Student Number: **Degree Programme:**

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2 MOCK EXAM

MAS1002

Optimisation and Linear Methods

Time allowed: 1 hour 30 minutes

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FIVE questions on this paper.

Write your answers on the exam paper, in the spaces provided.
Write rough work on the reverse of the pages.

No.	Mark
1	
2	
3	

No.	Mark
4	
5	
Total	

1. Let f be the function

$$f = \frac{x^5}{20} - \frac{\pi}{8}x^4 + \frac{1}{6}\left(\frac{3\pi^2}{4}+1\right)x^3 - \frac{\pi}{4}\left(\frac{\pi^2}{4}+1\right)x^2 + \left(\frac{\pi^4}{64}+\frac{\pi^2}{8}-1\right)x - \cos(x) + 1.$$

$f' =$

$$\frac{x^4}{4} - \frac{\pi}{2}x^3 + \frac{1}{2}\left(\frac{3\pi^2}{4}+1\right)x^2 - \frac{\pi}{2}\left(\frac{\pi^2}{4}+1\right)x + \left(\frac{\pi^4}{64}+\frac{\pi^2}{8}-1\right) + \sin(x).$$

Show that f has a stationary point at $x = \pi/2$.

$$\begin{aligned} f'(\pi/2) &= \frac{(\pi/2)^4}{4} - \frac{\pi}{2}(\pi/2)^3 + \frac{1}{2}\left(\frac{3\pi^2}{4}+1\right)(\pi/2)^2 \\ &\quad - \frac{\pi}{2}\left(\frac{\pi^2}{4}+1\right)(\pi/2) + \left(\frac{\pi^4}{64}+\frac{\pi^2}{8}-1\right) + \sin(\pi/2) \\ &= \frac{\pi^4}{16\times 4} - \frac{\pi^4}{16} + \left(\frac{3\pi^4}{16\times 2}+\frac{\pi^2}{8}\right) \\ &\quad - \left(\frac{\pi^4}{16}+\frac{\pi^2}{4}\right) + \left(\frac{\pi^4}{64}+\frac{\pi^2}{8}-1\right) + 1 \\ &= \frac{\pi^4}{16\times 4}(1-4+6-4+1) + \frac{\pi^2}{8}(1-2+1) - 1 + 1 \\ &= 0. \end{aligned}$$

Use the nth derivative test to determine whether or not f has a local extremum at $\pi/2$ and if so whether it's a local minimum, a local maximum.

Question 1 continued on next page

$$f''(x) = x^3 - \frac{3\pi}{2}x^2 + \left(\frac{3\pi^2}{4} + 1\right) x - \frac{\pi}{2}\left(\frac{\pi^2}{4} + 1\right) + \cos(x).$$

$$f''(\pi/2) = \frac{\pi^3}{8} - \frac{3\pi^3}{8} + \left(\frac{3\pi^3}{8} + \frac{\pi}{2}\right) - \frac{\pi}{2}\left(\frac{\pi^2}{4} + 1\right) = 0.$$

The second derivative test does not apply so we continue to differentiate.

$$f'''(x) = 3x^2 - 3\pi x + \left(\frac{3\pi^2}{4} + 1\right) - \sin(x).$$

$$f'''(\pi/2) = 3\frac{\pi^2}{4} - 3\frac{\pi^2}{2} + \left(\frac{3\pi^2}{4} + 1\right) - 1 = 0.$$

The next derivative is therefore required.

$$f^{(4)}(x) = 6x - 3\pi - \cos(x).$$

$$f^{(4)}(\pi/2) = 3\pi - 3\pi = 0.$$

The next derivative is therefore required.

$$f^{(5)}(x) = 6 + \sin(x) \text{ so } f^{(5)}(\pi/2) = 7 \neq 0.$$

From the derivative test, as 5 is odd, $f^{(i)}(\pi/2) = 0$, for $i = 1, \ldots 4$ and $f^{(5)}(\pi/2) \neq 0$ there is not a local extremum at $x = \pi/2$.

[15 marks]

2. Let f be the function from the interval $[0,3]$ to $\mathbb{R}$ given by

$$f = \ln(4-(1-x)^2), \text{ for } x \text{ such that } 0 \leq x \leq 3.$$

Compute $f'(x)$ and find the stationary points of f in $[0,3]$; showing your working.

$f'(x) =$

$$\left(\frac{1}{4-(x-1)^2}\right)(-2(x-1))$$

so $f'(x) = 0$ only if $x = 1$.

Stationary points: $x = 1$.

f' is defined at all points of $[0,3]$ except $x =$ 3.

Describe the behaviour of the function f as x approaches points where f' is not defined.

$f \to -\infty$ as $x \to 3^-$.

Evaluate f at all the points that are potential maxima or minima.

$f(0) = \ln(4-1) = \ln(3)$.

$f(1) = \ln(4)$.

Using the above say whether or not f has points of global minima and/or global maxima and list all such points.

f has no global minimum.

f has a global maximum at $x = 1$.

[15 marks]

3. Let

$$f(x_1, x_2) = x_1^2 - x_2^2.$$

Find the global extrema of f subject to the constraint

$$g(x_1, x_2) = x_1^2 + (x_2 - 1)^2 - 1$$

as follows. Write down the Lagrange function $L(x_1, x_2)$ of the problem:

$L(x_1, x_2) =$ $L(x_1, x_2) = x_1^2 - x_2^2 - \lambda(x_1^2 + (x_2 - 1)^2 - 1).$

Calculate the partial derivatives of L with respect to x_1 and x_2.

$\partial L/\partial x_1 =$ $2x_1 - 2\lambda x_1$

$\partial L/\partial x_2 =$ $-2x_2 - 2\lambda(x_2 - 1)$

Find the stationary points of $L(x_1, x_2)$ which satisfy $g = 0$.

Eliminate λ:

$$0 = (x_2 - 1) \times (\partial L/\partial x_1) - x_1 \times (\partial L/\partial x_2) = 2[x_1(x_2 - 1) + x_1 x_2] = 2x_1(2x_2 - 1).$$

Therefore $x_1 = 0$ or $x_2 = 1/2$.

If $x_1 = 0$ then $g = 0$ gives $(x_2 - 1)^2 - 1 = 0$ so $x_2 = 0$ or 2. This gives 2 stationary points $(0, 0)$ and $(0, 2)$ of L which satisfy the constraint.

If $x_2 = 1/2$ then $g = 0$ gives $x_1^2 = 3/4$ so $x_1 = \pm\sqrt{3}/2$. This gives another 2 stationary points $(\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$ of L which satisfy the constraint.

Question 3 continued on next page

Compare the values of the function at the points found above and at any other potential global extrema and so find the global maximum and minimum of f constrained by g.

As L is differentiable everywhere and all points satisfying the condition $g = 0$ are distance 1 from $(0, 1)$ the global max and min must occur at stationary points of L found above. Evaluating we have

$$f(0,0) = 0, \qquad f(0,2) = -4, \qquad f(\pm\sqrt{3}/2, 1/2) = 1/2.$$

Therefore there are global maxima at both $(\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$ and a unique global min at $(0, 2)$.

[20 marks]

4. A company manufactures a non-reflective paint using 3 colours A, B and C in ratios given by parameters x, y and z. The quantities x, y and z may be positive or negative but must satisfy the constraints that

$$x + y + 3z + 4 = 0 \text{ and } x - y + z - 2 = 0.$$

The reflective index of the paint is given by the formula

$$f(x, y, z) = x^2 + 2y^2 + 2z^2.$$

Use Lagrange's method to find the quantities x, y and z which result in the lowest reflective index.

The constraint functions are

$$g_1(x, y, z) = x + y + 3z + 4 = 0 \text{ and}$$

$$g_2(x, y, z) = x - y + z - 2 = 0.$$

The problem is to minimise f subject to these constraints.

The Lagrangean is

$$L(x, y, z) = x^2 + 2y^2 + 2z^2 - \lambda_1(x + y + 3z + 4) - \lambda_2(x - y + z - 2).$$

The resulting equations are

$$\partial L/\partial x = 2x - \lambda_1 - \lambda_2 = 0 \qquad \text{(i)}$$

$$\partial L/\partial y = 4y - \lambda_1 + \lambda_2 = 0 \qquad \text{(ii)}$$

$$\partial L/\partial z = 4z - 3\lambda_1 - \lambda_2 = 0 \qquad \text{(iii)}$$

First we eliminate the λ_i's.

$$\text{(i)} + \text{(ii)}: \; 2x + 4y - 2\lambda_1 = 0 \qquad \text{(iv)}$$

$$\text{(ii)} + \text{(iii)}: \; 4y + 4z - 4\lambda_1 = 0 \qquad \text{(v)}$$

Question 4 continued on next page

$2(\text{iv}) - (\text{v})$: $4x + 4y - 4z = 0$ iff $x + y - z = 0$ iff $z = x + y$ (vi)

Next use the constraints.

(vi) and g_1: $z + 3z + 4 = 0$ if and only if $z = -1$ (vii)

(vii) and g_2: $x - y - 3 = 0$ (viii)

(vi), (vii) and (viii): $y + 3 + y = -1$ so $y = -2$.

The only solution to the Lagrangean equations is

$(x, y, z) = (1, -2, -1)$.

L is differentiable everywhere and as $|(x, y, z)| \to \infty$ so $f \to \infty$. Hence f has a global minimum at $(1, -2, -1)$. Thus to minimise reflective index the company should set $(x, y, z) = (1, -2, -1)$.

[25 marks]

5. Use the simplex method to solve the following linear programming problem. Maximise

$$f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 4x_3,$$

subject to the constraints

$$\begin{aligned} 3x_1 + x_2 + 4x_3 &\leq 60 \\ x_1 + 2x_2 + 3x_3 &\leq 30 \\ 2x_1 + 2x_2 + 3x_3 &\leq 600 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Slack variables are x_4, x_5, x_6 where

$3x_1 + x_2 + 4x_3 + x_4 = 60$, $x_1 + 2x_2 + 3x_3 + x_5 = 30$ and

$2x_1 + 2x_2 + 3x_3 + x_6 = 600$.

The first dictionary is

$$x_4 = 60 - 3x_1 - x_2 - 4x_3$$

$$x_5 = 30 - x_1 - 2x_2 - 3x_3$$

$$x_6 = 600 - 2x_1 - 2x_2 - 3x_3$$

$$z = 3x_1 + 2x_2 + 4x_3.$$

The initial solution is $(0, 0, 0, 60, 30, 600)$ with $z = 0$.

Choosing x_1 from the possible vertices to leave the basis. From the first dictionary with $x_2 = x_3 = 0$:

as $x_4 \geq 0$, $x_1 \leq 20$,

as $x_5 \geq 0$, $x_1 \leq 30$ and

Question 5 continued on next page

as $x_6 \geq 0$, $x_1 \leq 300$.

Therefore x_4 joins the basis and x_1 becomes a slack variable. The basis is now x_2, x_3, x_4 and the slack variables are x_1, x_5, x_6. Using the first dictionary $x_1 = 20 - x_2/3 - 4x_3/3 - x_4/3$ so the second dictionary is:

$$x_1 = 20 - \frac{1}{3}(x_2 + 4x_3 + x_4)$$
$$x_5 = 30 - 20 + \frac{1}{3}(x_2 + 4x_3 + x_4)) - 2x_2 - 3x_3 = 10 - \frac{5x_2}{3} - \frac{5x_3}{3} + \frac{x_4}{3}$$
$$x_6 = 600 - 40 + \frac{2}{3}(x_2 + 4x_3 + x_4)) - 2x_2 - 3x_3 = 560 - \frac{1}{3}(4x_2 + x_3 - 2x_4)$$
$$z = 60 - x_2 - 4x_3 - x_4 + 2x_2 + 4x_3 = 60 + x_2 - x_4.$$

The second solution is $(20, 0, 0, 30, 560)$ with $z = 60$.

The only candidate to leave the basis is x_2.

$x_1 \geq 0$, $x_2 \leq 60$.

$x_5 \geq 0$, $x_2 \leq 6$.

$x_6 \geq 0$, $x_2 \leq 420$.

Therefore x_5 joins the basis. The basis is now x_3, x_4, x_5 and the slack variables are x_1, x_2, x_6. From the second dictionary

$$x_2 = \frac{3}{5}(10 - \frac{5x_3}{3} + \frac{x_4}{3} - x_5.$$

Question 5 continued on next page

The third dictionary is:

$$x_1 = 20 - \frac{1}{3}(6 - x_3 + \frac{x_4}{5} - \frac{3x_5}{5} + 4x_3 + x_4) = 18 - \frac{1}{3}(3x_3 + \frac{6x_4}{5} - \frac{3}{5})$$
$$x_2 = \frac{3}{5}(10 - \frac{5x_3}{3} + \frac{x_4}{3} - x_5$$
$$x_6 = 560 - \frac{4}{5}(10 - \frac{5x_3}{3} + \frac{x_4}{3} - x_5) - \frac{x_3}{3} + \frac{2x_4}{3} = 552 + x_3 + \frac{6x_4}{15} + \frac{4x_5}{5}$$
$$z = 60 + 6 - x - 3 + \frac{x_4}{5} - \frac{3x_5}{5} - x_4 = 66 - x_3 - \frac{4x_4}{5} - \frac{3x_5}{5}.$$

The objective function cannot now be increased by increasing a basis variable. The final solution is therefore $(18, 6, 0, 0, 0, 552)$. The original problem has solution $(x_1, x_2, x_3) = (18, 6, 0)$ with maximum value 66.

[**25 marks**]

THE END