Algebraic geometry over algebraic structures
Lecture 1

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based on joint results with Evelina Yu. Daniyarova[1] and
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Outline

Let $\mathcal{A}$ be an algebraic structure (group, monoid, ring etc.).

- Definition of algebraic geometry over $\mathcal{A}$;
- Universal algebraic geometry;
- References.

Unification Theorem A (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebraic structure $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_\exists(\mathcal{A}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

By $\mathcal{L}_\mathcal{A} = \mathcal{L} \cup \{a \mid a \in \mathcal{A}\}$ we denote the language $\mathcal{L}$ extended by elements from $\mathcal{A}$ as new constant symbols. We say that a structure $\mathcal{C}$ in the language $\mathcal{L}_\mathcal{A}$ is an $\mathcal{A}$-structure if the substructure of $\mathcal{C}$ generated by the constants $a \in \mathcal{A}$ is canonically isomorphic to $\mathcal{A}$.

Unification Theorem B (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated algebraic $\mathcal{A}$-structure $\mathcal{C}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{C})$, *i.e.*, $\mathcal{C} \equiv_{\forall,\mathcal{A}} \mathcal{A}$;
2. $\mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{C})$, *i.e.*, $\mathcal{C} \equiv_{\exists,\mathcal{A}} \mathcal{A}$;
3. $\mathcal{C}$ *$\mathcal{A}$-embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is $\mathcal{A}$-discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{A})$ *in the language* $\mathcal{L}_{\mathcal{A}}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in* $\mathcal{A}$.

Unification Theorem C (No coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebraic structure $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, *i.e.*, $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of coefficient-free equations.*

Unification Theorem D (With coefficients)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in the language $\mathcal{L}_{\mathcal{A}}$ (with no predicates in $\mathcal{L}$). Then for a finitely generated algebraic $\mathcal{A}$-structure $\mathcal{C}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}_{\mathcal{A}}(\mathcal{A})$, *i.e.,* $\mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{A}) = \mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}_{\mathcal{A}}(\mathcal{A})$;
3. *$\mathcal{C}$ $\mathcal{A}$-embeds into a direct power of $\mathcal{A}$;*
4. *$\mathcal{C}$ is $\mathcal{A}$-separated by $\mathcal{A}$;*
5. *$\mathcal{C}$ is a subdirect product of finitely many limit algebras over $\mathcal{A}$;*
6. *$\mathcal{C}$ is an algebra defined by a complete atomic type in the theory $\mathrm{Th}_{\mathrm{qi},\mathcal{A}}(\mathcal{A})$ in the language $\mathcal{L}_{\mathcal{A}}$;*
7. *$\mathcal{C}$ is the coordinate algebra of a non-empty algebraic set over $\mathcal{A}$ defined by a system of equations with coefficients in $\mathcal{A}$.*

- Dually equivalence theorem for the categories of algebraic sets and coordinate algebras;
- General theorems on dimension;
- Geometric equivalence;
- The process that Alexei talked about.

- Model-theoretic methods, more precisely, methods of atomic model theory;
- Methods of universal algebra.

1. Motivated by other sciences and branches of Mathematics;
2. Every scientific branch can sustain life while it has an excess of new problems;
3. It is to contribute to the unity of Mathematics.

1. Quantum polynomials (physics and quantum mechanics);
2. Min-Max algebraic structures;
   - Let $\mathcal{A}$ be an $\mathcal{L}$-algebra, extend $\mathcal{L}$ to $\mathcal{L}^* = \mathcal{L} \cup \{\min(x,y), \max(x,y)\}$;
   - For example, take $\langle \mathbb{R}, +, \min \rangle$;
   - The study of the latter is motivated by quantum mechanics and linear programming.
3. Take $\langle \mathbb{R}, +, -, \cdot, 0, 1, f(x) = y \rangle$, where f is an analytic function, e.g. $f = e^x$. This algebraic system is important in studying analytic varieties.
4. Extended Presburger arithmetic — for problems in computer science.

Let $\mathcal{L} = \mathcal{F} \cup \mathcal{C}$ be a first-order language with no predicates, consisting of a set $\mathcal{F}$ of symbols of operations F (given together with their arities n_F), and a set of constants $\mathcal{C}$.

Example

- The language of groups $\mathcal{L}_{\mathrm{g}}$ consists of a binary operation $\cdot$ (multiplication), a unary operation $^{-1}$ (inversion), and a constant symbol e (the identity). Every group G with a natural interpretation of the symbols of $\mathcal{L}_{\mathrm{g}}$ is an $\mathcal{L}_{\mathrm{g}}$-structure.
- The language of additive commutative monoids $\mathcal{L}_{\mathrm{m}}$ consists of a binary operation $+$ (addition) and a constant symbol 0 (the identity).
- The language $\mathcal{L}_{\mathrm{Lie}}$ of Lie algebras over fixed field k consists of two binary operations $+$ and $[,]$ (addition and multiplication), a set of unary operations F_α, $\alpha \in k$ (multiplication by $\alpha \in k$), and constant symbol 0.

Languages and formulas

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Let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set of variables.

Recall that terms in $\mathcal{L}$ in variables X are formal expressions defined recursively as follows:

T1) variables $x_1, x_2, \ldots, x_n, \ldots$ are terms;

T2) constants from $\mathcal{L}$ are terms;

T3) if $F(x_1, \ldots, x_n) \in \mathcal{L}$ is function and $t_1, \ldots, t_n$ are terms then $F(t_1, \ldots, t_n)$ is a term.

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By $\mathrm{T}_{\mathcal{L}}(X)$ we denote the set of all terms in $\mathcal{L}$.

The set of all atomic formulas $(t = s)$, $t, s \in \mathrm{T}_{\mathcal{L}}(X)$, we denote by $\mathrm{At}_{\mathcal{L}}(X)$.

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Formulas in $\mathcal{L}$ in variables X are defined recursively as follows:

F1) atomic formulas are formulas;

F2) if ϕ and ψ are formulas then $\neg\phi, (\phi \vee \psi), (\phi \wedge \psi), (\phi \to \psi)$ are formulas;

F3) If ϕ is a formula and x is a variable then $\forall x\phi$ and $\exists x\phi$ are formulas.

One of the principle results in mathematical logic states that any formula ϕ is equivalent to a formula ψ in the following form:

$$Q_1x_1 \dots Q_nx_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} \psi_{ij} \right),$$

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Recall that a universal formula in $\mathcal{L}$ is a formula of the type

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right),$$

and a quasi-identity is a universal formula of the type

$$\forall x_1 \dots \forall x_n \left((\bigwedge_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x})) \quad \rightarrow \quad (t(\bar{x}) = s(\bar{x})) \right),$$

where $t(\bar{x}), s(\bar{x}), t_i(\bar{x}), s_i(\bar{x}), w_{ij}(\bar{x}), v_{ij}(\bar{x})$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \dots, x_n)$.

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An $\mathcal{L}$-structure $\mathcal{A}$ is given by the following data:

- a non-empty set A called the universe of $\mathcal{A}$;
- a function $F^{\mathcal{A}} : A^{n_F} \to A$ of arity n_F for each $F \in \mathcal{F}$;
- an element $c^{\mathcal{A}} \in A$ for each $c \in \mathcal{C}$.

We use notation $A, B, C, \ldots$ to refer to the universes of the structures $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$.

Structures in a language with no predicates are termed algebras.

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Universal classes, quasivarieties

We denote by $\mathrm{Th}_\forall(\mathcal{A})$ the set of all universal formulas in $\mathcal{L}$ which hold on the algebra $\mathcal{A}$. Similarly $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ is the set of all quasi-identities in $\mathcal{L}$ which hold on $\mathcal{A}$.

The universal closure of $\mathcal{A}$ (**Ucl**$(\mathcal{A})$) is the class of all algebras in $\mathcal{L}$ which satisfy all formulas from $\mathrm{Th}_\forall(\mathcal{A})$. And quasivariety generated by $\mathcal{A}$ (**Qvar**$(\mathcal{A})$) is the class of all algebras in $\mathcal{L}$ which satisfy all formulas from $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$.

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Existential formulas and classes

Recall that a existential formula in $\mathcal{L}$ is a formula of the type

$$\exists x_1 \dots \forall x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right),$$

where $w_{ij}(\bar{x}), v_{ij}(\bar{x})$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \dots, x_n)$.

We denote by $\mathrm{Th}_\exists(\mathcal{A})$ the set of all existential formulas in $\mathcal{L}$ which hold on the algebra $\mathcal{A}$.

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Examples of universal classes

Abelian groups

Theorem

If A and B are two torsion-free abelian groups, then $\mathrm{Th}_\forall(A) = \mathrm{Th}_\forall(B)$. *Therefore,*

$$\mathbf{Ucl}(A) = \{\textit{torsion free abelian groups}\}.$$

1. If $A = \langle \mathbb{N}, + \rangle$, then $\mathbf{Ucl}(A)$ is understood;
2. Let F_1 and F_2 be free non-abelian group.

Theorem

Let F_1 and F_2 be free non-abelian groups, then $\mathrm{Th}_\forall(F_1) = \mathrm{Th}_\forall(F_2)$.

Let $X = \{x_1, \dots, x_n\}$ be a finite set of variables.

- Equation in the language $\mathcal{L}$ in variables X is an atomic formula $(t = s) \in \mathrm{At}_{\mathcal{L}}(X)$, where t, s are terms;
- Any subset $S \subseteq \mathrm{At}_{\mathcal{L}}(X)$ forms a system of equations in $\mathcal{L}$.

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Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

- The solution of a system of equations S over $\mathcal{A}$,

$$\mathrm{V}_{\mathcal{A}}(S) = \{\ (a_1, \ldots, a_n) \in A^n \mid t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n) \\ \forall\ (t = s) \in S\ \},$$

  is termed the algebraic set over $\mathcal{A}$.

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  is termed the algebraic set over $\mathcal{A}$.

If someone wants to investigate the Diophantine algebraic geometry over $\mathcal{A}$ then it is enough to take instead of $\mathcal{L}$ the language $\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup \{c_a \mid a \in A\}$, which is obtained from $\mathcal{L}$ by adding a new constant c_a for every element $a \in A$.

The structure $\mathcal{A}$ in obvious way is an $\mathcal{L}_{\mathcal{A}}$-structure.

We also refer to equations in the language $\mathcal{L}_{\mathcal{A}}$ as $\mathcal{A}$-equations.

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The equivalent problem is problem of classification of coordinate algebras of algebraic sets over $\mathcal{A}$.

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Theorem

Every algebraic set over equationally Noetherian structure $\mathcal{A}$ can be expressed as a finite union of irreducible algebraic sets (irreducible components). Furthermore, this decomposition is unique up to permutation of irreducible components and omission of superfluous ones.

Theorem

Let $\mathcal{B}$ be an equationally Noetherian $\mathcal{L}$-algebra. A finitely generated $\mathcal{L}$-algebra $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{B}$ if and only if it is a subdirect product of finitely many coordinate algebras of irreducible algebraic sets over $\mathcal{B}$.

Corollary

Classification of irreducible algebraic sets (or/and their coordinate structures) is the essential problem of algebraic geometry over equationally Noetherian structure.

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References

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