MAS1002 Optimisation and Linear Methods: Problems

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1 Functions of one variable

1.1 **Drug production.** A company has 2 processes for manufacturing its drug "zappo": process A and process B. Production costs are

$$C_A(x) \text{ for producing } x \text{ units of zappo using process A;}$$
$$C_B(x) \text{ for producing } x \text{ units of zappo using process B.}$$

Suppose D units of zappo are required. How should this be produced at minimal cost?

If we assume x units of zappo are produced using process A then $D - x$ units must be produced using process B. The total cost of producing D units of zappo is then

$$C(x) = C_A(x) + C_B(D - x).$$

(a) Assume we know that $C_A(x) = \frac{1}{4}(x^2 + x)$ and that $C_B(x) = x$ and suppose that $D = 3$. Write out the explicit formula for $C(x)$ in this case. Now find the global minimum of $C(x)$ with $0 \le x \le 3$. (Why is x in this range?)

(b) Now suppose that $D = 1$. Rerun the minimisation procedure and see if the answer is changed.

(c) Suppose now that

$$C_A(x) = 211x^5 + 9x^4 - 8x^3 + 32 \text{ and } C_B(x) = x^5 - 5x^2 + x + \cos(x) - 1$$

and $D = 10^4$. Write down the formula for $C(x)$ in this case. What is the minimisation problem to be solved? What equation needs to be solved to find the global minimum in this case and how much chance have you of solving it?

1.2 A rectangle is said to be *inscribed* in an ellipse if the four corners of the rectangle meet the ellipse; and there are no other points of intersection. Draw diagrams to illustrate your answers to the following questions.

(a) Find the dimensions of a rectangle of maximal area which can be inscribed in the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(b) Find the dimensions of a rectangle of maximal area which can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive real numbers.

1.3 Suppose that you have a holiday job at a fixed hourly rate of pay, and that you must decide how many hours to work. Write W for the number of hours you work in a day and P, for play, for the remaining hours, so

$$W + P = 24. \tag{1.1}$$

Let's say the wage rate is m pounds per hour. To keep things simple assume that you spend all the money that you earn, so your daily spending is S where

$$S = mW. \tag{1.2}$$

You decide that you can quantify the pleasure you derive from the combination of play and spending by means of a function of P and S. Call this function $U(P, S)$. In optimisation such functions are often called *utility functions*. We shall suppose that you use the utility function

$$U(P, S) = \frac{1}{3}\ln(S) + \frac{2}{3}\ln(P). \tag{1.3}$$

Now fix the number of hours you work to maximise $U(P, S)$. Complete this process in the following steps.

(a) First replace U by a function f of the single variable W; by setting $f(W) = U(P, S)$ and using (1.1) and (1.2) to eliminate P and S from $U(P, S)$.

(b) The problem now reduces to maximising $f(W)$. Note that for (1.3) to make sense you must have $S > 0$, so you cannot choose $W = 0$; no work. Therefore $W > 0$. Similarly, you must have $P > 0$ so $W < 24$. What happens to $f(W)$ as $W \to 0^+$ and as $W \to 24^-$?

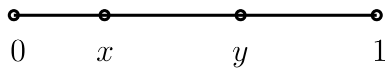
(c) Find the global maximum of f on $(0, 24)$ by using the previous part of the problem and the stationary points theorem.

(d) How does the wage rate m affect the number of hours that you decide to work? If your employer increases the hourly rate are you likely to work longer?

2 Functions of several variables

2.1 Find the global extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = e^{4xy}$.

2.2 **Ice-cream sellers on the beach.**
On a beach holidaymakers can buy ice-cream from 2 different vendors. We'll assume they always buy from the nearest of the 2 vendors. We'll also assume that the holidaymakers are distributed evenly along the beach - that is, uniformly distributed. If no regulations are imposed then a vendor who is furthest from the middle of the beach can increase market share by moving closer to the middle. The result is that both vendors end-up back-to-back at the middle of the beach. As beach manager you don't think this is very sensible; as it means holidaymakers have to walk further than necessary. You decide to tell the vendors where to position themselves so as to minimise the distance that the holidaymakers have to walk. That is you wish to minimise the average distance walked to buy ice-cream. We assume the beach is 1 unit long, for simplicity. You will place the vendors at distances x and y from the west end of the beach, where $0 \leq x \leq y \leq 1$, as shown below.



As holidaymakers are uniformly distributed along the beach the average distances walked to get an ice-cream are

- $x/2$, for people between 0 and x;
- $(y - x)/4$, for people between x and y;
- $(1 - y)/2$, for people between y and 1.

(a) Find the average distance $f(x, y)$ walked by holidaymakers to the nearest vendor.

(b) The problem is now to find values of x and y which minimise f, with $0 \leq x \leq y \leq 1$. Find the partial derivatives of f and the stationary points of f. Are there any points at which f is not differentiable?

(c) What can you say about $f(x, y)$ as $|(x, y)| \to \infty$? (Forget the constraints on the values of x and y for the moment.)

(d) Combine the above to find the global minima of f. Where should the vendors be placed on the beach?

(e) In this example do free market forces work to the benefit of everyone or is management required for the common good?

2.3 Let $g(x, y) = 3\ln(x) + 2\ln(y)$ and suppose that g is defined on the set $S \subseteq \mathbb{R}^2$ consisting of points (x, y) such that $0 < x$, $0 < y$ and $\frac{x}{2} + y - 5 < 0$. Sketch the region S. Find the global minimum and maximum of g on this region, if they exist.

3 Lagrange's Method

3.1 Find the global maxima and minima of the function f_0, subject to the constraint that $f_1 = 0$, in each part of the following question.

(a) $f_0(x_1, x_2) = x_1^2 + 12x_1x_2 + 2x_2^2$, $f_1(x_1, x_2) = 4x_1^2 + x_2^2 - 25$.

(b) $f_0(x_1, x_2, x_3) = x_1x_2^2x_3^3$, $f_1(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$.

3.2 A small boy has £5 to spend at the Hoppings. He wants to spend his money on ice-creams, which cost £1 each, and rides, which cost 50p each. The happiness he enjoys from the combination of x ice-creams and y rides is given by the Utility Function

$$U(x, y) = 3\ln(x) + 2\ln(y).$$

Determine how many ice-creams and rides he should buy to maximise his happiness.

3.3 A firm assembles a product, Q, using inputs of type X and type Y. The quantity of each type of input for each assembly is not fixed: if x of input X and y of input Y are used then the firm can produce q of output Q, where

$$q = 10\sqrt{x}\sqrt{y}.$$

If X costs £8 per unit and Y costs £16 per unit what is the most efficient way to produce 90 units of Q.

4 Linear Programming

4.1 Use the simplex algorithm to solve the following problems.

(a) Maximise $f(x_1, x_2) = 11x_1 + 10x_2$ subject to constraints

$$\begin{aligned} x_1 &\leq 80 \\ 3x_1 + 2x_2 &\leq 320 \\ x_1 + 3x_2 &\leq 240 \\ x_1 + 2x_2 &\leq 180 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(b) Maximise $f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 4x_3$ subject to constraints

$$\begin{aligned} 3x_1 + x_2 + 4x_3 &\leq 60 \\ x_1 + 2x_2 + 3x_3 &\leq 30 \\ 2x_1 + 2x_2 + 3x_3 &\leq 600 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$