Generalized tetrahedron groups and the Tits alternative

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The Tits alternative

A class $\mathcal{C}$ of groups is said to satisfy the **TITS ALTERNATIVE** if every group in $\mathcal{C}$ either contains a soluble subgroup of finite index, or a nonabelian free subgroup.
The Tits alternative is known to hold for

- linear groups (Tits)
- Coxeter groups (Margulis/Vinberg/Noskov)
- mapping class groups (McCarthy/Ivanov)
- $\mathrm{Out}(F_n)$ (Bestvina/Feighn/Handel)
- one-relator groups (Karrass/Solitar)
- 3-manifold groups (Perelman/Button)
- 2-knot groups (Hillman)

Generalized triangle groups

A **GENERALIZED TRIANGLE GROUP** is

$$\langle x, y | x^\ell = y^m = W(x,y)^p = 1 \rangle$$

Example: Triangle group

$$T(\ell, m, p) = \langle x, y | x^\ell = y^m = (xy)^p = 1 \rangle$$

Rosenberger Conjecture:
Generalized triangle groups satisfy the Tits alternative.

Known for ‘most’ generalized triangle groups (Rosenberger, Levin, Fine, Howie, Williams, Benyash-Krivets, . . .).

No known counterexamples.

Generalized tetrahedron groups

A **GENERALIZED TETRAHEDRON GROUP** is

$$\langle x, y, z | x^{\ell} = y^{m} = z^{n} = W_1(x, y)^p = W_2(y, z)^q = W_3(z, x)^r = 1 \rangle$$

Example: Tetrahedron group

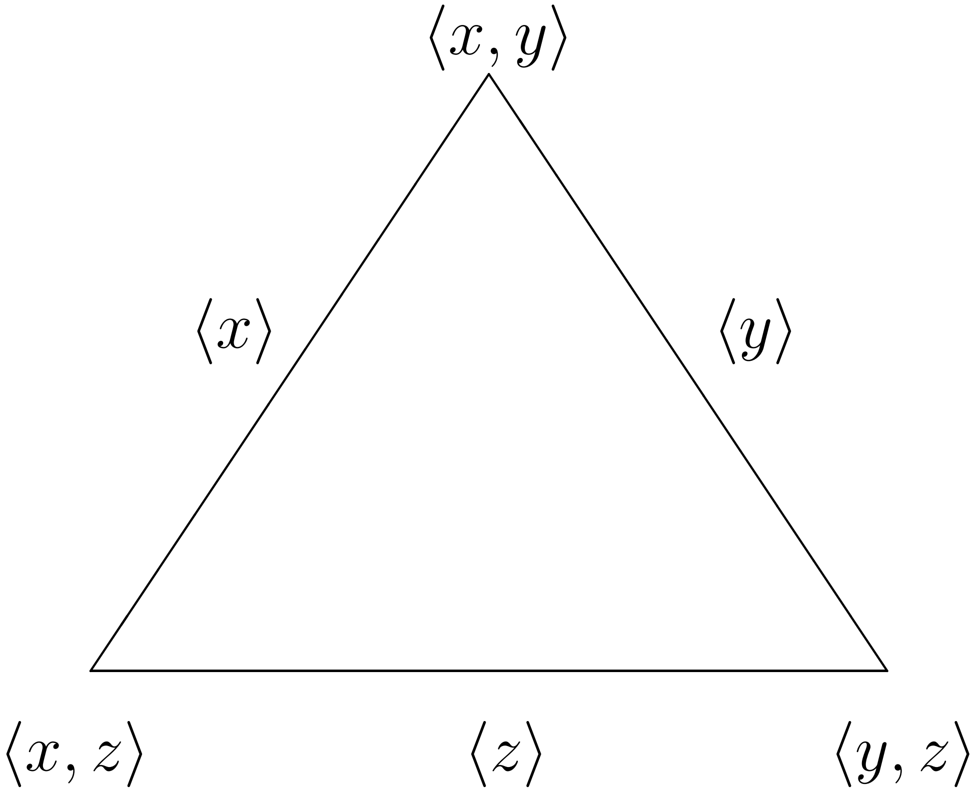
$$\langle x, y, z | x^{\ell} = y^{m} = z^{n} = (x^{-1}y)^p = (y^{-1}z)^q = (z^{-1}x)^r = 1 \rangle$$

Conjecture:

Generalized tetrahedron groups satisfy the Tits alternative.

Triangle of generalized triangle groups

A generalized tetrahedron group is the colimit of a triangle of generalized triangle groups:



Curvature

Gersten-Stallings angle. Let $A, B < G$. Then $(G; A, B) := \frac{\pi}{m}$ where $2m$ is the minimal free product length of a nontrivial element in $\mathrm{Ker}(A * B \to G)$.

A triangle is **negatively curved** if G-S angle sum is $< \pi$.
A triangle is **non-spherical** if G-S angle sum is $\leq \pi$.

THEOREM 1
(1) Negatively curved $\Rightarrow$ free subgroups.
(2) Non-spherical ‘sometimes’ $\Rightarrow$ free subgroups.

PROOF van Kampen diagrams $+$ curvature.

To classify triangles, we need to calculate G-S angles.

Spelling Theorem

THEOREM 2 In $G = \langle x, y | x^\ell = y^m = W(x,y)^p = 1 \rangle$, no nontrivial relation between x and y is shorter than W^p.

COROLLARY If $W = x^{a(1)} y^{b(1)} \cdots x^{a(k)} y^{b(k)}$, then $(G; \langle x \rangle, \langle y \rangle) = \frac{\pi}{kp}$.

Classify the tetrahedron groups into three types:

- Negatively curved (so $\exists$ free subgroups);
- Euclidean (non-spherical but not negatively curved);
- Spherical.

7 Euclidean types

E1 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^\eta z^\theta)^6 = 1 \rangle$

E2 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^4 = (x^\eta z^\theta)^4 = 1 \rangle$

E3 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^3 = (y^\gamma z^\delta)^3 = (x^\eta z^\theta)^3 = 1 \rangle$

E4 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^3 = 1 \rangle$

E5 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^4 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^2 = 1 \rangle$

E6 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^{\gamma_1} z^{\delta_1} y^{\gamma_2} z^{\delta_2})^2 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^2 = 1 \rangle$

E7 $\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2} x^{\eta_3} z^{\theta_3})^2 = 1 \rangle$

Non-spherical results

THEOREM 3 The only generalized tetrahedron group, given by non-spherical triangles, which do not contain free subgroups, are the euclidean wallpaper groups

$$\langle x, y, z | x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = 1 \rangle$$

with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

These are all abelian-by-finite.

COROLLARY Generalized tetrahedron groups given by non-spherical triangles satisfy the Tits alternative.

Methods of proof 1 - curvature

EXAMPLE In the negatively curved case, $\{(xyz)^N, (xzy)^N\}$ is a basis for a free subgroup for $N \gg 0$.

PROOF. A reduced van Kampen diagram Δ has negatively curved interior (so positively curved boundary).

A segment of $\partial\Delta$ labelled $(xyz)^N$ or $(xzy)^N$ has curvature $\leq -(\mathrm{constant}) \cdot N$.

Positive curvature at meeting of $(xyz)^N$ and $(xzy)^N$ segments is bounded.

Hence $\nexists$ reduced van Kampen diagram with boundary label $\mathrm{Word}((xyz)^N, (xzy)^N)$ for $N \gg 0$.

Methods of proof 2 - respresentations

$\exists$ representations $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ which are *essential* ($\rho(x)$, $\rho(y)$, $\rho(z)$ and the $\rho(W_i)$ all have the correct orders).

Most subgroups of $\mathrm{PSL}_2(\mathbb{C})$ contain free subgroups.

$\rho : \langle x, y | x^\ell = y^m = (x^{a(1)}y^{b(1)} \cdots x^{a(k)}y^{b(k)})^p = 1\rangle \to \mathrm{PSL}_2(\mathbb{C})$ determined up to conjugacy by a poly τ in $\mathrm{trace}(\rho(x))$, $\mathrm{trace}(\rho(y))$, and $\lambda = \mathrm{trace}(\rho(xy))$, of λ-degree k.

PROOF OF SPELLING THEOREM
If $V(x, y) = 1$, calculate entries of $\rho(V)$ under tautological representation

$$\rho : G \to \mathrm{PSL}_2\left(\mathbb{C}[\lambda]/\prod_j(\tau(\lambda) - 2\cos(j\pi/p))\right)$$

Finite generalized tetrahedron groups

Ordinary tetrahedron groups are index 2 subgroups of Coxeter groups. Classification of finite Coxeter groups $\Rightarrow$ classification of finite tetrahedron groups.

TSARANOV - classified finite generalized Coxeter groups $\Rightarrow$ classification of finite generalized tetrahedron groups of the form

$$\langle x, y, z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^p = (y^\gamma z^\delta)^q = (z^\varepsilon x^\zeta)^r = 1 \rangle$$

ROSENBERGER & SCHEER - completed classification of finite generalized tetrahedron groups, building on previous work of **ROSENBERGER, FINE, LEVIN, ROEHL, EDJVET, THOMAS, STILLE, METAFTSIS, HOWIE,LEVAI,SOUVINGNIER.**

3 Spherical types

S1 $\langle x,y,z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^2 = W_3(x,z)^r = 1\rangle$

S2 $\langle x,y,z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^\eta z^\theta)^r = 1\rangle$, $r = 3,4,5$

S3 $\langle x,y,z | x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^2 = 1\rangle$

THEOREM
The Tits alternative also holds for S2 and S3.

Proof Ad-hoc mixture of curvature, representation-theoretic, and other arguments.

Cases S2 & S3

All groups of type S2 or S3 contain free subgroups, except for 8 finite groups (10 presentations, up to equivalence), and the following 5 infinite, virtually soluble groups:

- $\langle x, y, z | x^3 = y^3 = z^2 = (xy)^2 = (yz)^3 = (xz)^3 = 1 \rangle$;
- $\langle x, y, z | x^2 = y^2 = z^3 = (xy)^2 = (yz)^3 = (xz)^4 = 1 \rangle$;
- $\langle x, y, z | x^2 = y^4 = z^2 = (xy)^2 = (yz)^3 = (xz)^4 = 1 \rangle$;
- $\langle x, y, z | x^2 = y^2 = z^4 = (xy)^2 = (yz)^3 = (xz)^3 = 1 \rangle$;
- $\langle x, y, z \,|\, x^2 = y^2 = z^3 = (xy)^2 = (yz)^3 = (xzxz^2)^2 = 1 \rangle$.

Case S1

$$G = \langle x, y, z | x^{\ell} = y^{m} = z^{n} = (x^{\alpha} y^{\beta})^2 = (y^{\gamma} z^{\delta})^2 = W_3(x, z)^r = 1 \rangle$$

WORK IN PROGRESS

FOR EXAMPLE:
Assume $W_3 = x^{e(1)} z^{f(1)} \dots x^{e(k)} y^{f(k)}$ with $k \geq 2$. Then the Tits alternative holds under any one of the following conditions:

- $r \geq 3$
- $m \geq 6$