Elimination Theory I: Unification Theorems and Limits of groups

Alexei Miasnikov

Alagna
December 15, 2008

Outline

This is a first talk in a series of three.

The goal of the series is to outline the main ideas of the **elimination theory** in groups.

It will provide some basics for the series of talks on:

Equations with rational constrains by V. Diekert,

Tarski's Problems by O. Kharlampovich

Equations in right angled Artin groups by M. Casal-Ruis and I. Kazachkov.

and give some examples in support of the series of lectures by V. Remeslennikov and E. Daniyarova on Universal algebraic geometry.

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- Gauss elimination in modules,
- Hall collection in nilpotent groups (or standard bases in polycyclic groups),
- Nielsen method

all three are now unified in the modern framework of modern **Grobner basis** theory or **Knuth-Bendix** method.

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The modern ideas of ET in groups stem from the theory of equations in free groups and semigroups.

Makanin-Razborov process plays an important part in this theory.

In the first lecture I am going to discuss the first level of the elimination theory in groups.
Its scope can be described as the group-theoretic counterpart of the basic **quantifier-free model theory**, or **universal algebraic geometry**.

In group theory it appears in the form of:

- universal theories of groups,
- algebraic geometry over groups,
- residual theory or discrimination of groups,
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Algebraic geometry over groups

G - a group generated by A,
$F(X)$ - free group on $X = \{x_1, x_2, \dots x_n\}$.

A **system of equations** $S(X, A) = 1$ in variables X and coefficients from G (viewed as a subset of $G * F(X)$).

A **solution** of $S(X, A) = 1$ in G is a tuple $(g_1, \dots, g_n) \in G^n$ such that $S(g_1, \dots, g_n) = 1$ in G.

$V_G(S)$, the set of all solutions of $S = 1$ in G, is called an **algebraic set** defined by S.

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The maximal subset $R(S) \subseteq G * F(X)$ with

$$V_G(R(S)) = V_G(S)$$

is the **radical** of $S = 1$ in G.

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the **coordinate group** of $S = 1$.

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Radicals and coordinate groups

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Zariski topology

The following conditions are equivalent:

- G is equationally Noetherian, i.e., every system $S(X) = 1$ over G is equivalent to some finite part of itself.
- the Zariski topology (formed by algebraic sets as a sub-basis of closed sets) over G^n is Noetherian for every n, i.e., every proper descending chain of closed sets in G^n is finite.
- Every chain of proper epimorphisms of coordinate groups over G is finite.

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Irreducible components

If the Zariski topology is Noetherian then every algebraic set can be uniquely presented as a finite union of its irreducible components:

$$V = V_1 \cup \ldots V_k.$$

Recall, that a closed subset V is irreducible if it is not a union of two proper closed (in the induced topology) subsets.

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Irreducible components

Embedding theorem

Let G be an equationally Noetherian. Then for every system of equations $S(X) = 1$ over G there are finitely many irreducible systems $S_1(X) = 1, \ldots, S_m(X) = 1$ (that determine the irreducible components of the algebraic set $V(S)$) such that

$$G_{R(S)} \hookrightarrow G_{R(S_1)} \times \ldots \times G_{R(S_m)}$$

Limit groups

Unification Theorems

Let G be a finitely generated group and $F \leq G$. Then the following conditions are equivalent:

1) G is the coordinate group of an irreducible variety over F.
2) G is discriminated by F, i.e. for any finite subset $M \subseteq G$ there exists an F-homomorphism $G \longrightarrow F$ injective on M.
3) G is universally equivalent to F;
4) G is a limit of free groups in Gromov-Hausdorff metric.
5) G is a Sela's limit group.

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Unification theorems for limits of free groups

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General Unification Theorem for groups: No coefficients

Let G be an equationally Noetherian group. Then for a finitely generated group H the following conditions are equivalent:

1. $\mathrm{Th}_\forall(G) \subseteq \mathrm{Th}_\forall(H)$, i.e., $H \in \mathbf{Ucl}(G)$;
2. $\mathrm{Th}_\exists(G) \supseteq \mathrm{Th}_\exists(H)$;
3. H embeds into an ultrapower $\prod G/D$ of G;
4. H is discriminated by G;
5. H is a limit group over G;
6. H is defined by a complete atomic type in the theory $\mathrm{Th}_\forall(G)$ in the first-order group language;
7. H is the coordinate algebra of an irreducible non-empty algebraic set over G defined by a system of coefficient-free equations.

Equational Noetherian Property is the key to the unification theorem.

An analog of the Hilbert Basis Theorem.

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Lyndon: introduced **free exponential groups** $F^{\mathbb{Z}[t]}$ over polynomials $\mathbb{Z}[t]$ (to describe solution sets of one-variable equations).
He showed also: $F^{\mathbb{Z}[t]}$ *is discriminated by* F.

M. and Remeslennikov

$F^{Z[t]}$ can be obtained as union of an infinite chain of extensions of centralizers:

$$F = G_0 < G_1 < \ldots < \ldots \cup G_i = F^{Z[t]}$$

where

$$G_{i+1} = \langle G_i, t_i \mid [C_{G_i}(u_i), t_i] = 1 \rangle.$$

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Structure of subgroups of $F^{\mathbb{Z}[t]}$

From Bass-Serre theory

Finitely generated subgroups of $F^{Z[t]}$ are fundamental groups of very particular graphs of groups.

The Second Embedding Theorem

The Second Embedding Theorem [Kharlampovich and M. 96]:

Let $S = 1$ be an irreducible system $S = 1$ over F. Then:

- There is an embedding of $F_{R(S)}$ into a group G_i which is obtained from F by finitely many extensions of centralizers. Such an embedding can be found effectively.
- There is an embedding of $F_{R(S)}$ into Lyndon's group $F^{\mathbb{Z}[t]}$. Such an embedding can be found effectively.

This allows one to study the coordinate groups of irreducible systems of equations (fully residually free groups) via their splittings into graphs of groups.

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Corollary

- Every finitely generated fully residually free group is finitely presented. There is an algorithm to find a finite presentation.
- For every non-abelian finitely generated fully residually free group one can effectively find its non-trivial splitting (as a free product, or an amalgamated product, or an HNN extension over a cyclic subgroup)
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Limits of hyperbolic groups

Theorem [Sela]

Torsion-free hyperbolic groups are equationally Noetherian.

Unification Theorem

Let H be a torsion-free hyperbolic group and G a finitely generated group with $H \leq G$. Then the following conditions are equivalent:

1) G is discriminated by H;
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Theorem [BMR]

Let H be a torsion-free hyperbolic group. Then

- $H^{\mathbb{Z}[t]}$ is a union of extension of centralizers.
- $H^{\mathbb{Z}[t]}$ is fully residually (discriminated by) H.

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Embeddings

The Embedding Theorem [Kharlampovich, Myasnikov]

Let H be a torsion-free hyperbolic group. Then:

- Every finitely generated fully residually H group embeds into into a group obtained from H by finitely many extensions of centralizers.
- Every finitely generated fully residually H group embeds into $H^{Z[t]}$.

All the standard corollaries follow.

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Limits of solvable groups

Definition

A series of normal subgroups of G

$$G = G_1 > G_2 > \ldots > G_n > G_{n+1} = 1 \quad (1)$$

is called *principal* if the factors G_i/G_{i+1} are abelian groups which do not have torsion as $\mathbb{Z}[G/G_i]$-modules, $i = 1, \ldots, n$.

Definition

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Uniqueness of the principal series

Theorem

- Every rigid group has only one principal series.
- The length of the principal series is equal exactly to the solvability class of G.

For a rigid group G with a principal series

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the group G/G_i also has a principal series

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Examples

Proposition

The following hold:

1) Rigid groups are torsion-free solvable groups.
2) Torsion-free abelian groups are rigid.
3) Subgroups of rigid groups are rigid.
4) Direct products of two groups, one of which is non-abelian, is not rigid.
5) Non-abelian groups with non-trivial center are not rigid.

Examples

Proposition

The following hold:

1) Free solvable groups are rigid.
2) If A is torsion-free abelian and B is rigid then the wreath product $A \wr B$ is rigid.

Let A_m be a free abelian group of rank m.
Put $W(m,0) = A_m$ and define $W(m,n)$ by induction

$$W(m,n) = A_m \wr W(m,n-1).$$

Definition

$W(m,n)$ is an iterated wreath product of n free abelian groups A_m.

The following result gives a nice characterization of finitely generated rigid groups.

Theorem

Let G be an n-rigid m-generated group. Then G embeds into $W(m,n)$. Conversely, the group $W(m,n)$ is rigid, so every finitely generated subgroup of $W(m,n)$ is rigid.

Put $\mathcal{W} = \cup_{n=1}^{\infty} W(n,n)$. Then

f.g. rigid groups = f.g. subgroups of $\mathcal{W}$.

The following result gives a nice characterization of finitely generated rigid groups.

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Theorem, [Gupta,Romanovskii, 2007]

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Let A be a rigid group. Then the coordinate groups of irreducible algebraic sets over A are rigid.

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- Finitely generated groups discriminated by a rigid group are rigid;
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Principal dimension

Let G be a rigid group with the principal series

$$G = G_1 > G_2 > \ldots > G_n > G_{n+1} = 1.$$

Then G_i/G_{i+1} is a torsion-free $\mathbb{Z}[G/G_i]$-module.

Since the group G/G_i is solvable and torsion-free the group ring $\mathbb{Z}[G/G_i]$ is an Ore domain (P.H.Kropholler, P.A.Linnell and J.A.Moody),

Hence $\mathbb{Z}[G/G_i]$ embeds into its ring of fractions $K_i(G)$ which is a division ring.

Since the G_i/G_{i+1} has no $\mathbb{Z}[G/G_i]$-torsion it embeds into its tensor completion $V_i(G) = G/G_i \bigotimes_{\mathbb{Z}[G/G_i]} K_i(G)$, which is a vector space over $K_i(G)$.

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Principal dimension

Put

$$r_i(G) = \dim_{K_i(G)}, \quad r(G) = (r_1(G), \ldots, r_n(G)).$$

The tuple $r(G)$ is the *principal dimension* of G.

Lemma

Let G be an n-rigid group. If G is generated by m elements then $r_1(G) \leq m$ and $r_i(G) \leq m - 1$ $(2 \leq i \leq n)$.

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Lemma

Let G and H be n-rigid groups. If $\varphi : G \to H$ is a proper epimorphism then $r(G) > r(H)$ in the left lexicographical order.

Theorem

Let A be a finitely generated n-rigid group. Then

1) Every strictly decreasing chain of irreducible closed sets in A^m has length at most $(m+1)^n$.
2) The Zariski dimension of any irreducible algebraic set from A^m does not exceed $(m+1)^n$. In particular, the Zariski dimension of A^m is finite for every m.

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Limits of nilpotent groups

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Let N be a finitely generated torsion-free nilpotent group. Then:

1) Every finitely generated N-limit group is a subgroup of $N^{\tilde{\mathbb{Z}}}$, where $\tilde{\mathbb{Z}} = \Pi\mathbb{Z}/D$ is an ultrapower of the ring $\mathbb{Z}$ over a non-principal ultrafilter D, and $N^{\tilde{\mathbb{Z}}}$ is the Hall completion of N over $\tilde{\mathbb{Z}}$.
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Prime ideals from a different view-point

Note:

- If k is algebraically closed then an ideal p of $k[X]$ is prime if and only if the k-algebra $k[X]/p$ is k-discriminated by k (hence k and $k[X]/p$ mutually discriminate each other) .
- Equivalently, p is prime iff $k[X]$ and $k[X]/p$ mutually discriminate each other (as k-algebras).
- Equivalently, p is prime iff $k[X]$ and $k[X]/p$ are universally equivalent (as k-algebras).

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We say that a normal subgroup N of a group H is a *prime ideal* in H if H and H/N are universally equivalent.

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Krull dimension $Kdim(H)$ of a G-group H is the supremum of all natural numbers k such that exists a chain

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of distinct prime ideals in H.

Example: $Kdim(\mathbb{Z}) = 1$.

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- If G is a free abelian group of rank n then $Kdim(G) = n$.
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In the case of free or hyperbolic groups much less is known.

If F is a free group then Zariski dimension of F^1 is equal to 2 (Appel, Lorents).

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