The Andrews-Curtis Conjecture and Finite Groups

Joint work with C. Bates, A. Lubotsky and A. G. Myasnikov

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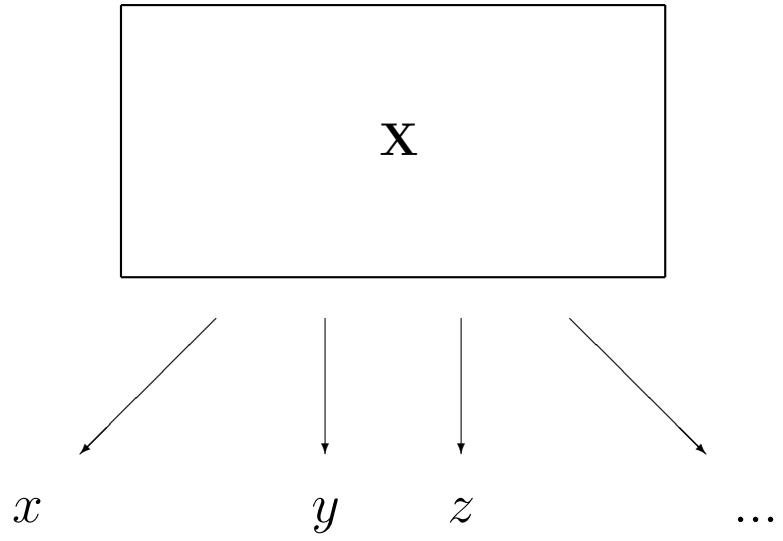
Newcastle, 29 June 2004

Black Box Group



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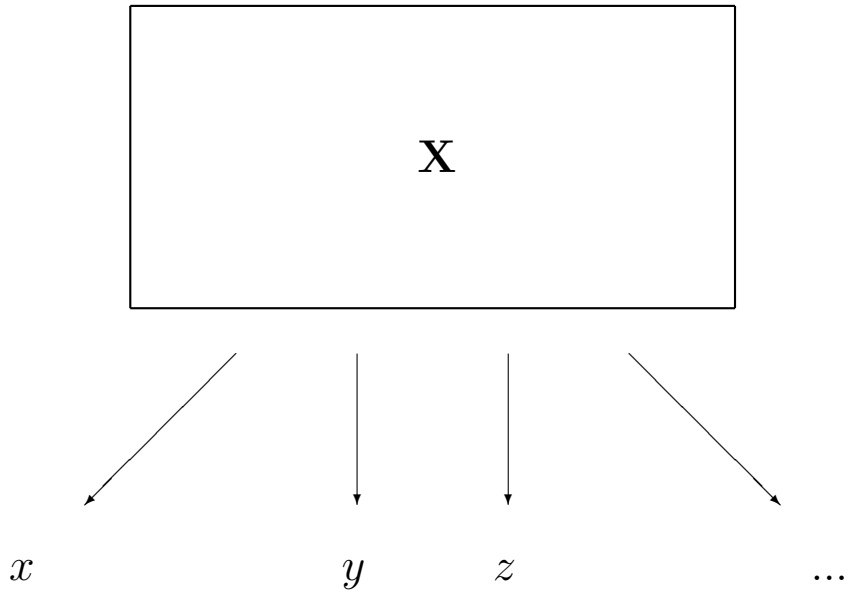
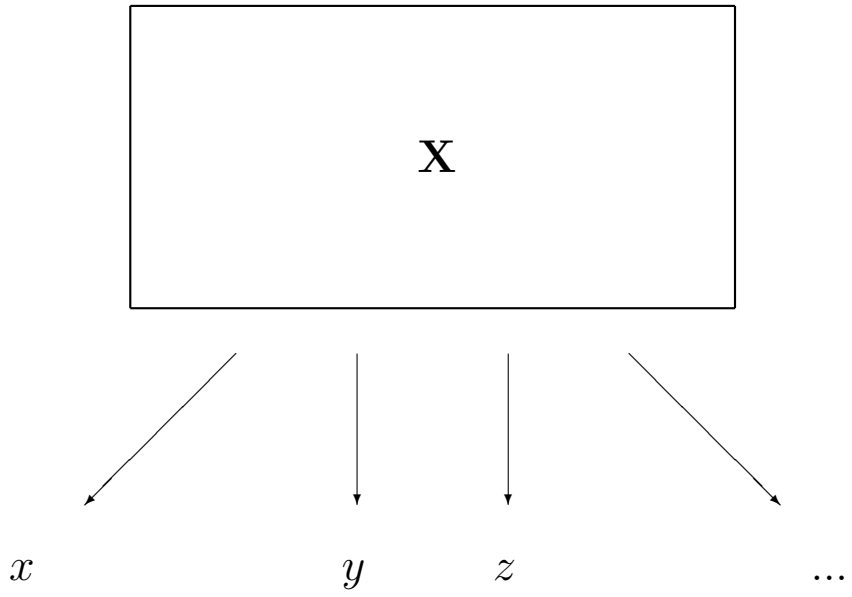
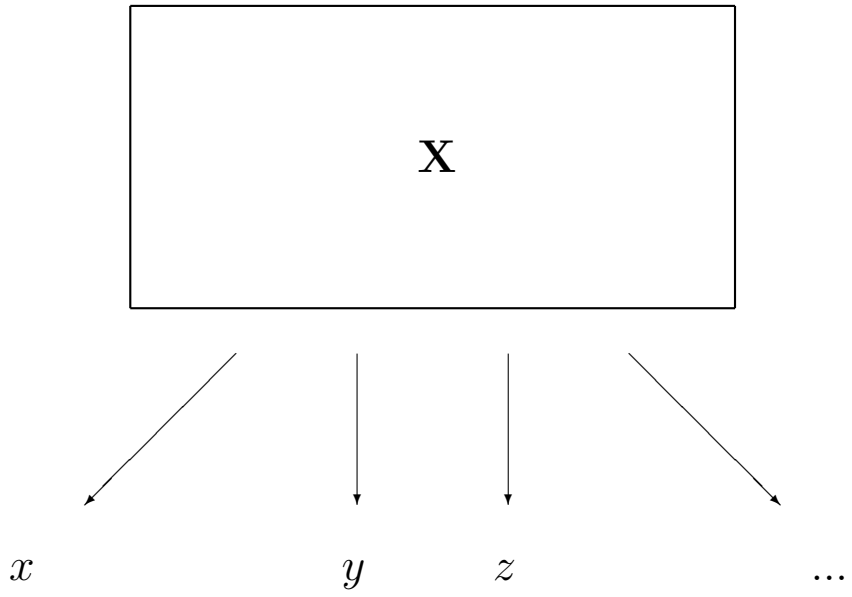
"Format": $n^2 \log q$



- We can compare: $x = y$? $\Downarrow$



- multiply, invert: $x \cdot y, \quad x^{-1}$ $\Downarrow$



- (sometimes) find orders: $|x|$

Aim: determine $\mathbf{X}$

Recognition of black box is highly technical and uses CFSG.

In this talk, I concentrate on a general set-up and relations to combinatorial group theory

Generation of random elements,

Leedham-Green et al.: ⇓

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graph with edges:

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Walk randomly over this graph and choose random g_i.

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$\Downarrow$

Random walk on $\Gamma_k(G)$

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Mixing time: number t of steps s.t. after t steps

$$\frac{1}{2} \sum_{v \in \Gamma} \left| P(\text{get at } v) - \frac{1}{\#\Gamma} \right| < \frac{1}{e}$$

Lubotzky–Pak:

If $Aut\, F_k$ satisfies Kazhdan property (T), then mixing time of a random walk on a component of $\Gamma_k(G)$

$$\mathsf{mix} \leqslant C(k) \cdot \log_2 |G|.$$

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Conjecture. For $k \geqslant 4$, $Aut\, F_k$ has (T). $\Downarrow$

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Kazhdan property (T)

G a topological group, $Q \subset G$ a compact set

$$K = \inf_{\rho} \inf_{v \neq 0} \max_{q \in Q} \frac{\|\rho(q)(v) - v\|}{\|v\|} > 0$$

ρ: all unitary representations without fixed non-zero vectors

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- How one can construct a good black box for the normal closure

$$\left\langle y_1^{\mathbf{X}}, \ldots, y_k^{\mathbf{X}} \right\rangle ?$$

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Edges:

$$(x_1, \ldots, x_k) \longrightarrow (x_1, \ldots, x_i x_j, \ldots, x_k),\ i \neq j$$

$$(x_1, \ldots, x_k) \longrightarrow (x_1, \ldots, x_i^{-1}, \ldots, x_k)$$

$$(x_1, \ldots, x_k) \longrightarrow (x_1, \ldots, x_i^w, \ldots, x_k),\ w \in G$$

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Conjecture. A random walk on $\Delta_k(G, N)$ provides a fast black box for N.

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$\Delta_k(F_k, F_k)$ is connected $\Downarrow$

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Myasnikov & Myasnikov: Some potential counterexamples (originating in topology) are killed by application of **genetic algorithms**, say, in $F = \langle x, y \rangle$,

$$(x^2y^{-3}, xyxy^{-1}x^{-1}y^{-1}) \sim (x, y)$$

(example by Akbulut and Kirbi, 1985).

Finitary AC Conjecture

- Can a counterexample to Andrews-Curtis be found at the level of finite groups?

Theorem (B, Lubotzky & Myasnikov): In a finite group G, the connected components of $\triangle_k(G, G)$ are those inherited from $G/[G, G]$. $\Downarrow$

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- No easy way to refute the AC conjecture using finite groups.

Let $G = \langle S \rangle$ and $N \lhd G$.

Restricted Andrews–Curtis graph $\overline{\Delta}_k(G, N)$:

the same vertices as in $\Delta_k(G, N)$:

$$\left\{ (h_1, \ldots, h_k) \mid \langle h_1^G, \ldots, h_k^G \rangle = N \right\}.$$

Two vertices are connected by one of the edges:

$$(\ldots, x_i, \ldots, x_j, \ldots) \longrightarrow (\ldots, x_i x_j^{\pm 1}, \ldots,), \ i \neq j$$

$$(\ldots, x_i, \ldots, x_j, \ldots) \longrightarrow (\ldots, x_j^{\pm 1} x_i, \ldots), \ i \neq j$$

$$(\ldots, x_i, \ldots) \longrightarrow (\ldots, x_i^s, \ldots), \ s \in S$$

$$(\ldots, x_i, \ldots) \longrightarrow (\ldots, x_i^{-1}, \ldots).$$

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- The **"YES"** answer would explain a good practical performance of the AC algorithm for generation random elements of normal subgroups.
- **Related question:** does the group $AC_k(F_k)$ generated by the (reduced) Andrews-Curtis transformations of the free group F_k has Kazhdan's Property (T)?

Groups generated by Nielsen moves

G a finite group.

FG_n the free group with n free generators $x_1, \ldots, x_n$ in the variety generated by G.

$\mathrm{NilFree}(G) < \mathrm{Aut}(FG_n)$ is generated by all Nielsen moves

$$\begin{aligned} L_{ij}^{\pm} &: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i, \ldots, x_i^{\pm 1} x_j, \ldots, x_n) \\ R_{ij}^{\pm} &: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i, \ldots, x_j x_i^{\pm 1}, \ldots, x_n) \\ \mathrm{Inv}_k &: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k^{-1}, \ldots, x_n) \end{aligned}$$

$\mathrm{Nil}^{+}\mathrm{Free}(G) = \left\langle L_{ij}^{\pm}, R_{ij}^{\pm} \right\rangle$

K the intersection of the kernels of all surjective homomorphisms $FG_n \longrightarrow G$

$\mathrm{Space}_n(G) = FG_n/K$.

$$\mathrm{NilFree}(G) \longrightarrow \mathrm{Aut}(\mathrm{Space}_n(G));$$

$GL_n(G)$ image of $\mathrm{NilFree}(G)$ in $\mathrm{Aut}(\mathrm{Space}_n(G))$

$SL_n(G)$ image of $\mathrm{Nil}^+\mathrm{Free}(G)$ in $\mathrm{Aut}(\mathrm{Space}_n(G))$

If $G = \mathbb{Z}/p\mathbb{Z}$ then $\mathrm{Space}_n(G) \simeq (\mathbb{Z}/p\mathbb{Z})^n$ and

$$SL_n(G) \simeq SL_n(\mathbb{F}_p).$$

Nielsen moves are transvections.

What can be said about $SL_n(G)$?

- If G is a solvable group, is it true that simple non-abelian composition factors of $SL_n(G)$ are groups $SL_k(p)$ for primes p dividing $|G/[G,G]|$ and appropriate dimensions k?

Non-abelian simple groups G.

$\mathrm{Space}_n(G)$ is the direct product of copies of G:

$$\mathrm{Space}_2(\mathrm{Alt}_5) \simeq \mathrm{Alt}_5^{19}$$

$SL_n(G)$ permutes the copies of G in $\mathrm{Space}_n(G)$.

A computer-friendly way to find the action (for small G).

- $\mathcal{G}_k(G)$ is the set of all k-tuples of elements of G which generate G.
- $\mathcal{H}_k(G)$ is the factor set of $\mathcal{G}_k(G)$ under the action of $\mathrm{Aut}(G)$.
- The Nielsen moves act on $\mathcal{H}_k(G)$ and generate a subgroup of $\mathrm{Sym}(\mathcal{H}_k(G))$; we denote it $\mathrm{Nie}_k(G)$.

Conjecture The restriction of $\mathrm{Nie}_k(G)$ to an orbit on $\mathcal{H}_k$ is the full symmetric or alternating group on this orbit.

Some experimental results (Chris Bates)

$\mathbf{Alt_5}$, rank 2

$|\mathrm{Alt}_5| = 60$

$H = \mathrm{Nie}_2(\mathrm{Alt}_5)$ acts on $\Omega = \mathcal{H}_2(\mathrm{Alt}_5)$ of cardinality 19.

Two orbits, of size 9 and 10.

The restrictions of H are Alt_9 and Sym_{10}.

$\mathbf{Alt_5}$, rank 3

$|\mathrm{Alt}_5| = 60$

The group $H = \mathrm{Nie}_3(\mathrm{Alt}_5)$ acts on $\Omega = \mathcal{H}_3(\mathrm{Alt}_5)$ of cardinality 1668 **transitively**

and is in fact isomorphic to Alt_{1668}.

$\mathbf{Alt_6}$, rank 2

$|\mathrm{Alt}_6| = 360$

$H = \mathrm{Nie}_2(\mathrm{Alt}_6)$ acts on $\Omega = \mathcal{H}_2(\mathrm{Alt}_6)$ of cardinality 53.

Four orbits of size 10, 12, 15 and 16.

The restrictions are Sym_{10}, Sym_{12}, Sym_{15} and Alt_{16}.

$\mathbf{L_3(2)}$, rank 2

$|L_3(2)| = 168$.

$H = \mathrm{Nie}_2(L_3(2))$ acts on $\Omega = \mathcal{H}_2(L_3(2))$ of cardinality 57.

Four orbits of size 7, 16, 16 and 18.

The restrictions of H are Sym_7, Alt_{16}, Alt_{16} and Sym_{18}.

S$L_2(8)$, **rank** 2

$H = \mathrm{Nie}_2(L_2(8))$ acts on $\Omega = \mathcal{H}_2(L_2(8))$ of cardinality 142.

Three orbits of size 18, 54, and 70.

The restrictions of H are Sym_{18}, Sym_{54} and Sym_{70}.

If G is simple, any tuple in

$$\Omega = G \times \cdots \times G \setminus \{(1, \ldots, 1)\}$$

generates G as a normal subgroup.

Conjecture

- The Andrews-Curtis moves generate the full symmetric or alternating group on Ω.

Root and parabolic subgroups

With every root $\rho = \epsilon_i - \epsilon_j$ of the root system

$$\Phi_{A_{n-1}} = \{\, \epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n,\ i \neq j \,\}$$

of type A_{n-1} one can associate the *root subgroup*

$$U_\rho(G) = U_{ij}(G) = \left\langle L^{\pm}_{i,j}, R^{\pm}_{i,j} \right\rangle$$

$$U_{ij}(G) \simeq G \circ G$$

(central product of two copies of G).

$U_{ij}(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$, in agreement with the standard notation for linear algebraic groups.

Unitriangular and triangular subgroups

$$\begin{aligned} UT_n^+(G) &= \langle\, U_{ij}(G) \mid i < j \,\rangle \\ &= \left\langle L_{ij}^{\pm}, R_{ij}^{\pm} \mid i < j \right\rangle \\ T_n^+(G) &= \left\langle L_{ij}^{\pm}, R_{ij}^{\pm}, \mathrm{Inv}_k \mid i < j \right\rangle \end{aligned}$$

- What is the intersection of **opposite** triangular subgroups

$$T_n^+(G) \cap T_n^-(G)?$$

Diagonal subgroup

For $g \in G$,

$$D_k(g) : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_k^g, \ldots, g_n)$$
$$\mathrm{Inv}_k : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_k^{-1}, \ldots, g_n)$$

$$\mathrm{Diag}_n(G) = \langle \mathrm{Inv}_k, D_k(g) \mid g \in G \rangle$$

The Andrews-Curtis group

$$AC_n(G) = \langle \mathrm{NielFree}_n(G), \mathrm{Diag}_n(G) \rangle$$

acts on $\Delta_n(G, G)$

Weyl group

$W = \mathrm{Sym}_n$ acting by permutation of components in $(g_1, \ldots, g_n)$.

Borel subgroup

$$B = B_n(G) = \langle UT_n(G), \mathrm{Diag}_n(G) \rangle$$

Monomial subgroup

$$N = \langle \mathrm{Diag}_n(G), W \rangle$$

(B, N) looks like a formal "non-commutative" analogue of a BN-pair in $AC_n(G)$.