Reflections on some groups of B. H. Neumann

Gilbert Baumslag

Most of the work discussed here is joint with C.F. Miller III.

Neumann's 1937 paper

- B. H. Neumann, Some remarks on infinite groups, JLMS (1937).

- Every presentation

$$G =< x_1, \ldots, x_m; R >$$

of a finitely presented group on a finite set of generators and a countably infinite set of relators contains a finite sub-presentation

$$G =< x_1, \ldots, x_m; r_1, r_2, \ldots, r_k >$$

$$(r_1, \ldots, r_k \in R, \ k < \infty)$$

.

There exist continuously many 2-generator groups. So there exist 2-generator groups which cannot be finitely presented.

• A whole new world of 2-generator groups - can be likened to transcendental numbers.

$$G =< x_1, \ldots, x_m; R >, \ m \ finite$$

Let N be the normal closure of R in F, the free group on the x_i. Let r_n be the number of elements of N of length at most n:

$$r_n =| \{w \in N \mid w \text{ of length} \leq n\}.$$

The r_n can be encoded as:

$$\rho(G) = \sum_{n=0}^{\infty} r_n x^n.$$

$$\nu(G) = \sum_{n=0}^{\infty} p_n^{-r_n}.$$

$\nu(G)$ is usually transcendental; if $\rho(G)$ is rational, G is a group with a solvable word problem. If G is a group with a solvable word problem, then ρ and ν are recursive. Almost nothing known, hard to compute. Analogous definitions when we work in the abelianizations of N, using instead of the length of w the mimimum length of an element in the coset $w[N, N]$, with corresponding functions ρ_{ab} and ν_{ab}. These turn out to be more amenable to examination. I will say a little more about this at the end of my talk.

The description of the groups B_S

Here S is any set of odd integers $n \geq 5$.

A_n is the alternating group on n symbols;

$O = \{2i+1 \mid i \geq 2\}$;

$Q = \prod_{n=2}^{\infty} A_n$ is the unrestricted direct product of all the alternating groups; $P = P_O = \prod_{n \in O} A_n$;

$P_S = \prod_{n \in S} A_n$;

ρ_S is the retraction of Q onto P_S.

$$\alpha = ((1\ 2\ 3), (1\ 2\ 3), (1\ 2\ 3), \ldots, (1\ 2\ 3), \ldots)$$

$$\tau = ((1\,2\,3\,4\,5), (1\,2\,3\,4\,5\,6\,7), \ldots, (1\,2\,3\ \ldots\ n), \ldots)$$

We will often view P_S as a subgroup of Q. So for each odd integer $j \geq 5$, we view $P_j = A_j$ also as a subgroup of Q.

$$B = B_O = gp(\alpha, \tau).$$

$$B_S = gp(\alpha_S = \rho_S(\alpha), \tau_S = \rho_S(\tau)).$$

Theorem A *(Neumann)*

$B_S \cong B_T$ if and only if $S = T$.

Corollary 1 *(Neumann) There exist continuously many non-isomorphic 2-generator groups.*

Simplest 2-generator group which is not finitely related (Baumslag and Strebel):

$$H = gp(A, B),$$

where

$$A = \begin{pmatrix} 2/3 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

• These groups are all recursively presentable. More difficult to give explicit examples of finitely generated groups which are not recursively presentable. Following theorem is largely due to Cannonito, Miller and me.

Theorem B *Let G and H be finitely generated groups. Then their wreath product $W = G \wr H$ is recursively presentable if and only if both G and H are recursively presentable and, if $G \neq 1$, then either H has a solvable word problem or G is abelian.*

Theorem C *The following are equivalent:*

1. B_S *is recursively presentable;*

2. $O \setminus S$ *is recursively enumerable;*

• This then is a direct way of concocting 2-generator groups which are not recursively presentable.

Theorem D B_S *is finitely presented if and only if* S *is finite.*

Theorem E *B is a recursively presentable group with a solvable word problem.*

Theorem F *1. S is recursively enumerable if and only if*

$$\{w(\alpha, \tau) \mid w \neq 1 \text{ in } B_S\}$$

is recursively enumerable.

2. S is recursive if and only if B_S has a solvable word problem.

• *If* $O \setminus S$ *is recursively enumerable, but* S *is not, then* B_S *is a recursively presentable, residually finite group with an unsolvable word problem.*

This is in contrast with the fact that finitely presented, residually finite groups have solvable word problem. Examples of this kind are well-known, with the first such example probably due to Verena Dyson.

Mortal and immortal words.

α has order 3; τ has infinite order.

A_n is generated by the two elements $a = (1\ 2\ 3)$ and $t_n = (1\ 2\ 3 \ldots\ n)$ when n is odd and is simple when $n \geq 5$.

Definition 2 *A word $w = w(\alpha, \tau)$ in α and τ is termed mortal if there exists an odd integer $k \geq 5$ such that $w(a_n, t_n) = 1$ for every $n > k$ and immortal if $w(a_n, t_n) \neq 1$ for every $n > k$.*

Key to presenting the various B_S is contained in the

Lemma 3 *Suppose that* $n \geq 5$ *is an odd integer. Suppose that* $w = w(\alpha, \tau)$ *is a cyclically reduced word in the free group on* α *and* τ *of length* λ *which has exponent sum 0 on* τ. *If* $n \geq \lambda + 3$, *then, setting* $a_n = a$,

1. $w(a, t_n)$ *is (freely) conjugate to a cyclically reduced word*

   $$\overline{w}(a, t_n)$$

   which moves only symbols in the range $1, \ldots, \lambda + 3$.

2. *If* $\overline{w}(a, t_n) = 1$ *then* $w(a, t_k) = 1$ *for every odd* $k \geq n$, *i.e.,* $w(\alpha, \tau)$ *is mortal.*

3. *If* $w(a, t_n) \neq 1$, *then* $\overline{w}(a, t_k) \neq 1$ *for every odd* $k \geq n$, *i.e.,* $w(\alpha, \tau)$ *is immortal.*

Not hard to deduce

Theorem G *B is a recursively presentable group with a solvable word problem.*

The structure of $B = B_O$

We denote the subgroup of Q generated by the A_j $(j \in O)$ by D and the normal closure of α in B by N. Then we find If $N = gp_B(\alpha)$, then

- B/N is infinite cyclic;

- $D \leq N$;

- N/D is isomorphic to the finitary alternating group on $\mathbb{Z}$;

- N is locally finite.

Theorem D. *If S is an infinite set of odd integers greater than or equal to 5, then B_S is not finitely presented.*

• B_S is an extension of a locally finite group by an infinite cyclic group.

Proof that B_S is not finitely presented follows from a more general theorem - special case of a variation of a theorem of Bieri and Strebel.

Theorem H *Suppose that G is a group having a locally finite, normal subgroup L with G/L infinite cyclic. If G is finitely presented, then L is finite.*

$$G = \langle t, b_1, \ldots, b_n \mid r_1 = 1, \ldots, r_m = 1 \rangle$$

here theb_i represent elements of L and the r_j have exponent sum 0 on t. Can arrange that the r_j are freely equal to words in the elements $\beta_{ik} = t^{-k} b_i t^k$ with $k \geq 0$. Since the r_j are finite in number there is a maximum value, say δ of k required so that all the r_j are freely equal to words in the β_{ik} for $k = 0, \ldots, \delta$ and $i = 1, \ldots, n$.

$$G = \langle t, \beta_{ik}\, (1 \le i \le n,\ 0 \le k \le \delta) \mid q_1 = 1, \ldots, q_m,$$

$$t^{-1}\beta_{ik}t = \beta_{i\ k+1}\, (1 \le i \le n,\ 0 \le k \le \delta{-}1)\rangle$$

where the q_j are the r_j expressed as words in the β_{ik}.

Let H be the subgroup generated by the β_{ik} - this is a finite group. Enlarge the set of relations q_j so that they give a presentation for H on the generators β_{ik}. Let

$$C_0 = gp(\beta_{ik}\ (1 \le i \le n,\ 0 \le k \le \delta - 1)$$

$$C_1 = gp(\beta_{ik}\ (1 \le i \le n,\ 1 \le k \le \delta).$$

C_0 and C_1. So G is an HNN extension of the finite group H.

H and $t^{-1}Ht$ generate their free product with amalgamation. Since L is finite, it follows $C_0 = C_1 = H = L$.

Presentations of finitely generated groups

There are criteria which ensure that finitely generated groups are not recursively presentable.

- the center of a recursively presentable group is a recursively presentable abelian group;

- the integral homology groups of finitely generated recursively presentable groups are recursively presentable;

• amalgamated products of finitely generated groups with subgroups which cannot be recursively enumerated are not recursively presentable.

Finitely generated groups which are not recursively presentable have received scant attention.

Liouville constructed transcendental numbers by proving that there exists a limit to which an algebraic number not rational can be approximated by rational numbers:

Theorem I *Let* α *be an algebraic number of degree* $n > 1$*. Then there exists* $c = c(\alpha) > 0$ *such that*

$$\mid \alpha - p/q \mid > c/q^n$$

for every rational number p/q*.*

Is there a group-theoretic version of this theorem?