On Systems of Equations over Partially Commutative Groups

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joint results with M. Casals-Ruiz

Partially commutative groups

A **partially commutative group** $\mathbb{G}$ is a group given by the presentation

$$\mathbb{G} = \langle a_1, \ldots, a_r \mid R \rangle,$$

where $R \subseteq \{[a_i, a_j] \mid 1 \leq i < j \leq r\}$.

Partially commutative groups are equationally Noetherian, hence we can develop Diophantine geometry (by the Unification theorems).

A bit of history

Given a system of equations S, the two main questions are:

1. Does S have a solution?
2. Can one describe the set of all solutions of S?

Free Groups

- R. Lyndon (1960) - one-variable equations;
- Yu. Hmelevskiĭ (1971, 1972) and Yu. Ozhigov (1983) - two variable equations;
- A. Razborov (1984) - no generalizations to 3 variables;
- A. Malcev (1962) - the commutator equation;
- Comerford-Edmunds and Grigorchuk-Kurchanov - general quadratic equations;
- G. Makanin (1977, 1982) - decidability of compatibility;
- A. Razborov (1985, 1987) - description of solutions.

A bit of history

Given a system of equations S, the two main questions are:

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Partially Commutative Groups

- V. Diekert and A. Muscholl (2006) - decidability of equations;
- V. Diekert and M. Lohrey (2008) - existential and positive theories;
- S. Shestakov (2005, 2006) - Wicks forms and the equation $x^2y^2 = g$.

The result

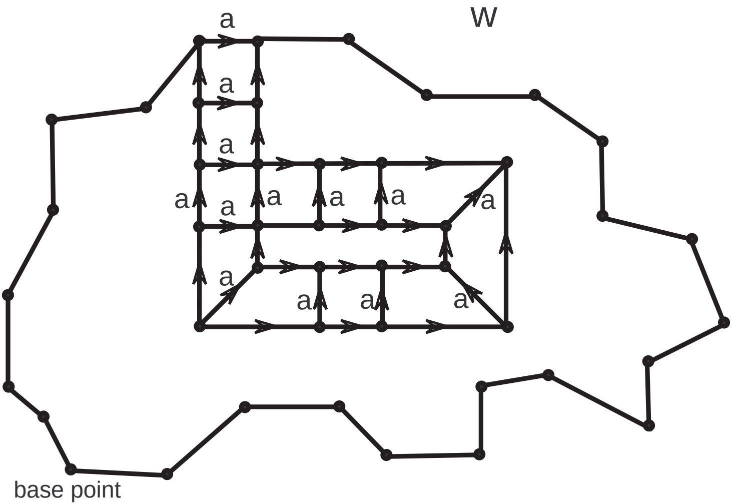
Theorem

The set of all $\mathbb{G}$-homomorphisms $\mathrm{Hom}(G, \mathbb{G})$ from a f.g. group G to $\mathbb{G}$ can be effectively described by a finite rooted tree. This tree is oriented from the root and all its vertices are labelled by coordinate groups. The leaves of the tree are labelled by $\mathbb{G}$-fully residually $\mathbb{G}$ partially commutative groups.
To each vertex group we assign a group of automorphisms. Each edge (except for the edges from the root and the edges to the leaves) in this tree is labelled by an epimorphism, and all the epimorphisms are proper. Every $\mathbb{G}$-homomorphism from G to $\mathbb{G}$ can be written as a composition of the $\mathbb{G}$-homomorphisms corresponding to the edges, automorphisms of the groups assigned to the vertices, and specialisations of free variables.

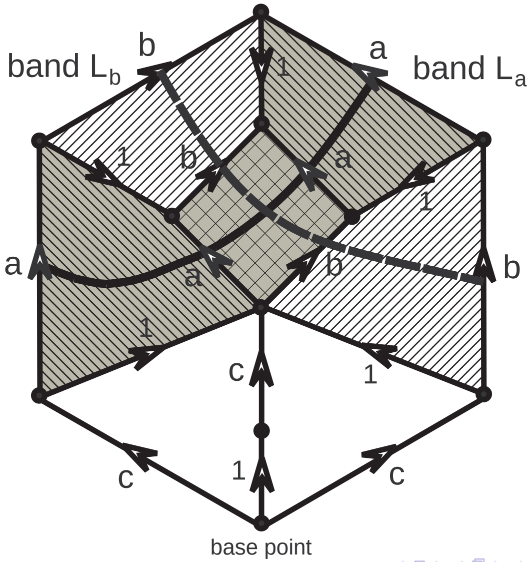
Ingredients of the proof

1. Reduction to graphical equations;
2. The Process.

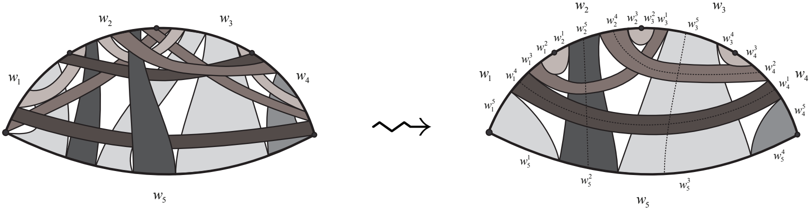
Cancellation schemes: reduction to partially commutative monoids



Cancellation schemes: reduction to partially commutative monoids



Cancellation schemes: reduction to partially commutative monoids



Reduction to graphical equalities

Hint: do not use lexicographical normal forms.

1. V. Diekert and A. Muscholl found a normal form that is stable under the involution.
2. They also proved that there exists only finitely many ways to take the product of two words to the normal form.

Can we run the Process?

Almost... We imposed commutation to the variables of the generalised equations.

1. It is important not to transfer the difficulty of the generalised equation onto constraints...
2. We don't, as the commutation is, in fact, "disjoint commutation".

Stopping the Process

- The process runs analogously to the free group;
- Problems arise, especially, when one uses Tietze transformations;
- For example, "linear" case may give an infinite branch;
- The automorphisms for linear and quadratic cases come from the process and they arise "naturally".
- Periodic structures: automorphisms are used to bound the exponent of periodicity. They don't come from the process, but from the structure of the coordinate group!

Output

First output

- We get a finite tree.
- What are the leaves?
- One can't take any specialisation and leaves are not fully residually $\mathbb{G}$.

Second output

Extend the leaves with finitely many fully residually $\mathbb{G}$ partially commutative groups, so that solutions are specialisations of variables in canonical parabolic subgroups.

Promotion

1. We obtained a direct analogue of Razborov's result for free groups - parametrisation of the solution set or the set of homomorphisms...
2. It was unexpected:
   - actions on $\mathbb{R}$-trees,
   - pc groups are not CSA, subgroups of pc groups are complicated,
   - no known JSJ (universal splittings) structure.

What can we try now?

1. Generalisations to other groups (not related to stable actions).
2. Razborov's process with particular constraints.
3. Understand the geometric counterpart: actions on trees or other spaces (Rips machine).
4. Structural results on f.g. fully residually $\mathbb{G}$-groups.