Structure Theorems for Subgroups of $Homeo(S^1)$

Francesco Matucci

(joint with C.Bleak and M.Kassabov)

Alagna Valsesia, December 19th, 2008

Introduction

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Loose idea: First study elements with fixed points. Then study the action of elements which have no fixed points. Understand the interaction of these two types of elements.

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Theorem (Farb-Franks)

Every finitely-generated, torsion-free nilpotent group is isomorphic to a subgroup of $Diff_+^1(S^1)$*.*

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Given $f \in Homeo_+(S^1)$, a **lift** F of f is a map $F : \mathbb{R} \to \mathbb{R}$ such that

- for all $x \in \mathbb{R}, F(x+1) = F(x) + 1$, and
- $f(x) = F(x) \pmod 1$.

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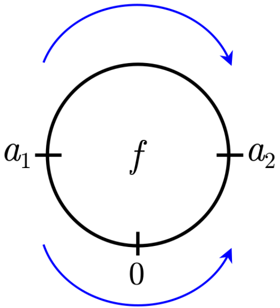

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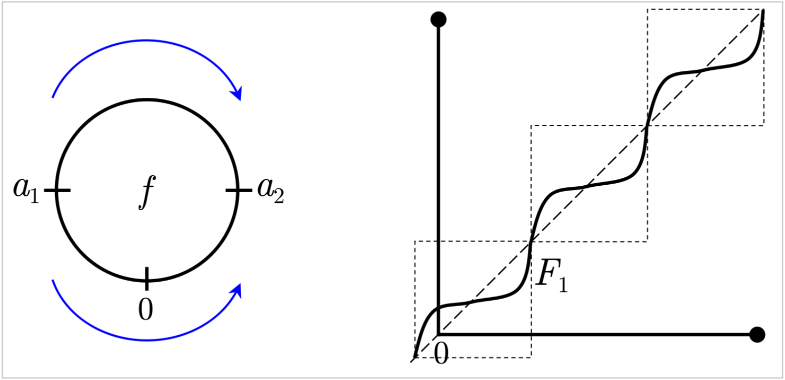

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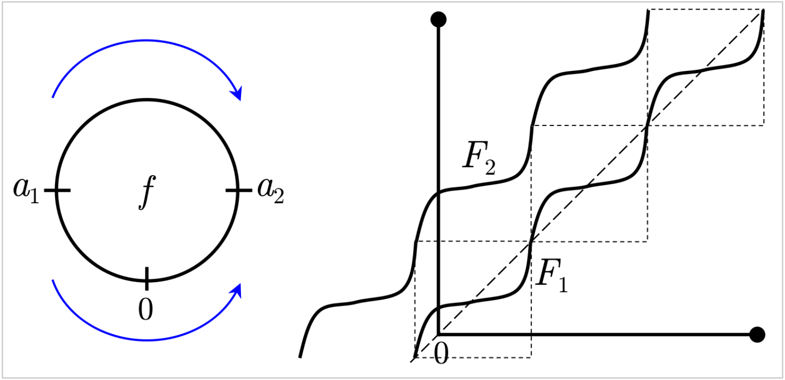

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The limit is independent of the choice of x and of the lift.

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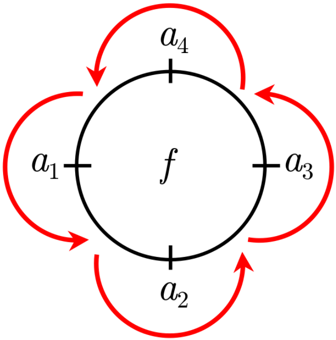

$$f([a_i, a_{i+1}]) = [a_{i+1}, a_{i+2}]$$

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Then f is conjugate to a rotation by an element in $Homeo_+(S^1)$.

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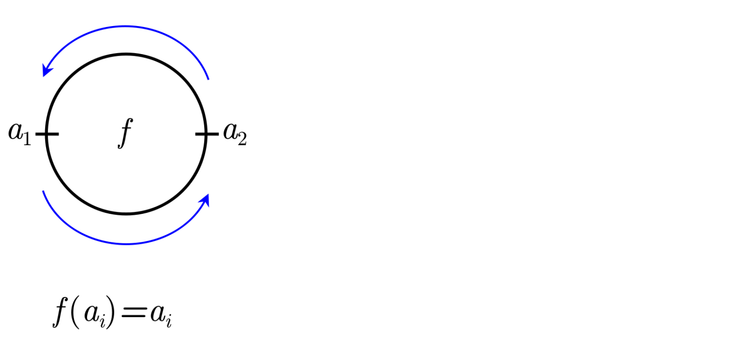

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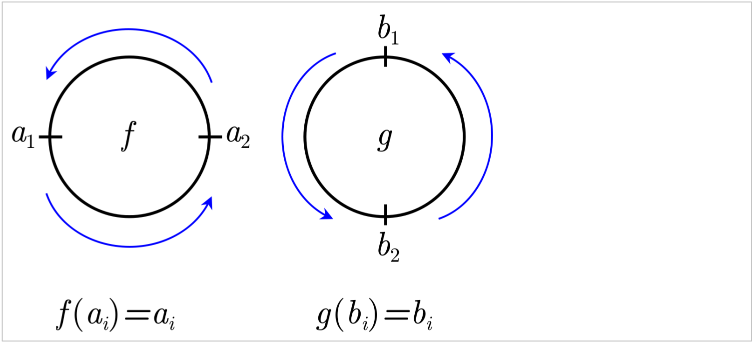

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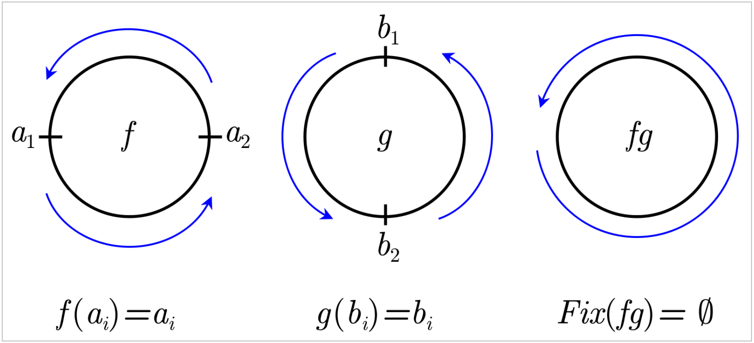

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Theorem (Ping-Pong)

Let G be a group of permutations on a set X. Let g_1 and g_2 be elements of G. If there are non-empty, disjoint sets X_1 and X_2 contained in X, where for all $n \neq 0$ and $i \neq j$, we have $X_i g_j^n \subset X_j$, then $\langle g_1, g_2 \rangle \leq G$ is isomorphic to a free group on two generators.

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- *The subset* G_0 *is a subgroup.*
- *The map* $rot : G \to \mathbb{R}/\mathbb{Z}$ *is a homomorphism,*
- $\ker(rot) = G_0$,
- $G/G_0 \cong rot(G) \leq \mathbb{R}/\mathbb{Z}$.

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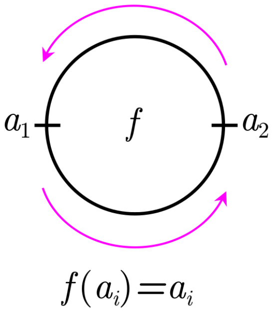

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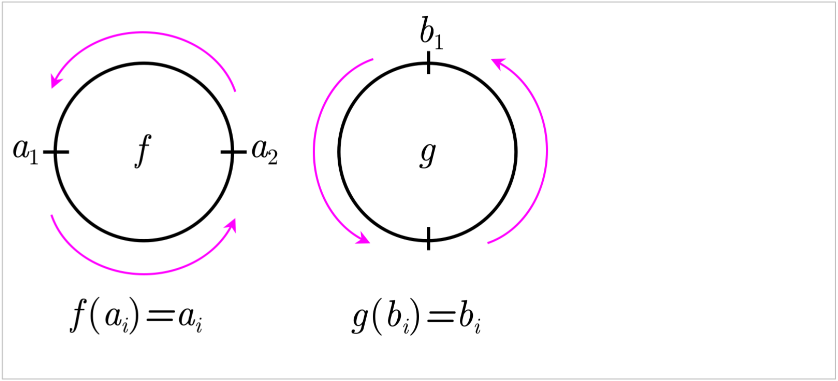

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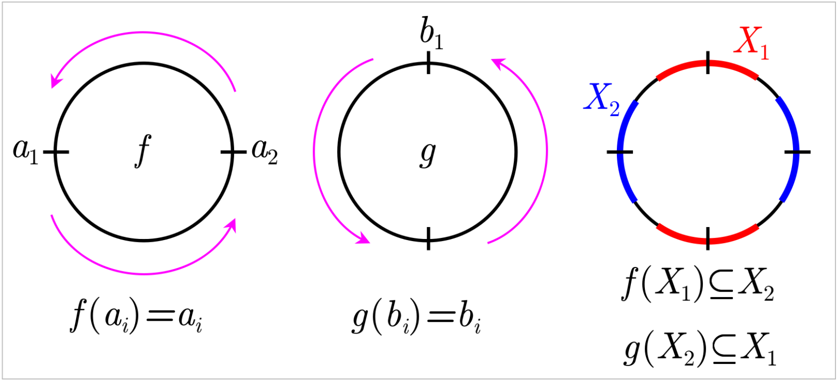

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- *For every finitely generated $H \leq G_0$, we have $Fix(H) \neq \emptyset$ (finite intersection property).*
- *G_0 admits a global fixed point (by compactness of S^1).*

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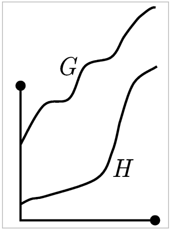

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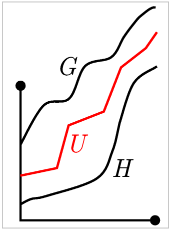

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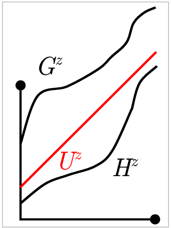

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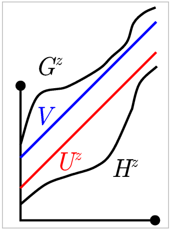

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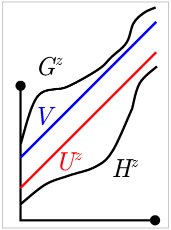

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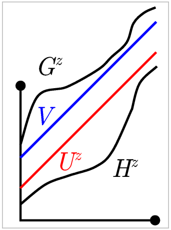

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Corollary

The commutator subgroup $[G, G]$ lies inside G_0.

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$$rot(fg) = rot(f) + rot(g). \square$$

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If $a = g(s), b = h(s)$, then it becomes

$$= rot(h) - rot(g) \qquad \square$$

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Proof: Let $s \in Fix(G_0)$. By Denjoy's Theorem, the orbit $s^G \subseteq Fix(G_0)$ is dense in S^1. So $Fix(G_0) = S^1$ and $G \leq \mathbb{R}/\mathbb{Z}$. $\square$

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D is a fundamental domain for G acting on the intervals of $S^1 \setminus \overline{s^G}$.

We consider H_0 restriction of G_0 to the domain $D = \bigcup I_i$ and define it the identity otherwise.

Theorem (BKM)

Suppose G admits no non-abelian free subgroups, then

- *G embeds in $\mathbb{R}/\mathbb{Z}$, or*
- *G embeds in $H_0 \wr K$, unrestricted wreath product, where $K := G/G_0$ is isomorphic to a subgroup of $\mathbb{R}/\mathbb{Z}$ (at most countable) and $H_0 \leq \prod Homeo_+(I_i)$ has no non-abelian free subgroups.*

Recall: $H_0 \wr K = K \ltimes \prod_{k\in K} H_0^k$.

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Theorem (BKM)

For every $K \leq \mathbb{Q}/\mathbb{Z}$, there is an embedding of the restricted wreath product $F \wr K$ into T, where F and T are the respective Thompson's groups.

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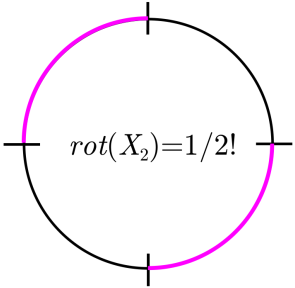

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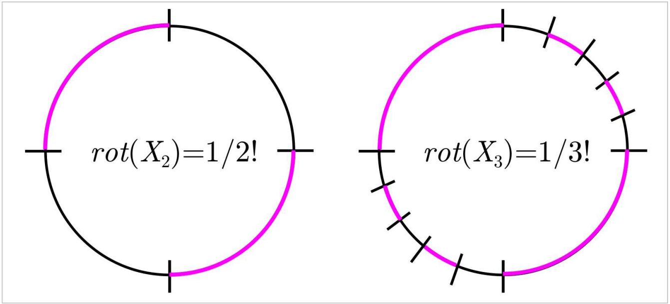

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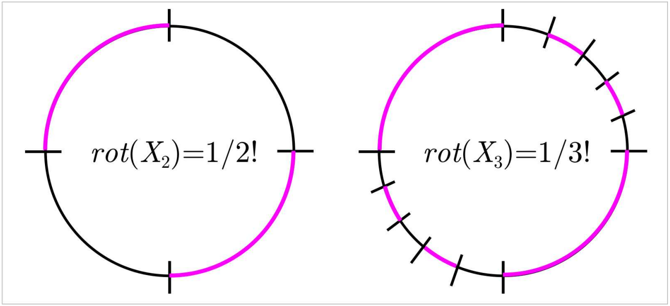

We have $(X_n)^n = X_{n-1}$ and the domain D is an interval.

How we started: Solvable subgroups of $PL_+(S^1)$

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- *embeds in $G_n \wr K$, restricted wreath product, for some K subgroup of $\mathbb{Q}/\mathbb{Z}$ and some positive integer n.*

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Theorem (BKM)

A subgroup H of $PL_+(S^1)$ either

- *contains a non-abelian free subgroup on two generators, or*
- *contains a copy of W, or*
- *is solvable.*