Algebraic geometry over algebraic structures
Lecture 3

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Outline

Major problem of algebraic geometry

One of the major problems of algebraic geometry over algebra $\mathcal{A}$ consists in classifying algebraic sets over the algebra $\mathcal{A}$ with accuracy up to isomorphism.

Due to Theorem on dual equivalence of the category of algebraic sets and the category of coordinate algebras this problem is equivalent to the problem of classifications of coordinate algebras.

And Unifications Theorems are useful for solving this problem.

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And Unifications Theorems are useful for solving this problem.

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.,* $\mathcal{C} \in$ **Ucl**$(\mathcal{A})$;
2. $\mathrm{Th}_\exists(\mathcal{A}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem C (No coefficients)

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1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, *i.e.*, $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
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Zariski Topology and Irreducible Sets

Zariski topology

There are three perspectives for investigation in algebraic geometry over algebra $\mathcal{A}$: algebraic, geometrical and logic. Geometrical approach is connected with examination of affine space A^n as topological space.

We define Zariski topology on A^n, where algebraic sets over $\mathcal{A}$ form a subbase of closed sets, i.e., closed sets in this topology are obtained from the algebraic sets by finite unions and (arbitrary) intersections.

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Irreducible sets

A subset $Y \subseteq A^n$ is called irreducible if for all closed subsets $Y_1, Y_2 \subseteq A^n$ inclusion $Y \subseteq Y_1 \cup Y_2$ involves $Y \subseteq Y_1$ or $Y \subseteq Y_2$; otherwise, it is called reducible. For example, any singleton set $\{p\}$ $(p \in A^n)$ is irreducible.

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Every algebraic set $Y \subseteq A^n$ is a union of maximal with respect to including irreducible algebraic sets over $\mathcal{A}$.

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A finitely generated $\mathcal{L}$-algebra $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{A}$ if and only if it is a subdirect product of coordinate algebras of irreducible algebraic sets over $\mathcal{A}$.

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Let $\mathcal{C}_i$, $i \in I$, be an $\mathcal{L}$-algebras. Denote by p_j, $j \in I$, the canonical projections $\prod_{i\in I} \mathcal{C}_i \to \mathcal{C}_j$.

Recall, that a subalgebra $\mathcal{C}$ of direct product $\prod_{i\in I} \mathcal{C}_i$ is a subdirect product of the algebras $\mathcal{C}_i$, $i \in I$, if $p_j(\mathcal{C}) = \mathcal{C}_j$ for all $j \in I$.

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Equationally Noetherian Algebras

Definition

An $\mathcal{L}$-algebra $\mathcal{A}$ is equationally Noetherian, if for any positive integer n and any system of equations $S \subseteq \mathrm{At}_{\mathcal{L}}(x_1, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}(S) = \mathrm{V}(S_0)$.

Equationally Noetherian algebras

Example

The following algebras are equationally Noetherian in the language with constants ($\mathcal{L}_{\mathcal{A}}$ for $\mathcal{A}$):

- any Noetherian commutative ring;
- any linear group over Noetherian ring (in particular, free groups, polycyclic groups, finitely generated metabelian groups) [Bryant, Guba, Baumslag, Myasnikov, Remeslennikov];
- any torsion-free hyperbolic group [Sela];
- any free solvable group [Gupta, Romanovskii];
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- $A \wr B$, where A is non-abelian and B is infinite [Baumslag, Myasnikov, Roman'kov];
- the Grigorchuk group Γ [Grigorchuk, Sapir];
- the Min-Max structures $\mathcal{M}_{\mathbb{R}} = \langle \mathbb{R}; \max, \min, \cdot, +, -, 0, 1 \rangle$ and $\mathcal{M}_{\mathbb{N}} = \langle \mathbb{N}; \max, \min, +, 0, 1 \rangle$ [Dvorjestky, Kotov].

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Equationally Noetherian algebras

Open Problems

Are the following algebras equationally Noetherian:

- *free Lie algebra?*
- *free anti-commutative algebra?*
- *free associative algebra?*
- *free products of equationally Noetherian groups?*

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Lemma

Let $\mathcal{A}$ be an equationally Noetherian algebra. Then the following algebras are equationally Noetherian too:

1. *every subalgebra of $\mathcal{A}$;*
2. *every filterpower (ultrapower, direct power) of $\mathcal{A}$;*
3. *coordinate algebra $\Gamma(Y)$ of an algebraic set Y over $\mathcal{A}$;*
4. *every algebra separated by $\mathcal{A}$;*
5. *every algebra discriminated by $\mathcal{A}$;*
6. *every algebra from $\mathbf{Qvar}(\mathcal{A})$;*
7. *every algebra from $\mathbf{Ucl}(\mathcal{A})$;*
8. *every limit algebra over $\mathcal{A}$;*
9. *every algebra defined by a complete atomic type in the theory $\mathrm{Th}_{\forall}(\mathcal{A})$ in $\mathcal{L}$.*

Equationally Noetherian algebras

Lemma

For an algebra $\mathcal{A}$ the following conditions are equivalent:

1. *$\mathcal{A}$ is equationally Noetherian;*
2. *for any positive integer n Zariski topology on A^n is Noetherian (i.e., it satisfies the descending chain condition on closed subsets);*
3. *every chain of proper epimorphisms*

$$\mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow \mathcal{C}_3 \longrightarrow \dots$$

of coordinate algebras of algebraic sets over $\mathcal{A}$ is finite.

Theorem

Every algebraic set over equationally Noetherian algebra $\mathcal{A}$ can be expressed as a finite union of irreducible algebraic sets (irreducible components). Furthermore, this decomposition is unique up to permutation of irreducible components and omission of superfluous ones.

Theorem

Let $\mathcal{A}$ be an equationally Noetherian $\mathcal{L}$-algebra. A finitely generated $\mathcal{L}$-algebra $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{A}$ if and only if it is a subdirect product of finitely many coordinate algebras of irreducible algebraic sets over $\mathcal{A}$.

Corollary

Classification of irreducible algebraic sets (or/and their coordinate algebras) is the essential problem of algebraic geometry over equationally Noetherian algebra.

Equationally Noetherian algebras

Irreducible coordinate algebras

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Classification of irreducible algebraic sets (or/and their coordinate algebras) is the essential problem of algebraic geometry over equationally Noetherian algebra.

Unification Theorems

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_\exists(\mathcal{A}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
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Unification Theorem B (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated $\mathcal{A}$-algebra $\mathcal{C}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{C})$, *i.e.,* $\mathcal{C} \equiv_{\forall,\mathcal{A}} \mathcal{A}$;
2. $\mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{C})$, *i.e.,* $\mathcal{C} \equiv_{\exists,\mathcal{A}} \mathcal{A}$;
3. *$\mathcal{C}$ $\mathcal{A}$-embeds into an ultrapower of $\mathcal{A}$;*
4. *$\mathcal{C}$ is $\mathcal{A}$-discriminated by $\mathcal{A}$;*
5. *$\mathcal{C}$ is a limit algebra over $\mathcal{A}$;*
6. *$\mathcal{C}$ is an algebra defined by a complete atomic type in the theory $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A})$ in the language $\mathcal{L}_{\mathcal{A}}$;*
7. *$\mathcal{C}$ is the coordinate algebra of a non-empty irreducible algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in $\mathcal{A}$.*

Unification Theorem C (No coefficients)

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Unification Theorems

Remark

As follows from proof of Theorems A and C, some of them arrows hold for arbitrary algebra $\mathcal{A}$ (not necessary equationally Noetherian), namely:

$$\textit{Theorem A:}\quad \{4 \Leftrightarrow 7\} \quad\longrightarrow\quad \{1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 5 \Leftrightarrow 6\};$$

$$\textit{Theorem C:}\quad \{5\} \quad\longrightarrow\quad \{1 \Leftrightarrow 6\} \quad\longleftarrow\quad \{2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 7\}.$$

Examples

Min-Max structures

Example (Dvorjetsky, Kotov)

The structure $\mathcal{M}_{\mathbb{R}} = \langle \mathbb{R}; \max, \min, \cdot, +, -, 0, 1 \rangle$ with obvious interpretation of the symbols from signature on $\mathbb{R}$ is an example of so-called Min-Max structure.

Theorem

A set $Y \subseteq \mathbb{R}^n$ is algebraic over $\mathcal{M}_{\mathbb{R}}$ if and only if it is closed in topology induced by Euclidean metric on $\mathbb{R}^n$.

Abelian groups

Example (Myasnikov, Remeslennikov)

Let A be a fixed abelian group and $\mathcal{L}$ be the language of abelian groups with constants from A: $\mathcal{L} = \{+, -, 0, \ c_a, a \in A\}$.

Theorem

Let C be a finitely generated A-group. Then C is the coordinate group of an algebraic set over A if and only if the following conditions holds:

1. $C \simeq A \oplus B$, *where B is a finitely generated abelian group;*
2. $e(A) = e(C)$ *and* $e_p(A) = e_p(C)$ *for every prime number p.*

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Example (Myasnikov, Remeslennikov)

Recall, that the period of an abelian group A is the minimal positive integer m, if it exists, such that $mA = 0$; and ∞ otherwise.

Let $T(A)$ be the torsion part of A and $T(A) \simeq \oplus_p T_p(A)$ be the primary decomposition of $T(A)$ (here and below in this Example p is a prime number).

Denote by $e(A)$ the period of A, and by $e_p(A)$ the period of $T_p(A)$.

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Example (Myasnikov, Remeslennikov)

Now it is easy to describe algebraic set Y, corresponding to coordinate group $C = A \oplus B$ from Theorem above. Fix a primary cyclic decomposition of the group B:

$$B \simeq \langle a_1 \rangle \oplus \ldots \oplus \langle a_r \rangle \oplus \langle b_1 \rangle \oplus \ldots \oplus \langle b_t \rangle,$$

here a_i-s are generators o infinite cyclic groups and b_j-s are generators of finite cyclic groups of orders $p_j^{m_j}$.

For positive integer n denote by $A[n]$ the set $\{a \in A \mid na = 0\}$. Then

$$Y = \underbrace{A \oplus \ldots \oplus A}_{r} \oplus A[p_1^{m_1}] \oplus \ldots \oplus A[p_t^{m_t}].$$

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Example (Myasnikov, Remeslennikov)

For a positive integer k and a prime number p we denote by $\alpha_{p^k}(A)$ the dimension, if it exists, of factor-group $A[p^k]/A[p^{k-1}]$ as vector-space over finite field with p elements; and ∞ otherwise.

Theorem

Let C be a finitely generated A-group. Then C is the coordinate group of an irreducible algebraic set over A if and only if the following conditions holds:

1. $C \simeq A \oplus B$*, where B is a finitely generated abelian group;*
2. $e(A) = e(C)$ *and* $e_p(A) = e_p(C)$ *for every prime number p;*
3. $\alpha_{p^k}(A) = \alpha_{p^k}(C)$ *for each prime number p and positive integer k.*

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Free Lie algebra

Example (Daniyarova, Remeslennikov)

Let L be a free Lie algebra of finite rank over a field k.

- An algebraic set Y is called bounded if it enters into some finite dimensional subspace of L^n as k-linear space.
- Examples of bounded algebraic sets are so-called n-parallelepipeds.
- Under n-parallelepiped $\mathbf{V}$ we mean a Cartesian product of a n-tuple of finite dimensional subspaces $V_1, \ldots, V_n$ of L:

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Example (Daniyarova, Remeslennikov)

1-parallelepipeds are finite dimensional subspaces in L. For linear subspace in L with basis $v_1, \dots, v_m$ we have

$$s_1(x) = [x, v_1],$$

$$s_2(x) = [[x, v_1], [v_2, v_1]],$$

$$\dots$$

$$s_m(x) = [s_{m-1}(x), s_{m-1}(v_m)].$$

So that $\mathrm{V}(s_m) = \mathrm{lin}_k\{v_1, \dots, v_m\}$.

Example (Daniyarova, Remeslennikov)

- The dimension of n-parallelepiped $\mathbf{V}$ is

$$\dim(\mathbf{V}) = \dim(V_1) + \ldots + \dim(V_n).$$

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Free Lie algebra

Example (Daniyarova, Remeslennikov)

Theorem

Let $\mathbf{V}$ *be a n-parallelepiped over the free Lie algebra L over a field* k*. Algebraic sets over the algebra L, bounded by parallelepiped* $\mathbf{V}$*, lie in one-to-one correspondence with algebraic sets over the field* k*, which conform to systems of equations in* $\dim(\mathbf{V})$ *variables.*

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Let $\mathbf{V} = V_1 \times \ldots \times V_n$ and $\{v_1^i, \ldots, v_{m_i}^i\}$ — a basis of the linear space $V_i,\ i = \overline{1,n}$.

Then the correspondence (reference to the previous theorem) is set by the rule:

$$\begin{array}{ccc} \{\alpha_1^1 v_1^1 + \ldots + \alpha_{m_1}^1 v_{m_1}^1,\ \ldots,\ \alpha_1^n v_1^n + \ldots + \alpha_{m_n}^n v_{m_n}^n\} \subseteq \mathbf{V} \\ \searrow \qquad\qquad\qquad\qquad \nearrow \\ \{\alpha_1^1, \ldots, \alpha_{m_1}^1,\ \ldots,\ \alpha_1^n, \ldots, \alpha_{m_n}^n\} \subseteq k^{\dim(\mathbf{V})}. \end{array}$$

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