Algebraic geometry over algebraic structures
Lecture 2

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Workshop

December 16, 2008, Alagna Valsesia, Italia

Outline

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebraic structure $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_\exists(\mathcal{A}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Elements of Model Theory

Languages and algebras

Let $\mathcal{L} = \mathcal{F} \cup \mathcal{C}$ be a first-order language with no predicates, consisting of a set $\mathcal{F}$ of symbols of functions F, given together with their arities n_F, and a set of constants $\mathcal{C}$.

An $\mathcal{L}$-structure $\mathcal{A}$ is given by the following data:

- a non-empty set A called the universe of $\mathcal{A}$;
- a function $F^{\mathcal{A}} : A^{n_F} \to A$ of arity n_F for each function $F \in \mathcal{L}$;
- an element $c^{\mathcal{A}} \in A$ for each constant $c \in \mathcal{L}$.

We use notation $A, B, C, \ldots$ to refer to the universes of the structures $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$.

Structures in a language with no predicates are termed algebras. As usual, one can define the notions of $\mathcal{L}$-homomorphism, $\mathcal{L}$-isomorphism, $\mathcal{L}$-embedding, $\mathcal{L}$-epimorphism between $\mathcal{L}$-algebras.

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Example

- The language of groups $\mathcal{L}_{\mathrm{g}}$ consists of a binary operation $\cdot$ (multiplication), a unary operation $^{-1}$ (inversion), and a constant symbol e (the identity). Every group G with a natural interpretation of the symbols of $\mathcal{L}_{\mathrm{g}}$ is an $\mathcal{L}_{\mathrm{g}}$-structure.
- The language of additive commutative monoids $\mathcal{L}_{\mathrm{m}}$ consists of a binary operation $+$ (addition) and a constant symbol 0 (the identity).
- The language $\mathcal{L}_{\mathrm{Lie}}$ of Lie algebras over fixed field k consists of two binary operations $+$ and $[,]$ (addition and multiplication), a set of unary operations F_α, $\alpha \in k$ (multiplication by $\alpha \in k$), and constant symbol 0.

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Let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set of variables.

Recall that terms in $\mathcal{L}$ in variables X are formal expressions defined recursively as follows:

T1) variables $x_1, x_2, \ldots, x_n, \ldots$ are terms;

T2) constants from $\mathcal{L}$ are terms;

T3) if $F(x_1, \ldots, x_n) \in \mathcal{L}$ is function and $t_1, \ldots, t_n$ are terms then $F(t_1, \ldots, t_n)$ is a term.

By $\mathrm{T}_{\mathcal{L}}(X)$ we denote the set of all terms in $\mathcal{L}$ in variables X. The set of all atomic formulas $(t = s)$, $t, s \in \mathrm{T}_{\mathcal{L}}(X)$, we denote by $\mathrm{At}_{\mathcal{L}}(X)$.

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If given an $\mathcal{L}$-algebra $\mathcal{A}$ then every term $t(x_1, \ldots, x_n) \in \mathrm{T}_{\mathcal{L}}(X)$ defines a function $t^{\mathcal{A}} : A^n \to A$ via recursion by definition of t.

For example, when studying algebraic geometry over groups in the language of groups $\mathcal{L}_{\mathrm{g}}$ we may think about terms as words of free group, generated by X. And any atomic formula $(t = s)$ is equivalent to atomic formula of specific form $(t \cdot s^{-1} = e)$.

So, examining commutative associative rings in the language $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0\}$, we may think about terms as polynomials in variables X over the ring $\mathbb{Z}$. And any atomic formula $(t = s)$ is equivalent to atomic formula $(t - s = 0)$.

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Elements of Algebraic Geometry

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of variables.

- Equation in the language $\mathcal{L}$ in variables X is an atomic formula $(t = s) \in \mathrm{At}_{\mathcal{L}}(X)$, where t, s are terms;
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Let $\mathcal{A}$ be an $\mathcal{L}$-algebra.

- The solution of a system of equations S over $\mathcal{A}$,

$$\mathrm{V}_{\mathcal{A}}(S) = \{ (a_1, \ldots, a_n) \in A^n \mid t^{\mathcal{A}}(a_1, \ldots, a_n) = s^{\mathcal{A}}(a_1, \ldots, a_n) \\ \forall\, (t = s) \in S \},$$

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If someone wants to investigate the Diophantine algebraic geometry over $\mathcal{A}$ then it is enough to take instead of $\mathcal{L}$ the language $\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup \{c_a \mid a \in A\}$, which is obtained from $\mathcal{L}$ by adding a new constant c_a for every element $a \in A$.

The $\mathcal{L}$-algebra $\mathcal{A}$ in obvious way is an $\mathcal{L}_{\mathcal{A}}$-algebra.

Sometimes, to emphasize that formulas are from $\mathcal{L}$ we call such equations (and systems of equations) coefficient-free equations, meanwhile, in the case when $\mathcal{L} = \mathcal{L}_{\mathcal{A}}$, we refer to such equations as equations with coefficients in algebra $\mathcal{A}$.

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It is recognized two directions in papers on algebraic geometry over concrete algebraic structures: with coefficients and with no coefficients.

For instance, if G is some group, then it is said about algebraic geometry over G with no coefficients, when studying equations in the language of groups $\mathcal{L}_{\mathrm{g}}$ and corresponding algebraic sets over G. If one consider equations in the extended language $\mathcal{L}_{\mathrm{g},G}$, then it is said about algebraic geometry over G with coefficient in G. In this case equations are called an G-equations, coordinate groups are G-groups, etc.

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- The set of atomic formulas

$$\mathrm{Rad}(S) = \{\ (t = s) \in \mathrm{At}_{\mathcal{L}}(X) \mid t^{\mathcal{A}}(a_1, \ldots, a_n) = s^{\mathcal{A}}(a_1, \ldots, a_n) \\ \forall\ (a_1, \ldots, a_n) \in \mathrm{V}(S)\ \}$$

is termed the radical of the algebraic set $\mathrm{V}(S)$.

- The factor-algebra

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The set $\mathrm{T}_{\mathcal{L}}(X)$ of all terms in $\mathcal{L}$ in variables X with a natural interpretation of the symbols of $\mathcal{L}$ form absolutely free $\mathcal{L}$-algebra or termal algebra $\mathcal{T}_{\mathcal{L}}(X)$ with basis X.

Let Δ be a congruent set of atomic formulas. Then it defines congruence $\sim_\Delta$ on the algebra $\mathcal{T}_\mathcal{L}(X)$:

$$t \sim_\Delta s \quad \Longleftrightarrow \quad (t = s) \in \Delta, \quad t, s \in \mathrm{T}_\mathcal{L}(X).$$

More precisely, $\sim_\Delta$ is an equivalence relation on the set of terms $\mathrm{T}_\mathcal{L}(X)$, which preserves all functions from $\mathcal{L}$ such that factor-set $\mathrm{T}_\mathcal{L}(X)/\sim_\Delta$ has a natural interpretation of all of the symbols from $\mathcal{L}$. Resulting $\mathcal{L}$-structure with universe $\mathrm{T}_\mathcal{L}(X)/\sim_\Delta$ is termed factor-algebra. We denote it by $\mathcal{T}_\mathcal{L}(X)/\Delta$.

A set of atomic formulas $\Delta \subseteq \mathrm{At}_{\mathcal{L}}(X)$ is congruent if and only if it satisfies the following conditions:

1. $(t = t) \in \Delta$ for any term $t \in \mathrm{T}_{\mathcal{L}}(X)$;
2. if $(t_1 = t_2) \in \Delta$ then $(t_2 = t_1) \in \Delta$ for any terms $t_1, t_2 \in \mathrm{T}_{\mathcal{L}}(X)$;
3. if $(t_1 = t_2) \in \Delta$ and $(t_2 = t_3) \in \Delta$ then $(t_1 = t_3) \in \Delta$ for any terms $t_1, t_2, t_3 \in \mathrm{T}_{\mathcal{L}}(X)$;
4. if $(t_1 = s_1), \ldots, (t_{n_F} = s_{n_F}) \in \Delta$ then $(F(t_1, \ldots, t_{n_F}) = F(s_1, \ldots, s_{n_F})) \in \Delta$ for any terms $t_i, s_i \in \mathrm{T}_{\mathcal{L}}(X)$, $i = 1, \ldots, n_F$, and any function $F \in \mathcal{L}$.

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Elements of algebraic geometry

Major problem

One of the major problems of algebraic geometry over $\mathcal{L}$-algebra $\mathcal{A}$ consists in classifying algebraic sets over the algebra $\mathcal{A}$ with accuracy up to isomorphism.

One can classify algebraic sets by means of three languages, which are equivalent to each other:

1. in geometric language, by describing algebraic sets directly;
2. in the language of radical ideals;
3. and in algebraic language, by classifying coordinate algebras of algebraic sets.

Every algebraic set may be restored in unique manner from its radical and it may be restored from its coordinate structure just up to isomorphism.

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The category of coordinate algebras

We introduce two categories: the category $\mathbf{AS}(\mathcal{A})$ of algebraic sets over $\mathcal{A}$ and the category $\mathbf{CA}(\mathcal{A})$ of coordinate algebras of algebraic sets over $\mathcal{A}$.

Objects of $\mathbf{CA}(\mathcal{A})$ are all coordinate algebras of algebraic sets over $\mathcal{A}$. Morphism here are $\mathcal{L}$-homomorphisms.

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The category of algebraic sets

Objects of $\mathbf{AS}(\mathcal{A})$ are all algebraic sets over $\mathcal{A}$. To define morphisms in $\mathbf{AS}(\mathcal{A})$ we need the notion of a term-map. A map $\Pi : A^n \to A^m$ is called a term-map if there exist terms $t_1, \ldots, t_m \in \mathrm{T}_{\mathcal{L}}(x_1, \ldots, x_n)$ such that

$$\Pi(b_1, \ldots, b_n) = (\, t_1^{\mathcal{A}}(b_1, \ldots, b_n)\,,\ \ldots\ ,\ t_m^{\mathcal{A}}(b_1, \ldots, b_n)\,)$$

for all $(b_1, \ldots, b_n) \in A^n$. For two non-empty algebraic sets $Y \subseteq A^n$ and $Z \subseteq A^m$ a map $\Pi : Y \to Z$ is called a term-map if it is a restriction on Y of some term-map $\Pi : A^n \to A^m$ such that $\Pi(Y) \subseteq Z$.

The theorem on dual equivalence

As usual, one can define the notion of a isomorphism in the categories $\mathbf{CA}(\mathcal{A})$ and $\mathbf{AS}(\mathcal{A})$.

Theorem

The category $\mathbf{AS}(\mathcal{A})$ of algebraic sets over algebra $\mathcal{A}$ and the category $\mathbf{CA}(\mathcal{A})$ of coordinate algebras of algebraic sets over $\mathcal{A}$ are dually equivalent.

Corollary

Two algebraic sets Y and Z over algebra $\mathcal{A}$ are isomorphic if and only if $\Gamma(Y) \cong \Gamma(Z)$.

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The category $\mathbf{AS}(\mathcal{A})$ of algebraic sets over algebra $\mathcal{A}$ and the category $\mathbf{CA}(\mathcal{A})$ of coordinate algebras of algebraic sets over $\mathcal{A}$ are dually equivalent.

Corollary

Two algebraic sets Y and Z over algebra $\mathcal{A}$ are isomorphic if and only if $\Gamma(Y) \cong \Gamma(Z)$.

Corresponding

How algebraic sets and coordinate algebras correspond to each other?

If $Y \subseteq A^n$ an algebraic set over $\mathcal{A}$, then his coordinate algebra $\Gamma(Y)$ is corresponds to him.

Otherwise, if we known that $\mathcal{C}$ is a coordinate algebra of some algebraic set Y over $\mathcal{A}$, then we can white algebraic set, isomorphic to Y. It is the set $\mathrm{Hom}(\mathcal{C}, \mathcal{A})$ of all homomorphisms from $\mathcal{C}$ to $\mathcal{A}$. More detailed, as

$$\mathcal{C} = \mathcal{T}_{\mathcal{L}}(x_1, \ldots, x_n)/\mathrm{Rad}(S),$$

every homomorphism $h \in \mathrm{Hom}(\mathcal{C}, \mathcal{A})$ is uniquely defined by the images of elements $x_i/\mathrm{Rad}(S) \in A$, $i = \overline{1, n}$. These tuples form appropriate algebraic set over $\mathcal{A}$.

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Unification Theorems

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_\exists(\mathcal{A}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem B (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated $\mathcal{A}$-algebra $\mathcal{C}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{C})$, *i.e.*, $\mathcal{C} \equiv_{\forall,\mathcal{A}} \mathcal{A}$;
2. $\mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{C})$, *i.e.*, $\mathcal{C} \equiv_{\exists,\mathcal{A}} \mathcal{A}$;
3. $\mathcal{C}$ *$\mathcal{A}$-embeds into an ultrapower of $\mathcal{A}$;*
4. $\mathcal{C}$ *is $\mathcal{A}$-discriminated by $\mathcal{A}$;*
5. $\mathcal{C}$ *is a limit algebra over $\mathcal{A}$;*
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A})$ *in the language $\mathcal{L}_{\mathcal{A}}$;*
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in $\mathcal{A}$.*

Unification Theorem C (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, *i.e.,* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$*;*
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$*;*
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$*;*
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$*;*
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$*;*
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ *in* $\mathcal{L}$*;*
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem D (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebra in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated $\mathcal{A}$-algebra $\mathcal{C}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}_{\mathcal{A}}(\mathcal{A})$, *i.e.*, $\mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}_{\mathcal{A}}(\mathcal{A})$;
3. $\mathcal{C}$ *$\mathcal{A}$-embeds into a direct power of $\mathcal{A}$;*
4. $\mathcal{C}$ *is $\mathcal{A}$-separated by $\mathcal{A}$;*
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over $\mathcal{A}$;*
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{A})$ *in the language $\mathcal{L}_{\mathcal{A}}$;*
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in $\mathcal{A}$.*

Formulas in $\mathcal{L}$ in variables X are defined recursively as follows:

F1) atomic formulas are formulas;

F2) if Φ and Ψ are formulas then $\neg\Phi, (\Phi \vee \Psi), (\Phi \wedge \Psi), (\Phi \to \Psi)$ are formulas;

F3) If Φ is a formula and x is a variable then $\forall x\, \Phi$ and $\exists x\, \Phi$ are formulas.

One of the principle results in mathematical logic states that any formula Φ is equivalent to a formula Ψ in the following prenex form:

$$Q_1 x_1 \dots Q_n x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} \Psi_{ij} \right),$$

where $Q_i \in \{\forall, \exists\}$ and Ψ_{ij} is an atomic formula or its negation.

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Recall that a universal formula in $\mathcal{L}$ is a formula of the type

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right),$$

and a quasi-identity is a universal formula of the type

$$\forall x_1 \dots \forall x_n \left((\bigwedge_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x})) \quad \rightarrow \quad (t(\bar{x}) = s(\bar{x})) \right),$$

where $t(\bar{x}), s(\bar{x}), t_i(\bar{x}), s_i(\bar{x}), w_{ij}(\bar{x}), v_{ij}(\bar{x})$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \dots, x_n)$.

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We denote by $\mathrm{Th}_{\forall}(\mathcal{A})$ the set of all universal formulas in $\mathcal{L}$ which hold on the algebra $\mathcal{A}$. Similarly, $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ is the set of all quasi-identities in $\mathcal{L}$ which hold on $\mathcal{A}$.

The universal closure of $\mathcal{A}$ (denoted by **Ucl**$(\mathcal{A})$) is the class of all algebras in $\mathcal{L}$ which satisfy all formulas from $\mathrm{Th}_{\forall}(\mathcal{A})$. And quasivariety generated by $\mathcal{A}$ (denoted by **Qvar**$(\mathcal{A})$) is the class of all algebras in $\mathcal{L}$ which satisfy all formulas from $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$.

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Recall that a existential formula in $\mathcal{L}$ is a formula of the type

$$\exists x_1 \ldots \exists x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right),$$

where $w_{ij}(\bar{x}), v_{ij}(\bar{x})$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \ldots, x_n)$.

We denote by $\mathrm{Th}_\exists(\mathcal{A})$ the set of all existential formulas in $\mathcal{L}$ which hold on the algebra $\mathcal{A}$.

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We denote by $\mathrm{Th}_{\exists}(\mathcal{A})$ the set of all existential formulas in $\mathcal{L}$ which hold on the algebra $\mathcal{A}$.

By **Pvar**($\mathcal{A}$) we denote the prevariety, generated by algebra $\mathcal{A}$, i.e., the least class of $\mathcal{L}$-algebras, closed under direct products and subalgebras, and containing algebra $\mathcal{A}$.

The definitions of direct product and subalgebras for $\mathcal{L}$-algebras is the same as for groups. So, subalgebra of algebra $\mathcal{A}$ is any subset of universe $B \subseteq A$, closed under all functions from $\mathcal{L}$ and containing all constants from $\mathcal{L}$.

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We say that an $\mathcal{L}$-algebra $\mathcal{B}$ is $\mathcal{L}$-separated by $\mathcal{L}$-algebra $\mathcal{A}$ if for any distinct elements $b_1, b_2 \in B$ there is a $\mathcal{L}$-homomorphism $h : \mathcal{B} \to \mathcal{A}$ such that $h(b_1) \neq h(b_2)$.

We say that an $\mathcal{L}$-algebra $\mathcal{B}$ is $\mathcal{L}$-discriminated by $\mathcal{L}$-algebra $\mathcal{A}$ if for any finite set W of elements from B there is a $\mathcal{L}$-homomorphism $h : \mathcal{B} \to \mathcal{A}$ whose restriction onto W is injective.

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