MAS1002 Optimisation and Linear Methods: Assignment Solutions

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1 Functions of one variable

1.1 Procedure (1.9) is used to answer this question.

$$S'(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ -1/2, & \text{if } 1 \le x \le 2 \\ (3-x)e^{-(x-2)}, & \text{if } 2 < x \end{cases}.$$

$S'(x) = 0$ if and only if $x = 3$, so S has unique stationary point $x = 3$ and $S(3) = e^{-1} + 1$. S is non-differentiable at $x = 1$ and $x = 2$ and $S(1) = 1$ and $S(2) = 1/2$. Also $f \to 1^+$ as $x \to \infty$. Therefore S has global maximum at $x = 3$. That is: 3 parts of hardener should be used.

1.2 (a)

$$f(s) = (4s+1)\left(s + \frac{2000}{s^2} + 4\right).$$

(b)

$$\begin{aligned} f'(s) &= 4\left(s + \frac{2000}{s^2} + 4\right) + (4s+1)\left(1 - \frac{4000}{s^3}\right) \\ &= 8s + 17 - \frac{8000}{s^2} - \frac{4000}{s^3}. \end{aligned}$$

If $s > 0$ then $f'(s) = 0$ if and only if $s^3 \times f'(s) = 0$ if and only if

$$8s^4 + 17s^3 - 8000s - 4000 = 0.$$

(c) Maple solves the equation above to give $s = 9.51$. (A file containing the appropriate code can be found at `www.mas.ncl.ac.uk/~najd2/teaching/mas1002/`.)

(d) The function f is differentiable at all points $s > 0$. From the previous part of the question f has a unique stationary point at 9.51 and $f(9.51) = 1390.76$. When $s \to \infty$ then $f \to \infty$ and when $s \to 0^+$ then $f \to \infty$. Therefore f has a global minimum at $s = 9.51$. When $s = 9.51$ we have $h = 2000/9.51^2 = 22.11$. The cartons should have dimensions (approximately) 9.5×22.1 cm to minimise the amount of card used.

1.3 See the solution at `www.mas.ncl.ac.uk/~najd2/teaching/mas1002/`.

(vi) To find the equation of the tangent use Procedure 2.3. $\partial f/\partial x = 2x - y$ and $\partial f/\partial y = -x$. Evaluating at $(2, 1)$ we obtain $m = 3$ and $n = -1$. Therefore the tangent at $(2, 1)$ is

$$z = 3(x-2) - 2(y-1) + 3.$$

2 Part 2, May 11th 2007

2.1

$$L = x_1^2 + 2x_2^2 + 3x_3^2 - \lambda_1(x_1 + x_2 + x_3 - 1) - \lambda_2(2x_1 - x_2 - 3).$$

Therefore we obtain equations

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 2\lambda_2 = 0, \qquad \text{(i)}$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - \lambda_1 + \lambda_2 = 0 \quad \text{and} \qquad \text{(ii)}$$

$$\frac{\partial L}{\partial x_3} = 6x_3 - \lambda_1 = 0. \qquad \text{(iii)}$$

First eliminate λ_1 and λ_2:

$$\text{(i)} - \text{(iii)}: 2x_1 - 6x_3 - 2\lambda_2 = 0. \qquad \text{(iv)}$$

$$\text{(ii)} - \text{(iii)}: 4x_2 - 6x_3 + \lambda_2 = 0. \qquad \text{(v)}$$

$$\text{(iv)} + 2 \times \text{(v)}: 2x_1 + 8x_2 - 18x_3 = 0. \qquad \text{(vi)}$$

From the second constraint $x_2 = 2x_1 - 3$ and substitution of this in the first constraint gives

$$3x_1 + x_3 - 4 = 0 \text{ if and only if } x_3 = 4 - 3x_1.$$

Substitution of these expressions for x_2 and x_3 in (vi) gives

$$2x_1 + 8(2x_1 - 3) - 18(4 - 3x_1) = 72x_1 - 96 = 0 \text{ if and only if } x_1 = 4/3,$$

so $x_2 = -1/3$ and $x_3 = 0$. Therefore L has a unique admissable stationary point at $(4/3, -1/3, 0)$.

L is differentiable everywhere. The solutions (x_1, x_2, x_3) to the constraint equations satisfy $x_2 = 2x_1 - 3$ and $x_3 = 4 - 3x_1$ and substituting these expressions into f gives $x_1^2 + 2(2x_1 - 3)^2 + 3(4 - 3x_1)^2$ which tends to ∞ as x_1 tends to $\pm\infty$. Therefore the constrained problem has no global maximum but has a global minimum at an admissable stationary point. Hence the global minimum is at $(4/3, -1/3, 0)$.

2.2 Slack variables: $x_5, x_6 \geq 0$.

Dictionary 1.

$$\begin{aligned} x_5 &= 5 - x_1 - 2x_2 - 3x_3 - x_4 \\ x_6 &= 3 - x_1 - x_2 - 2x_3 - 3x_4 \\ z &= 5x_1 + 6x_2 + 9x_3 + 8x_4. \end{aligned}$$

Choose x_1 to become non-zero. From the 1st dictionary, with $x_2 = x_3 = x_4 = 0$:

$$0 \le x_5 = 5 - x_1 \Rightarrow x_1 \le 5$$
$$0 \le x_6 = 3 - x_1 \Rightarrow x_1 \le 3$$

We must keep $x_1 \le 3$. Setting $x_1 = 3$ gives $x_6 = 0$. Now x_6 becomes a basis variable and x_1 becomes a slack variable. From the 1st dictionary $x_1 = 3 - x_2 - 2x_3 - 3x_4 - x_6$ and substitution in the expressions for x_5 and z gives

Dictionary 2.

$$\begin{aligned} x_1 &= 3 - x_2 - 2x_3 - 3x_4 - x_6 \\ x_5 &= 5 - 3 + x_2 + 2x_3 + 3x_4 + x_6 - 2x_2 - 3x_3 - x_4 = 2 - x_2 - x_3 + 2x_4 + x_6 \\ z &= 15 - 5x_2 - 10x_3 - 15x_4 - 5x_6 + 6x_2 + 9x_3 + 8x_4 = 15 + x_2 - x_3 - 7x_4 - 5x_6. \end{aligned}$$

To increase z we must now choose x_2 to leave the basis. From the 2nd dictionary with $x_3 = x_4 = x_6 = 0$:

$$0 \le x_1 = 3 - x_2 \Rightarrow x_2 \le 3$$
$$0 \le x_5 = 2 - x_2 \Rightarrow x_2 \le 2.$$

We must keep $x_2 \le 2$. Setting $x_2 = 2$ gives $x_5 = 0$ so x_5 becomes a basis variable and x_2 becomes a slack variable. From the 2nd dictionary $x_2 = 2 - x_3 + 2x_4 - x_5 + x_6$ and substitution in the expressions for x_1 and z gives

Dictionary 3.

$$\begin{aligned} x_2 &= 2 - x_3 + 2x_4 - x_5 + x_6 \\ x_1 &= 3 - (2 - x_3 + 2x_4 - x_5 + x_6) - 2x_3 - 3x_4 - x_6 = 1 - x_3 - 5x_4 + x_5 - 2x_6 \\ z &= 15 + 2 - x_3 + 2x_4 - x_5 + x_6 - x_3 - 7x_4 - 5x_6 = 17 - 2x_3 - 5x_4 - x_5 - 4x_6. \end{aligned}$$

The coefficients of variables in z are now all non-positive so we have reached a global maximum at $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = x_6 = 0$. The solution to the original problem is $(x_1, x_2, x_3, x_4) = (1, 2, 0, 0)$.

2.3 See the solution at `www.mas.ncl.ac.uk/~najd2/teaching/mas1002/`.