Generic and measurable subsets in free groups

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Let $G = A *_C B$ be an amalgamated product of groups (or HNN extension of some group). Free products with amalgamation and HNN extensions are among the most studied classical constructions in algorithmic and combinatorial group theory. Methods developed to study the Word and Conjugacy Problems in these groups became the classical models much imitated in other areas of group theory.

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The striking undecidability results of this sort scared away any general research on the word and conjugacy problems in amalgamated free products and HNN extensions. The classical tools of amalgamated products have been abandoned and replaced by methods of hyperbolic groups in works of M. Bestvina and M. Feighn, I. Kapovich and A. Myasnikov, K. Mikhajlovski and A. Yu. Olshanskii, or automatic groups (G. Baumslag, S. Gersten, M. Shapiro and H. Short, D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson and W. Thurston), or relatively hyperbolic groups (by I. Bumagina, D. Osin).

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Our approach treats both decidable and undecidable cases simultaneously.
In this way we subdivide the set of inputs of algorithms onto two parts of "good" and "bad" inputs. On such a "good" part our algorithm works fast (for example, in polynomial time) and on "bad" it work slowly or doesn't work at all. Main idea of this stratification of algorithm's inputs is to show, that the "good" strata is large enough (i.e. generic) in the set of all inputs and the bad one is very small set (or so called negligible set).

Except of Word and Conjugacy problem itself, there are a lot of algorithmic problems has considered in this series of works below was researched (new algorithm for word, conjugacy problems, for membership, malnormality problem; algorithms for constructions of all type of normal forms). important For most of them we also need to transform one normal form to another and also to generate the random normal form for the elements of G.

Next step we have to do is a attempt to estimate the asymptotic behavior of such a forms during rewriting processes and stratify the sets of random generated forms.

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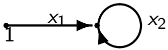
Subgroup graphs and coset graphs

Let $F = F(X)$ be a free group with basis $X = \{x_1, \ldots, x_n\}$. Let $C = \langle h_1, \ldots, h_m \rangle$ be a subgroup of F generated by finitely many elements $h_1, \ldots, h_m \in F$.
One can associate with C *the subgroup graph* $\Gamma = \Gamma_C$. For our purposes, the crucial property of Γ is that it is a directed finite connected graph with edges labelled by elements from X. For example, this is the graph for the subgroup generated by $x_1 x_2 x_1^{-1}$:



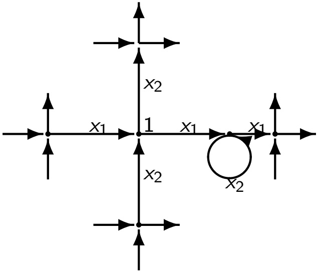
Pic. 1.

Reading labels on all paths in Γ without backtracking which start and end at 1, we get all reduced words which belong to C. There is an obvious way to translate this construction into the language of (deterministic) finite state automata. Another important property of Γ is that it is unique up to isomorphism of directed X-labelled graphs with distinguished vertices 1.

Subgroup graphs and coset graphs

The *extended subgroup graph* Γ^* of Γ is made from Γ by the following procedure: for every vertex v of Γ and every letter $x \in X$ such that there is no edge in Γ which starts at v and has label x, we attach to v an edge labelled x which leads outside of Γ, and then continue this edge by attaching a tree with every possible label in such a way that the resulting graph is folded.

Here is the fragment of the graph Γ^* for the previous example $C = \left\langle x_1 x_2 x_1^{-1} \right\rangle$:



Pic. 2.

Subgroup graphs and coset graphs

Notice that $\Gamma = \Gamma^*$ if and only if the subgroup C has finite index in F. Indeed, $\Gamma = \Gamma^*$ if and only if for every vertex v of Γ and every label $x \in X$, there is an edge in Γ labelled by x which exits from v, and an edge with label x which enters v, but this is precisely the characterization of subgroups of finite index in F.

A spanning subtree T of Γ with the root at the vertex 1 is called *geodesic* if for every vertex $v \in V(\Gamma)$ the unique path in T from 1 to v is a geodesic path in Γ. For a given graph Γ one can effectively construct a geodesic spanning subtree T.
From now on we fix an arbitrary spanning subtree T of Γ. The tree T uniquely extends to a spanning subtree T^* of Γ^*.

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Subgroup graphs and coset graphs

Let $V(\Gamma^*)$ be the set of vertices of Γ^*. For a subset $V_0 \subseteq V(\Gamma^*)$ and a subgraph Δ of Γ^*, we define the *language accepted by Δ* and V_0 as the set $L(\Delta, V_0, 1)$ of all words formed by the labels on edges in all paths in Δ which start at 1, end at one of the vertices in V_0, and have no backtracking.
Notice that all words in $V_0 \subseteq V(\Gamma^*)$ are reduced since the graph Γ^* is folded.

Proposition

In the notations above the following statements hold.

- $F = L(\Gamma^*, V(\Gamma^*), 1)$.
- $C = L(\Gamma, \{1\}, 1) = L(\Gamma^*, \{1\}, 1)$.
- $S_T = L(T^*, V(\Gamma^*), 1)$ *is a set of representatives of the right cosets of C in F.*

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Subgroup graphs and coset graphs

Let S be a (right) transversal of C. A representative $s \in S$ is called *internal* if it belongs to $V(\Gamma)$. Since Γ is uniquely defined by C, internal representatives are well-defined. By S_{int} we denote the set of all internal representatives in S. Elements from $S_{ext} = S \setminus S_{int}$ are called *external* representatives in S.

The fact that S_T is a transversal of C allows to view the extended graph Γ^* as the coset graph of C. Recall that the *coset graph* $\Gamma^\circ = \Gamma_H^\circ$ of a subgroup H of F is a connected labelled digraph which consists of the set of all (right) cosets of H in F as the set of vertices and such that there is an edge from Hu to Hv with label $x \in X^{\pm 1}$ if and only if $Hux = Hv$.

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Subgroup graphs and coset graphs

Proposition

Let C be a finitely generated subgroup of F. Then:

1. $\Gamma_C^* = \Gamma_c^\circ$.
2. S_T *is a* Shreier *transversal of C (i.e., every initial segment of a representative from S_T again belongs to S_T).*
3. *For every Shreier transversal S of C there exists a spanning subtree T' of Γ_c° such that $S = S_{T'}$.*
4. *For every Shreier transversal S' of C there exists a spanning subtree T of Γ such that $S' = S_T$.*

Corollary

For a finitely generated subgroup C of a free group F the number of different Shreier systems coincides with the number of spanning subtrees for a subgroup graph Γ_C.

It is allows to identify elements from a given Shreier transversal S of C with the vertices of the graph Γ^*. We will use this frequently in the sequel.

Special systems of representatives

We introduce various types of representatives for a finitely generated subgroup $C < F$.

Definition

Let S be a transversal of C.

- A representative $s \in S$ is called *geodesic* if it has minimal possible length in its coset Cs. The transversal S is geodesic if every $s \in S$ is geodesic. If it also Shreier, we would call such set of representatives a *special system of representatives*.
- A representative $s \in S$ is called *singular* if it belongs to the generalized normalizer $N_F^*(C) = \{f \in F | f^{-1}Cf \bigcap C \neq 1\}$ of C. All other representatives from S are called *regular*. By S_{sing} and, respectively, S_{reg} we denote the subsets of singular and regular representatives in S.
- A representative $s \in S$ is called *stable* if $sc \in S$ for any $c \in C$. By S_{st} we denote the set of all stable representatives in S, and the set of all *unstable* elements $S_{\mathrm{nst}} = S \setminus S_{\mathrm{st}}$.

Special systems of representatives

In the following lemma we collect some basic properties of various types of representatives. Recall that a *cone* defined by (or based at) an element $u \in F$ is a set of all reduced words in F that start with u.

Proposition

Let S be a Shreier transversal for C, so $S = S_T$ for some spanning subtree T of Γ. Then the following statements hold:

1) $|S_{\rm int}| = |V(\Gamma)|$.

2) $S_{\rm ext}$ *is the union of finitely many coni that are centered at vertices in* $\Gamma^* \smallsetminus \Gamma$ *which are immediate neighbours of vertices from* Γ.

3) $S_{\rm sing}$ *is contained in a finite union of double cosets* $Cs_1 s_2^{-1} C$ *of* C, *where* s_1, s_2 *are initial segments of some Nielsen base of* C.

4) $S_{\rm nst}$ *is contained in a finite union of left cosets of* C *of the type* $s_1 s_2^{-1} C$, *where* $s_1, s_2 \in S_{\rm int}$.

Generic and negligible sets

Consider a certain set F, and let $\mathcal{P}(F)$ is a set of all subsets of F and $\mathcal{A} \subset \mathcal{P}(F)$. A real non-negative additive function $\mu : \mathcal{A} \to \mathbb{R}^+$ called *pseudo-measure on F* if $\mathcal{A}$ is closed under complements.

If $\mathcal{A}$ is a subalgebra of $\mathcal{P}(F)$ then μ is a *measure*.
Let $F = F(X)$ be a free group. Denote by S_n and B_n correspondingly the sphere and the ball of radius n in F. Let μ be an atomic pseudo-measure on F. Recall, that measure μ on the countable set P called *atomic*, if every subset $Q \subseteq P$ is measurable; it is also holds $\mu(Q) = \sum\limits_{q \in Q} \mu(q)$.

Generic and negligible sets

For a set $R \subseteq F$ we define its *spherical asymptotic density of R with respect to μ* as the following limit (if it exists):

$$s\rho_\mu(R) = \limsup_{n \to \infty} s\rho_n(R)$$

where $s\rho_n(R) = \dfrac{\mu(R \bigcap S_n)}{\mu(S_n)}$.

Similarly, we define *ball asymptotic density of R with respect to μ* as the limit (if it exists):

$$b\rho_\mu(R) = \limsup_{n \to \infty} b\rho_n(R)$$

where $b\rho_n(R) = \dfrac{\mu(R \bigcap B_n)}{\mu(B_n)}$.

For example, if μ is the cardinality function, i.e. $\mu(A) = |A|$, then we obtain standard asymptotic density functions on F.

Generic and negligible sets

Lemma

Let μ be a pseudo-measure on F and $R \subseteq F$. Suppose that $\lim_{n\to\infty} \mu(B_n) = \infty$. Then for any subset R, if the spherical asymptotic density $s\rho_\mu(R)$ exists, the ball asymptotic density $b\rho_\mu(R)$ also exists and

$$b\rho_\mu(R) = s\rho_\mu(R).$$

Further, let μ be a pseudo-measure on F.

Definition

We say that a subset $R \subseteq F$ is *generic with respect to μ*, if $\mu(R) = 1$. If $s\rho_\mu(R)$ $(b\rho_\mu(R))$ is an asymptotic density of R with respect to μ, then we say that R is *strongly generic with respect to $s\rho_\mu(R)$ $(b\rho_\mu(R))$*, if the corresponding limit exists and $s\rho_n(R)$ $(b\rho_n(R))$ converges to 1 exponentially fast.

Definition

The complement $\overline{R}$ to a generic set R is called *negligible with respect to $s\rho_\mu(R)$ $(b\rho_\mu(R))$* set. $\overline{R}$ is also *strongly negligible*, if in addition relative frequencies for R converges to 1 exponentially fast.

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Free product of subsets

Consider a group A, given by (finite) presentation $A =< Y | R_A = 1 >$, and also a group $B =< Z | R_B = 1 >$ with the condition $Y \bigcap Z = \emptyset$.

Definition

Let $G = A * B$ be a free product of finitely generated groups A and B, and R is a nonempty subset of A, S is a nonempty subset of B. Then a *free product* $R * S$ is a set of all elements in G having a form

$$f = f_1 f_2 \dots f_k,$$

where $k = 1, 2, \dots;\ f_i \in R \bigcup S,\ i = \{1, \dots k\},\ f_i \neq 1$ if $i \geq 2$ and for all $i \in \{1,\ k-1\}$ elements f_i and f_{i+1} belong to different multipliers of G.

The number $k = s(f)$ is called an *syllable length* of f.

Free product of subsets

Let μ_A and μ_B are atomic pseudo-measures on A and B correspondingly and $\mu_A(1) = \mu_B(1)$. Let we also fix a probability distribution $\theta : \mathbb{N} \to \mathbb{R}^+$, i.e. $\sum\limits_{n=1}^{\infty} \theta(n) = 1$. We define an atomic measure μ on G in the following way.

Let

$$f = f_1 \ldots f_k, \text{ where} \tag{1}$$

$f_i \in A \bigcup B,\ i = 1, \ldots, k,\ f_i \neq 1$, if $i \geq 2$ and for all $i \in 1, \ldots, k-1$ elements f_i and f_{i+1} belong to different factors of G.

Set by definition

$$\mu(f) = \frac{1}{2}\theta(k)\mu_{F_1}(f_1) \ldots \mu_{F_k}(f_k), \tag{2}$$

where $F_i = A$, if $f_i \in A$, or $F_i = B$, if $f_i \in B$; $i = 1, \ldots, k$.

If $R \subseteq G$ then

$$\mu(R) = \sum_{f \in R} \mu(f).$$

We will say, that R is a *$\mu-$measurable* set, if $\mu(R) < \infty$. Denote by M_φ the set of all $\mu-$measurable subsets of G :

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Free product of subsets

Lemma

1) *If μ_A, μ_B are atomic pseudo-measures on A and B correspondingly, then measure μ on G, defined above is atomic pseudo-measure and $M_{\mu_A} \subset M_\mu, M_{\mu_B} \subset M_\mu$.*
2) *If μ_A, μ_B are atomic probability measures on A and B correspondingly, then μ is an atomic probability measure on G.*

Example. Let μ_A, μ_B are pseudo-measures on A and B, defined by cardinality functions on A and B and $\theta(n) = \frac{6}{\pi^2 n^2}$ is a probability distribution on $\mathbb{N}$. Then $M_{\mu_A} = \mathcal{F}(A)$ and $M_{\mu_B} = \mathcal{F}(B)$, where $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are sets of all finite subsets of A and B. But $M_\mu \supset \mathcal{F}(G)$ is a strict inclusion (where $\mathcal{F}(G)$ is a set of all finite subsets of G. Indeed, let $R \subseteq F$ and $R_k = R \bigcap G_k$. Then $R \in M_\mu$ iff row $\sum \frac{|R_k|}{k^2}$ converges.

Bidimensional asymptotic density

Let $T = R * S \subset G = A * B$. For a pair of natural numbers (n,k) we define the (n,k)-*ball*:

$$T_{n,k} = \{f = f_1 \dots f_l \in T : l \leq k, |f_i| \leq n, i \in (1, \dots k)\}.$$

We call $T = \bigcup_{k=1,n=1}^{\infty} T_{n,k}$ *bidimensional decomposition* for T.

Bidimensional decomposition helps us to analyze asymptotic behavior of $Q \subset T$.
A function $(n,k) \to \mu(Q \bigcap T_{n,k})$ called the growth function of Q on T, and a function $(n,k) \to \rho_{\mu,T}^{n,k}(Q)$ called the frequency function of Q with respect to T, where

$$\rho_{\mu,T}^{n,k}(Q) = \frac{\mu(Q \bigcap T_{n,k})}{\mu(T_{n,k})}.$$

Further we will consider a *bidimensional asymptotic density* of Q with respect to T, which determines as following limit (if it exists):

$$\nu\rho_T(Q) = \limsup_{\substack{n \to \infty \\ k \to \infty}} \rho_{\mu,T}^{n,k}(Q).$$

Bidimensional asymptotic density

Theorem

Let $G = A * B$ be a free product of free groups A and B of finite rank, μ is a measure on G, defined by formula (2) above and $R_0 \subseteq R \subseteq A$, $S_0 \subseteq S \subseteq B$. Then:

1) *Let μ_A, μ_B are probability measures on A, B correspondingly. Let there is the number $0 < q < 1$, such that $\frac{\mu_A(R_0 \bigcap S_n)}{\mu_A(R \bigcap S_n)} < q$ for all $n \geq n_0$ or $\frac{\mu_B(S_0 \bigcap S_n)}{\mu_B(S \bigcap S_n)} < q$ for all $n \geq n_0$
Then $R_0 * S_0$ is a strongly $\mu-$negligible with respect to $R * S$.*

2) *Let μ_A, μ_B are pseudo-measures on A and B defined by cardinality functions and numbers $\nu\rho_R(R_0)$ and $\nu\rho_S(S_0)$ exist and one of them less then 1.
Then set $R_0 * S_0$ is strongly $\mu-$negligible with respect to $R * S$.*

Bidimensional Cesaro asymptotic density

Let $\{\mu_s,\ 0 < s < 1\}$ is a family of probabilistic distributions on a free group F of finite rank m, where

$$\mu_s(w) = \frac{s(1-s)^{|w|}}{2m(2m-1)^{|w|-1}} \quad \text{for } \ w \neq 1, \ \ |w| \text{ is a length of w,} \\ \text{and} \quad \mu_s(1) = s. \tag{3}$$

If $R \subseteq F$ by $\mu_s(R) = \sum_{w \in R} \mu_s(w)$ we will denote the measure of the subset R in F.

Bidimensional Cesaro asymptotic density

Denote by $f_k = \dfrac{|R \bigcap S_k|}{|S_k|}$ relative frequencies for R.
In [3] was shown, that $\mu_s(R) = s \sum f_k(1-s)^k$, and an average length of words in F, distributed according to μ_s is equal to $l = \dfrac{1}{s} - 1$. This length is finite and $l \to \infty$ while $s \to 0^+$. Thus measures can be parametrized by number l : $\{\mu_l | \ 1 < l < \infty\}$.
For a subset R of F in [3] was defined the limit measure $\mu_0(R)$:

$$\mu_0(R) = \lim_{s \to 0^+} \mu_s(R) = \lim_{s \to 0^+} s \sum_{k=0}^{\infty} f_k (1-s)^k. \tag{4}$$

If the limit (4) exists for the set R, we would call this set *$C-$measurable subset of F.*

Bidimensional Cesaro asymptotic density

We would say that $R \subseteq F$ is *$C-$generic* if

$$\limsup_{l \to \infty} \mu_l(R) = 1;$$

vice versa, if this limit is equal to zero, the set R is called *$C-$negligible*.
If, in addition, this limit exists and $\mu_l(R)$ converges to 1 exponentially fast, then we say that R is a *strongly $C-$generic*.

Lemma

Let $\mu = \{\mu_l\}$ be the family of atomic probability measures as above. Then for every subset $R \subseteq F$ the following holds:

i) $\mu_0 = \lim_{l \to \infty} \mu_l(R)$ *is a pseudo-measure on* F;

ii) *R is $C-$generic iff $\mu_0(R) = 1$, and vice versa R is $C-$negligible iff $\mu_0(R) = 0$.*

Bidimensional Cesaro asymptotic density

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ii) R *is* $C-$*generic iff* $\mu_0(R) = 1$, *and vice versa* R *is* $C-$*negligible iff* $\mu_0(R) = 0$.

Bidimensional Cesaro asymptotic density

Let $\{\mu_{A,l}\}$ and $\{\mu_{B,l}\}$, $1 < l < \infty$ are families of atomic probability measures on free groups A and B correspondingly. Let also μ_l be an atomic probability measure on F generated by $\mu_{A,l}$ and $\mu_{B,l}$ and $R * S$ is a free product of $R \subseteq A, \quad S \subseteq B$.
A family of probability measures $\{\mu_l | \ 1 < l < \infty\}$ on $G = A * B$, where μ_l generated by $\mu_{A,l}$ and $\mu_{B,l}$ allow us to define relative bidimensional Cesaro asymptotic density for a subsets of $T \subseteq G$:

Definition

Let $Q \subseteq T \subseteq G$. The function from $\mathbb{N}^2 \rightarrow \mathbb{R}^+$ defined in the following way:

$$(l, k) \rightarrow \frac{\mu_l((A * B)^k \bigcap Q)}{\mu_l((A * B)^k \bigcap T)} = \rho_T^{(l,k)}(Q)$$

is called the *relative frequency function* for Q with respect to T in G.
The bidimensional Cesaro asymptotic density is

$$C\rho_T(Q) = \lim_{\substack{l \rightarrow \infty \\ k \rightarrow \infty}} \rho_T^{(l,k)}(Q).$$

Bidimensional Cesaro asymptotic density

Notions of $C-$genericity and $C-$negligibility, and also their strict analogs carry out on our situation in natural way. Since our definition of bidimensional Cesaro asymptotic density can be applied for a subset T of G itself, we will write $C\rho(T)$ instead of $C\rho_G(T)$ when we deal with such a situation.

Theorem

Let $G = A * B$ is a a free product of two free groups A and B of finite ranks with given families of atomic probability measures $\mu_A = \{\mu_{A,l}\}, \ \mu_B = \{\mu_{B,l}\}, 1 < l < \infty$ and induced family of atomic measures $\{\mu_l\}$. Let $R_0 \subseteq R \subseteq A$ and $S_0 \subseteq S \subseteq B$. If there is a number $0 < q < 1$, such that for all $1 < l < \infty$ $\dfrac{\mu_l(R_0)}{\mu_l(R)} < q$ or $\dfrac{\mu_l(S_0)}{\mu_l(S)} < q$, then the set $R_0 * S_0$ is a strongly $C-$negligible with respect to $R * S$.

Reduced product of two sets

Let $G = A * B$ as above and μ is a pseudo-measure on G. Let R_1 and R_2 be two subsets from G as well. The product $R_1 R_2$ called *reduced* with respect to syllable length function s and denoted by $R_1 \circ R_2$, if for all $r_1 \in R_1$ and $r_2 \in R_2$ we have $s(r_1 r_2) = s(r_1) + s(r_2)$.

Suppose that $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ and asymptotic densities $\rho_{\mu,R_1}(S_1), \rho_{\mu,R_2}(S_2)$ are also defined. What can we say about asymptotic density $\rho_{\mu,R_1 \circ R_2}(S_1 \circ S_2)$?
We will answer this question with some complementary restriction on multipliers, which are fulfilled for many useful applications.

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Reduced product of two sets

Proposition

Let $G = A * B$ sets R_1, R_2 be subsets of G;
$S_1 \subseteq R_1$, $S_2 \subseteq R_2$ and $R_1 \subseteq A \bigcup B$, i.e. all words from R_1 have a syllable length 1.
Denote $T = R_1 \circ R_2$ and $Q = S_1 \circ S_2$.

(i) *Let μ be an atomic multiplicative pseudo-measure on G. Then asymptotic density $\rho_{\mu,T}(Q)$ exists and not greater than $\rho_{\mu,R_2}(S_2)$.*

(ii) *Let μ be a pseudo-measure on G, defined by μ_A and μ_B; let ρ_μ is corresponding asymptotic density. If $\rho_{\mu,R_2}(S_2)$ exists, than $\rho_{\mu,T}(Q)$ also exists and not greater than $\rho_{\mu,R_2}(S_2)$.*

(iii) *Let $s\rho$ be a spherical (or $b\rho$ be a ball) asymptotic density on G. Suppose that S_2 is $s-$negligible ($b-$negligible) with respect to R_2. Then Q is $s-$negligible ($b-$negligible) with respect to T.*

Spherical asymptotic density

Let $G = A * B$ be a free product of free groups A and B of finite ranks.

If $R \subset A$, then spherical $s\rho_A(R)$ and ball $b\rho_A(R)$ asymptotic densities are defined, if limits of corresponding relative frequencies are exist. Moreover, if R is $s-$measurable, then R is also $b-$measurable and this pseudo-measures are coincide for R. All over above is valid for B.

In addition, if $R_0 \subseteq R \subseteq A$ (and $S_0 \subseteq S \subseteq B$), then also exist $s\rho_R(R_0),\ b\rho_R(R_0)$ (and $s\rho_S(S_0),\ b\rho_S(S_0)$) if correspondent limits exist. We promote the following conjecture about asymptotic behaviour of the set $R_0 * S_0$ in $R * S$.

Spherical asymptotic density

Observe, that the estimation of the spherical asymptotic density of free product $R_0 * S_0$ in $R * S$ is more complicated then other methods to estimate different types of asymptotic density for different subsets of F, which was introduced earlier.

Conjecture

[MFR] Let $R_0 \subseteq R \subseteq A$ and $S_0 \subseteq S \subseteq B$ and R_0 (S_0) is a strongly $s-$negligible set with respect to R (S). Then the set $R_0 * S_0$ is a strongly $s-$negligible set with respect to $R * S$.

It follows from definitions that exists such a number $q_0 \in (0, 1)$ and a natural number m, such as for all $n \geq m$

$$s\rho_{n,R}(R_0) < q_0^n \text{ and } s\rho_{n,S}(S_0) < q_0^n. \quad (5)$$

We'll confirm this conjecture in three most important for applications cases:

1) $m = 1$;
2) $m > 1$, $|S_n(R_0)| < |S_n(R)|$ and $|S_n(S_0)| < |S_n(S)|$;
3) Sets R_0, R are regular in A and S_0, S are regular in B.

Multiplicative measures and strongly negligible subsets

The studying of measures introduced below gives us the wishfull result about negligibility of singular and unstable representatives.

Theorem

Let C be a finitely generated subgroup of infinite index in F. Then following statements are hold:

1) *The generalized normalizer $N_F^*(C)$ of C in F is a strongly negligible in F.*
2) *The set of singular representatives S_{sing} of C in F is a strongly negligible in F and therefore the set S_{reg} is a strongly generic in F.*
3) *The set S_{nst} of unstable representatives is a strongly negligible in F and therefore the set S_{st} of stable representatives is a strongly generic in F.*

The END number one.

Multiplicative measures and $\lambda-$measurable subsets

Let R be a subset of the free group F of finite rank n, S_k is the sphere of radius k in F and

$$f_k(R) = \frac{|R \cap S_k|}{|S_k|}$$

is the relative frequency of elements from R among the words of length k in F.

The atomic measure λ on F is defined on singleton sets $\{w\}$, $w \in F$, by

$$\lambda(w) = \frac{1}{2n(2n-1)^{|w|-1}}, \quad \text{if} \quad w \neq 1, \quad \text{and} \quad \lambda(1) = 1,$$

and extended to all subset in F by countable additivity:

$$\lambda(R) = \sum_{w \in R} \lambda(w) = \sum_{k=0}^{\infty} f_k(R).$$

Multiplicative measures and $\lambda-$measurable subsets

A set $R \subseteq F$ is called *generic* (or *negligible*) if $\rho(R) = 1$ (or $\rho(R) = 0$), where the *asymptotic density* $\rho(R)$ is defined by

$$\rho(R) = \limsup_{k \to \infty} f_k(R).$$

If, in addition, there exists a positive constant $\delta < 1$ such that $1 - \delta^k < f_k(R) < 1$ ($f_k(R) < \delta^k$) for all sufficiently large k then R is called *strongly generic* (or *strongly negligible*).

Remark

Obviously, the previous definition coincides to notation s$-$generic (s$-$negligible) set with respect to spherical asymptotic density, defined by cardinality function μ.

A set $R \subseteq F$ is called λ-*measurable* if $\lambda(R) < \infty$. It easy to see that for the set R the condition "to be $\lambda-$measurable" is equivalent to the condition "to be negligible".

Multiplicative measures and $\lambda-$measurable subsets

It is convenient sometimes to use the relativized version of the notions described above. To this end, let R, T be subsets of F. Denote by

$$f_k(R, T) = \frac{|R \cap S_k|}{|T \cap S_k|}$$

the relative frequency of elements from R among the words of length k in T. We say that R is *negligible with respect to T* if

$$\sum_{k=1}^{\infty} f_k(R, T) < \infty.$$

It is convenient to modify the atomic measure λ and define for $w \in F$

$$\lambda^*(w) = \frac{2n}{2n-1}\lambda(w) = \frac{1}{(2n-1)^{|w|}}.$$

The new measure λ^* is *multiplicative*, i.e., if $w = u \circ v$ then

$$\lambda^*(w) = \lambda^*(u)\lambda^*(v).$$

Multiplicative measures and $\lambda-$measurable subsets

Below we collect some basic facts about negligible and strongly negligible sets in F.

Proposition

Let R_1 and R_2 be subsets of F. Then the following statements hold:

1) *If $R_1 \subseteq R_2$ and R_2 is negligible (strongly negligible) then so is R_1.*
2) *If R_1 and R_2 are negligible (strongly negligible) then so is the set*

$$R_1 \circ R_2 = \{r_1 \circ r_2 \mid r_i \in R_i\}.$$

3) *If R_1 and R_2 are negligible (strongly negligible) then so is the set*

$$R_1 \underset{t}{*} R_2 = \{r_1 r_2 : \ \forall r_1 \in R_1, \forall r_2 \in R_2 \ \ |r_1| + |r_2| - |r_1 r_2| \leq t\}.$$

4) *A regular subset of F is generic if and only if its prefix closure contains a cone.*
5) *Every negligible regular set of F is strongly negligible.*

Multiplicative measures and $\lambda-$measurable subsets

Definition

Let R_1 and R_2 be subsets of F and $f : R_1 \to R_2$ a map. Then:

- f is called *quasi-metric* if there exists a constant d such that for any element $w \in R_1$
$$| \, |f(w)| - |w| \, | \leq d.$$
- f is called *with bounded fibers* if there exists a constant k such that $|f^{-1}(w)| \leq k$ for any $w \in R_2$.

Proposition

Let R_1 and R_2 be subsets of F. Then the following statements hold:

1) *Let $f : R_1 \to R_2$ be a surjective quasi-metric map. Then if R_1 is negligible (strongly negligible) then so is R_2.*
2) *Let $f : R_1 \to R_2$ be a quasi-metric map with bounded fibers. Then if R_2 is negligible (strongly negligible) then so is R_1.*

Multiplicative measures and $\lambda-$measurable subsets

Proposition

If C is a finitely generated subgroup of infinite index in the free group F then the set S of special representatives is regular and generic.

Proposition

Let C be a finitely generated subgroup of infinite index in F. Then the following statements hold:

1) *C is strongly negligible in F and the constant δ can be effectively computed from the subgroup graph Γ_C of C.*
2) *Every coset of C in F is strongly negligible in F.*

Proposition

Let C be a finitely generated subgroup of infinite index in F and S is a special system of representatives of C in F. If $S_0 \subseteq S$ is a strongly negligible subset of F then the set $P = \bigcup_{s \in S_0} Cs$ is strongly negligible in F.

Multiplicative measures and $\lambda-$measurable subsets

Proposition

Let C be a finitely generated subgroup of infinite index in F. Then the following statements hold:

1) $C^* = \bigcup\limits_{f \in F} C^f$ *is strongly negligible in* F.
2) *For every* $c \in C,\ c \neq 1$ *the set* $c^F = \{f^{-1}cf | f \in F\}$ *is strongly negligible in* F.

Proposition

Let A and B be finitely generated subgroups of infinite index in F. Then for any $w \in F$ the double coset AwB is strongly negligible in F.

The final END.

Multiplicative measures and $\lambda-$measurable subsets

Now we are restate the main result about measurable subsets of F, formulated in Conjecture16:

Theorem

Let $R_0 \subseteq R \subseteq A$ and $S_0 \subseteq S \subseteq B$, where A and B are finitely generated free groups. Let also R_0, S_0 are regular sets, which in addition are strongly negligible with respect to R or S correspondingly, and sets R, S are prefix closed and regular. Then the set $R_0 * S_0$ is a strongly $s-$negligible with respect to $R * S$.

Multiplicative measures and $\lambda-$measurable subsets

Description of applied methods:
In [3] was proven the following results.

Theorem

The measure $\mu^*(L(A))$ (and hence the probability measure $\mu(L(A))$) of a regular subset of F is a rational function in t (and hence in s) with rational coefficients, where $\mu = \mu_s$ and $\mu_s^*(w) = t^{|w|}$, where $t = \dfrac{1-s}{2m-1}$.

Corollary

Every regular set in $F(X)$ is either (strongly) generic or strongly negligible set (or strongly $\lambda-$measurable set).

Multiplicative measures and $\lambda-$measurable subsets

Theorem

Let $\mathbf{R}$ *be a regular subset of* F*. Then* $\mathbf{R}$ *is (strongly) generic if and only if its prefix closure* $\overline{\mathbf{R}}$ *contains a cone.*

We have generalized results from [3] in the following manner: the group $F = F(X)$ is replaced by the prefix closed set L and the notion of cone was replaced by relativized definitions.

A *$L-$cone* $C = C_L(w)$ with vertex w is a set of all elements in L containing the given word w as initial segment. Obviously, $L-$cones are regular sets.
Further we considered relative frequencies

$$f_{L,n}(\mathbf{R}) = \frac{|\mathbf{R} \bigcap S_n|}{|L \bigcap S_n|},$$

where S_n is a sphere of radius n in F.

Multiplicative measures and $\lambda-$measurable subsets

We call $\mathbf{R}$ *$L-$measurable*, if the sum of series $\sum f_{L,n}(\mathbf{R}) < \infty$. The set $\mathbf{R}$ is called *strongly $L-$measurable*, if in addition $f_{L,n}(\mathbf{R}) < q^n$, where $0 < q < 1$. It is clear, that the property "to be strongly $L-$ measurable" is equivalent to the property "to be strongly negligible with respect to L" in terminology above.
Further, the $L-$cone $C = C(w)$ we will call a *small cone*, if C is a strongly $L-$measurable. In other case we would call such a cone a *big $L-$cone*. So, we can reformulate our theorem:

Theorem

Let $\mathbf{R}$ be a regular subset of a prefix closed regular set L in free group $F(X)$. Then either $\overline{\mathbf{R}}$ contains a big $L-$cone or $\overline{\mathbf{R}}$ is a strongly negligible with respect to L (or $\overline{\mathbf{R}}$ is a strongly $L-$measurable subset).

Indeed, since R, S are regular prefix closed subsets in A and B correspondingly, then $R * S$ is a prefix closed in $F(X)$. By conditions on R_0 and S_0 the set $R_0 * S_0$ doesn't contain any $R * S-$cone and so by statement of Theorem 31 we have the alternative situation, i.e. $R_0 * S_0$ is a strongly negligible with respect to $R * S$.

Multiplicative measures and $\lambda-$measurable subsets

Suppose that **R** doesn't contain a big $L-$cone. We proved that **R** is a strongly $L-$measurable set. Let $\mathcal{A}$ is an automaton which recognize L, i.e. $L = L(\mathcal{A})$. Based on automaton $\mathcal{A}$, we constructed an automaton $\mathcal{B}$, recognizing the set $\mathbf{R} \subseteq L$. We need to recall the theorem of Myhill-Nerode about regular languages:

Theorem

Given a language **R** *over an alphabet* A*, consider the equivalence relation on* A^* *defined as follows: two strings* w_1 *and* w_2 *are equivalent if, for each string* u *over* A*,* $w_1 u \in \mathbf{R}$ *iff* $w_2 u \in \mathbf{R}$*. Then* **R** *is regular if and only if there are only finitely many equivalence classes.*

We generalized this theorem in following form:

Lemma

Given two languages $\mathbf{R} \subseteq L$ *over an alphabet* $X \bigcup X^{-1}$*, where* L *is prefix closed and regular, consider the equivalence relation on* L *defined as follows: two words* w_1 *and* w_2 *from* L *are equivalent if, for each word* u *over* $X \bigcup X^{-1}$*,* $w_1 u \in \mathbf{R}$ *iff* $w_2 u \in \mathbf{R}$*. Then* **R** *is regular iff there are only finitely many equivalence classes.*

Multiplicative measures and $\lambda-$measurable subsets

By the construction of automaton $\mathcal{B}$ it was necessary to show,that
If $\overline{\mathbf{R}}$ contains no big $L-$cone then it is strongly $L-$measurable, that is

$$\lambda(R) = \sum_{w \in R} \lambda(w)$$

is finite. Here $\overline{\mathbf{R}}$ is a prefix closure of $\mathbf{R}$ in L.
Last statement was shown with a help of complementary transformations of automaton $\mathcal{B}$ and analysis of its properties and also properties of corresponding languages.

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