with $x, y > 0$ and the values of x, y are bounded the global maximum must occur at this stationary admissable point. We have $f_0(6, 2) = 3\ln(6) + 2\ln(2) = \ln(6^3 2^2) = \ln(864) \approx 6.76$. If $x = 0$ then $f_1 = 0$ implies $y = 5$ and $f_0(0, 5) = 2\ln(5) \approx 3.22$. If $y = 0$ then $x = 10$ and $f_0(10, 0) = 3\ln(10) \approx 2.3$. Therefore the global maximum occurs at $(x, y) = (6, 2)$ and the boy should buy 6 ice-creams and go on 2 rides. (He must enjoy being sick.)

3.3 We wish to minimise the cost of producing 90 units of output. The cost is

$$f_0(x, y) = 8(x + 2y).$$

The relation between output and x and y is given by $q = 10\sqrt{x}\sqrt{y}$; that is $90 = 10\sqrt{x}\sqrt{y}$ in our case. We can simplify this to $9 = \sqrt{x}\sqrt{y}$, and note that we must have $x, y > 0$. It turns out to make the differentiation easier if we take logarithms of both sides of this equation which then becomes $\ln(9) = \ln(\sqrt{x}) + \ln(\sqrt{y}) = (\ln(x) + \ln(y))/2$. Therefore we set

$$f_1(x, y) = \ln(x) + \ln(y) - 2\ln(9), \text{ where } x, y > 0.$$

We have

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = 8(x + 2y) - \lambda_1(\ln(x) + \ln(y) - 2\ln(9)).$$

Therefore

$$\frac{\partial L}{\partial x} = 8 - \frac{\lambda_1}{x} \quad \text{and} \tag{i}$$

$$\frac{\partial L}{\partial y} = 16 - \frac{\lambda_1}{y}. \tag{ii}$$

Then

$$\text{(i)} \times \frac{1}{y} - \text{(ii)} \times \frac{1}{x} = \frac{8}{y} - \frac{16}{x} = 0$$

if and only if $x = 2y$ (as $x, y > 0$). Using the constraint

$$f_1(2y, y) = \ln(2y) + \ln(y) - 2\ln(9) = \ln(2y^2) - 2\ln(9).$$

Hence $f_1(2y, y) = 0$ if and only if $\ln(2y^2) - 2\ln(9) = 0$ if and only if $2y^2 = 81$ if and only if $y = 9/\sqrt{2}$. This gives $x = 9\sqrt{2}$ and there is only one admissable stationary point of L.

L is differentiable at all points (x, y) where $x, y > 0$. If $|(x, y)| \to \infty$ and $f_1 = 0$ then either $x \to \infty$ and $y \to 0^+$ or $y \to \infty$ and $x \to 0^+$. In either case $f_0(x, y) \to \infty$ and it follows that f_0, restricted to admissable points, has a global minimum. Therefore the unique global minimum is at $(9\sqrt{2}, 9/\sqrt{2})$. Thus the firm should use $9\sqrt{2}$ of input X and $9/\sqrt{2}$ of input Y to produce 90 units of output.

4 Linear Programming

4.1 **Solutions.**

(a) Slack variables: $x_3, x_4, x_5, x_6 \geq 0$.
Dictionary 1.

$$\begin{aligned} x_3 &= 80 - x_1 \\ x_4 &= 320 - 3x_1 - 2x_2 \\ x_5 &= 240 - x_1 - 3x_2 \\ x_6 &= 180 - x_1 - 2x_2 \\ z &= 11x_1 + 10x_2. \end{aligned}$$

Choose x_1 to become non-zero. From the 1st dictionary, with $x_2 = 0$:

$$\begin{aligned} 0 &\leq x_3 = 80 - x_1 \Rightarrow x_1 \leq 80 \\ 0 &\leq x_4 = 320 - 3x_1 \Rightarrow x_1 \leq 320/3 \\ 0 &\leq x_5 = 240 - x_1 \Rightarrow x_1 \leq 240 \\ 0 &\leq x_6 = 180 - x_1 \Rightarrow x_1 \leq 180 \end{aligned}$$

We must keep $x_1 \leq 80$. Setting $x_1 = 80$ gives $x_3 = 0$. Now x_3 becomes a basis variable and x_1 becomes a slack variable. From the 1st dictionary $x_1 = 80 - x_3$ and substition in the expressions for x_4, x_5, x_6 and z gives
Dictionary 2.

$$\begin{aligned} x_1 &= 80 - x_3 \\ x_4 &= 320 - 240 + 3x_3 - 2x_2 = 80 + 3x_3 - 2x_2 \\ x_5 &= 240 - 80 + x_3 - 3x_2 = 160 + x_3 - 3x_2 \\ x_6 &= 180 - 80 + x_3 - 2x_2 = 100 + x_3 - 2x_2 \\ z &= 880 - 11x_3 + 10x_2. \end{aligned}$$

To increase z we must now choose x_2 to leave the basis. From the 2nd dictionary with $x_3 = 0$:

$$\begin{aligned} 0 &\leq x_1 = 80 \text{ and this imposes no constraint on } x_2 \\ 0 &\leq x_4 = 80 - 2x_2 \Rightarrow x_2 \leq 40 \\ 0 &\leq x_5 = 160 - 3x_2 \Rightarrow x_2 \leq 160/30 \\ 0 &\leq x_6 = 100 - 2x_2 \Rightarrow x_2 \leq 50. \end{aligned}$$

We must keep $x_2 \leq 40$. Setting $x_2 = 40$ gives $x_4 = 0$ so x_4 becomes a basis variable and x_2 becomes a slack variable. From the 2nd dictionary $x_2 = (80 +$

$3x_3 - x_4)/2$ and substition in the expressions for x_1, x_5, x_6 and z gives
Dictionary 3.

$$\begin{aligned} x_1 &= 80 - x_3 \\ x_2 &= (80 + 3x_3 - x_4)/2 \\ x_5 &= 160 + x_3 - (240 + 9x_3 - 3x_4)/2 = 40 - 7x_3/2 + 3x_4/2 \\ x_6 &= 100 + x_3 - 80 - 3x_3 + x_4 = 20 - 2x_3 + x_4 \\ z &= 880 - 11x_3 + 400 + 15x_3 - 5x_4 = 1280 + 4x_3 - 5x_4. \end{aligned}$$

To increase z we must now choose x_3 to leave the basis. From the 3rd dictionary with $x_4 = 0$:

$$\begin{aligned} 0 \le x_1 &= 80 - x_3 \Rightarrow x_3 \le 80 \\ 0 \le x_2 &= (80 + 3x_3)/2 \Rightarrow x_3 \le \infty \\ 0 \le x_5 &= 40 - 7x_3/2 \Rightarrow x_3 \le 80/7 \\ 0 \le x_6 &= 20 - 2x_3 \Rightarrow x_3 \le 10. \end{aligned}$$

We must keep $x_3 \le 10$. Setting $x_3 = 10$ gives $x_6 = 0$ so x_6 becomes a basis variable and x_3 becomes a slack variable. From the 3rd dictionary $x_3 = 10 - x_6/2$ and substition in the expressions for x_1, x_2, x_5 and z gives
Dictionary 4.

$$\begin{aligned} x_1 &= 80 - 10 + x_6/2 = 70 + x_6/2 \\ x_2 &= (80 + 30 - 3x_6/2 - x_4)/2 = (220 - 3x_6 - 2x_4)/4 \\ x_3 &= 10 - x_6/2 \\ x_5 &= 40 - 35 + 7x_6/4 + 3x_4/2 = 5 + 7x_6/4 + 3x_4/2 \\ z &= 1280 + 40 - 2x_6 - 5x_4 = 1320 - 2x_6 - 5x_4. \end{aligned}$$

The coefficients of variables in z are now all zero so we have reach a global maximum with $x_1 = 70$, $x_2 = 55$, $x_3 = 10$, $x_4 = x_6 = 0$ and $x_5 = 5$. The solution to the original problem is $x_1 = 70$, $x_2 = 55$ giving a value of 1320.

(b) Same method. Solution $(x_1, x_2, x_3) = (18, 6, 0)$ and value 66.