Algebraic geometry over algebraic structures
Lecture 4

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based on joint results with Evelina Yu. Daniyarova[1] and
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Outline

1. Atomic Types

2. Unification Theorems

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem C (No coefficients)

Let $\mathcal{A}$ *be an equationally Noetherian algebra in a language* $\mathcal{L}$ *(with no predicates). Then for a finitely generated algebra* $\mathcal{C}$ *of* $\mathcal{L}$ *the following conditions are equivalent:*

1. $\mathcal{C} \in$ **Qvar**$(\mathcal{A})$, *i.e.,* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in$ **Pvar**$(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Atomic Types

Atomic types

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of variables and T is a theory.

Definition

A set p of atomic or negations of atomic formulas in variables X is called an atomic type relative to a theory T if $p \cup T$ is consistent. A maximal atomic type in in variables X with respect to inclusion is termed a complete atomic type of T.

If p is a complete atomic type in variables X then for every atomic formula $\varphi \in \mathrm{At}_{\mathcal{L}}(X)$ either $\varphi \in p$ or $\neg\varphi \in p$.

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Atomic types

Example

Let $\mathcal{A}$ be an $\mathcal{L}$-algebra and $\bar{a} = (a_1, \ldots, a_n) \in A^n$. Then the set $\mathrm{atp}^{\mathcal{A}}(\bar{a})$ of atomic or negations of atomic formulas in variables X that are true in $\mathcal{A}$ under an interpretation $x_i \mapsto a_i$, $i = 1, \ldots, n$, is a complete atomic type relative to any theory T such that $\mathcal{A} \in \mathrm{Mod}(T)$.

Atomic types

Every complete atomic type p in variables X define congruence $\sim_p$ on the free $\mathcal{L}$-algebra $\mathcal{T}_\mathcal{L}(X)$

$$t \sim_p s \quad \Longleftrightarrow \quad (t = s) \in p, \quad t, s \in \mathrm{T}_\mathcal{L}(X).$$

We denote by $\mathcal{T}_\mathcal{L}(X)/p$ corresponding factor-algebra of $\mathcal{T}_\mathcal{L}(X)$.

Definition

Let p be a complete atomic type in variables X. Then the factor-algebra $\mathcal{T}_\mathcal{L}(X)/p$ of the free $\mathcal{L}$-algebra $\mathcal{T}_\mathcal{L}(X)$ is termed the algebra defined by the type p.

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Atomic types

Lemma

Let T be a universally axiomatized theory in $\mathcal{L}$. Then for any finitely generated $\mathcal{L}$-algebra $\mathcal{A}$ the following conditions are equivalent:

1) $\mathcal{A} \in \mathrm{Mod}(T)$;
2) $\mathcal{A} = \mathcal{T}_{\mathcal{L}}(X)/p$ *for some complete atomic type p in T.*

Unification Theorems

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem B (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated $\mathcal{A}$-algebra $\mathcal{C}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{C})$, *i.e.,* $\mathcal{C} \equiv_{\forall,\mathcal{A}} \mathcal{A}$;
2. $\mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{C})$, *i.e.,* $\mathcal{C} \equiv_{\exists,\mathcal{A}} \mathcal{A}$;
3. $\mathcal{C}$ *$\mathcal{A}$-embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is $\mathcal{A}$-discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A})$ *in the language* $\mathcal{L}_{\mathcal{A}}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in* $\mathcal{A}$.

Unification Theorem C (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, *i.e.*, $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem D (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated $\mathcal{A}$-algebra $\mathcal{C}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}_{\mathcal{A}}(\mathcal{A})$, *i.e.*, $\mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}_{\mathcal{A}}(\mathcal{A})$;
3. $\mathcal{C}$ *$\mathcal{A}$-embeds into a direct power of $\mathcal{A}$;*
4. $\mathcal{C}$ *is $\mathcal{A}$-separated by $\mathcal{A}$;*
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over $\mathcal{A}$;*
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory $\mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{A})$ in the language $\mathcal{L}_{\mathcal{A}}$;*
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in $\mathcal{A}$.*

Unification Theorems

Remark

As follows from proof of Theorems A and C, some of them arrows hold for arbitrary algebra $\mathcal{A}$ (not necessary equationally Noetherian), namely:

$$\textit{Theorem A:}\quad \{4 \Leftrightarrow 7\} \quad \longrightarrow \quad \{1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 5 \Leftrightarrow 6\};$$

$$\textit{Theorem C:}\quad \{5\} \quad \longrightarrow \quad \{1 \Leftrightarrow 6\} \quad \longleftarrow \quad \{2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 7\}.$$

References

1. E. Daniyarova, A. Miasnikov, V. Remeslennikov, *Unification theorems in algebraic geometry*, Journal of Algebra and ..., 2008, and on arxiv.org.
2. E. Daniyarova, A. Miasnikov, V. Remeslennikov, *Algebraic geometry over algebraic structures II: Foundations*, to be appear.