Deciding endo-fixedness

Enric Ventura

Departament de Matemàtica Aplicada III

Universitat Politècnica de Catalunya

Alagna, December 17, 2008

Outline

1. Some history

2. Algorithmic results

3. Needed tools

4. The proof

Outline

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (*n* letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\operatorname{Aut}(F_n) \subseteq \operatorname{Mono}(F_n) \subseteq \operatorname{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\operatorname{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \operatorname{End}(F_n)$ then
  $\operatorname{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \operatorname{Fix}(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{Mono}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}\,(F_n)$ then
  $\mathrm{Fix}\,(S) = \{x \in F_n \mid x\phi = x \;\forall \phi \in S\} = \cap_{\phi \in S}\mathrm{Fix}\,(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\text{Aut}(F_n) \subseteq \text{Mono}(F_n) \subseteq \text{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\text{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \text{End}(F_n)$ then
  $\text{Fix}(S) = \{x \in F_n \mid x\phi = x \; \forall \phi \in S\} = \cap_{\phi \in S}\text{Fix}(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\text{Aut}(F_n) \subseteq \text{Mono}(F_n) \subseteq \text{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\text{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \text{End}(F_n)$ then
  $\text{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S}\text{Fix}(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (*n* letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{Mono}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}\,(F_n)$ then
  $\mathrm{Fix}\,(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S}\mathrm{Fix}\,(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).

- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.

- F_n is the free group on A.

- $\text{Aut}\,(F_n) \subseteq \text{Mono}\,(F_n) \subseteq \text{End}\,(F_n)$.

- I let endomorphisms $\phi\colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.

- $\text{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

- If $S \subseteq \text{End}\,(F_n)$ then
  $\text{Fix}\,(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S}\text{Fix}\,(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{Mono}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}(F_n)$ then
  $\mathrm{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S}\mathrm{Fix}(\phi) \leqslant F_n$.

Fixed subgroups are complicated

$$\begin{aligned}
\phi \colon F_3 &\rightarrow F_3 \\
a &\mapsto a \\
b &\mapsto ba \\
c &\mapsto ca^2
\end{aligned}$$

$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{aligned}
\varphi \colon F_4 &\rightarrow F_4 \\
a &\mapsto dac \\
b &\mapsto c^{-1}a^{-1}d^{-1}ac \\
c &\mapsto c^{-1}a^{-1}b^{-1}ac \\
d &\mapsto c^{-1}a^{-1}bc
\end{aligned}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}$.

Fixed subgroups are complicated

$$\phi \colon F_3 \rightarrow F_3$$
$$a \mapsto a$$
$$b \mapsto ba$$
$$c \mapsto ca^2$$

$\text{Fix } \phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\varphi \colon F_4 \rightarrow F_4$$
$$a \mapsto dac$$
$$b \mapsto c^{-1}a^{-1}d^{-1}ac$$
$$c \mapsto c^{-1}a^{-1}b^{-1}ac$$
$$d \mapsto c^{-1}a^{-1}bc$$

$\text{Fix } \varphi = \langle w \rangle, \text{ where...}$

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}$.

Fixed subgroups are complicated

$$\begin{aligned}
\phi\colon F_3 &\rightarrow F_3 \\
a &\mapsto a \\
b &\mapsto ba \\
c &\mapsto ca^2
\end{aligned}$$

$\text{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{aligned}
\varphi\colon F_4 &\rightarrow F_4 \\
a &\mapsto dac \\
b &\mapsto c^{-1}a^{-1}d^{-1}ac \\
c &\mapsto c^{-1}a^{-1}b^{-1}ac \\
d &\mapsto c^{-1}a^{-1}bc
\end{aligned}$$

$\text{Fix}\,\varphi = \langle w \rangle, \text{where...}$

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}$.

Fixed subgroups are complicated

$$
\begin{aligned}
\phi \colon F_3 &\rightarrow F_3 \\
a &\mapsto a \\
b &\mapsto ba \\
c &\mapsto ca^2
\end{aligned}
$$

Fix $\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$
\begin{aligned}
\varphi \colon F_4 &\rightarrow F_4 \\
a &\mapsto dac \\
b &\mapsto c^{-1}a^{-1}d^{-1}ac \\
c &\mapsto c^{-1}a^{-1}b^{-1}ac \\
d &\mapsto c^{-1}a^{-1}bc
\end{aligned}
$$

Fix $\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}$.

Fixed subgroups are complicated

$$\phi \colon F_3 \rightarrow F_3$$
$$a \mapsto a$$
$$b \mapsto ba$$
$$c \mapsto ca^2$$

Fix $\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\varphi \colon F_4 \rightarrow F_4$$
$$a \mapsto dac$$
$$b \mapsto c^{-1}a^{-1}d^{-1}ac$$
$$c \mapsto c^{-1}a^{-1}b^{-1}ac$$
$$d \mapsto c^{-1}a^{-1}bc$$

Fix $\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

Theorem (Turner, 96)

Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n - 1$.

Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

Theorem (Turner, 96)

Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n - 1$.

Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

Theorem (Turner, 96)

Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n-1$.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert *if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.*

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that   1-mono-fixed   =   1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that   1-mono-fixed   =   1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that   1-mono-fixed   =   1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that   1-mono-fixed   =   1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that   1-mono-fixed   =   1-auto-fixed.

Relations between them

$$
\boxed{1 - auto - fixed} \quad \subseteq \quad \boxed{1 - endo - fixed}
$$

$$
\cap| \qquad\qquad\qquad\qquad \cap|
$$

$$
\boxed{auto - fixed} \quad \subseteq \quad \boxed{endo - fixed}
$$

Relations between them

$$\boxed{1 - auto - fixed} \quad \begin{array}{c} \subseteq \\ \neq \end{array} \quad \boxed{1 - endo - fixed}$$

$$\cap| \qquad\qquad\qquad\qquad \cap|$$

$$\boxed{auto - fixed} \quad \begin{array}{c} \subseteq \\ \neq \end{array} \quad \boxed{endo - fixed}$$
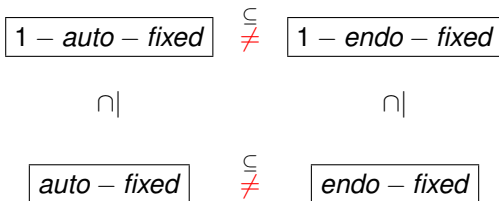
Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c \rangle$ and $H = \langle b, cacbab^{-1}c^{-1} \rangle \leqslant F_3$. Then,
$H = Fix(a \mapsto 1,\ b \mapsto b,\ c \mapsto cacbab^{-1}c^{-1})$, but H is NOT the fixed
subgroup of any set of automorphism of F_3.

Relations between them

$$
\boxed{1 - auto - fixed} \quad \begin{array}{c} \subseteq \\ \neq \end{array} \quad \boxed{1 - endo - fixed}
$$

$$
\cap|\ \ \|\,? \qquad\qquad\qquad \cap|\ \ \|\,?
$$

$$
\boxed{auto - fixed} \quad \begin{array}{c} \subseteq \\ \neq \end{array} \quad \boxed{endo - fixed}
$$
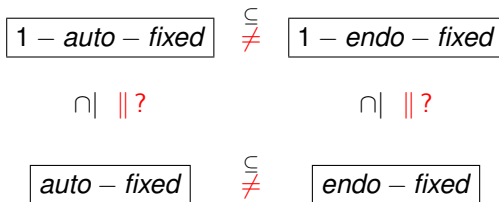
Theorem (Martino-V., 00)

Let $S \subseteq End(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $Fix(S) \leqslant_{\mathrm{ff}} Fix(\phi)$.

But... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

Outline

1. Some history

2. Algorithmic results

3. Needed tools

4. The proof

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it is an algorithm!

Conjecture

Fixed subgroups of endomorphisms of F_n are computable.

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it is an algorithm!

Conjecture

Fixed subgroups of endomorphisms of F_n are computable.

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it is an algorithm!

Conjecture

Fixed subgroups of endomorphisms of F_n are computable.

Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\text{fg}} F_n$, one can algorithmically decide whether

- i) *H is auto-fixed or not,*

- ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $\text{Fix}(S) = H$.

Conjecture

Given $H \leqslant_{\text{fg}} F_n$, one can algorithmically decide whether

- i) *H is 1-auto-fixed or not,*

- ii) *H is 1-endo-fixed or not,*

and in the affirmative case, find one automorphism (endomorphism) ϕ of F_n such that $\text{Fix}(\phi) = H$.

Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

  i) *H is auto-fixed or not,*

  ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $Fix(S) = H$.

Conjecture

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

  i) *H is 1-auto-fixed or not,*

  ii) *H is 1-endo-fixed or not,*

and in the affirmative case, find one automorphism (endomorphism) ϕ of F_n such that $Fix(\phi) = H$.

Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

  i) *H is auto-fixed or not,*

  ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $Fix(S) = H$.

Conjecture

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

  i) *H is 1-auto-fixed or not,*

  ii) *H is 1-endo-fixed or not,*

and in the affirmative case, find one automorphism (endomorphism) ϕ of F_n such that $Fix(\phi) = H$.

Outline

Fixed closures

Definition

Given $H \leqslant_{\mathrm{fg}} F_n$, we define the (auto- and endo-) stabilizer of H, respectively, as

$$Aut_H(F_n) = \{\phi \in Aut(F_n) \mid H \leqslant Fix(\phi)\} \leqslant Aut(F_n)$$

and

$$End_H(F_n) = \{\phi \in End(F_n) \mid H \leqslant Fix(\phi)\} \leqslant End(F_n)$$

Definition

Given $H \leqslant F_n$, we define the auto-closure and endo-closure of H as

$$a\text{-}Cl(H) = Fix(Aut_H(F_n)) \geqslant H$$

and

$$e\text{-}Cl(H) = Fix(End_H(F_n)) \geqslant H$$

Fixed closures

Definition

Given $H \leqslant_{\mathrm{fg}} F_n$, we define the (auto- and endo-) stabilizer of H, respectively, as

$$Aut_H(F_n) = \{\phi \in Aut(F_n) \mid H \leqslant Fix(\phi)\} \leqslant Aut(F_n)$$

and

$$End_H(F_n) = \{\phi \in End(F_n) \mid H \leqslant Fix(\phi)\} \leqslant End(F_n)$$

Definition

Given $H \leqslant F_n$, we define the auto-closure and endo-closure of H as

$$a\text{-}Cl(H) = Fix(Aut_H(F_n)) \geqslant H$$

and

$$e\text{-}Cl(H) = Fix(End_H(F_n)) \geqslant H$$

Main result

Theorem

For every $H \leqslant_{\text{fg}} F_n$, a-Cl(H) and e-Cl(H) are finitely generated and one can algorithmically compute bases for them.

Corollary

Auto-fixedness and endo-fixedness are decidable.

Observe that $e\text{-}Cl(H) \leqslant a\text{-}Cl(H)$ but, in general, they are not equal.

Main result

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-Cl(H) and e-Cl(H) are finitely generated and one can algorithmically compute bases for them.

Corollary

Auto-fixedness and endo-fixedness are decidable.

Observe that $e\text{-}Cl(H) \leqslant a\text{-}Cl(H)$ but, in general, they are not equal.

Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho \colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leqslant F_n$ is a retract or not.

Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho \colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leqslant F_n$ is a retract or not.

Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho \colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leqslant F_n$ is a retract or not.

The stable image

Definition

Let $\phi \in End(F_n)$. The *stable image* of ϕ is $F_n\phi^\infty = \cap_{i=1}^\infty F_n\phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi\colon F_n \to F_n$,

  i) $F_n\phi^\infty$ is ϕ-invariant,

  ii) the restriction $\phi\colon F_n\phi^\infty \to F_n\phi^\infty$ is an isomorphism,

  iii) $F_n\phi^\infty$ is a retract.

Example: For $\phi\colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2 \rangle$, $F_2\phi^2 = \langle a, b^4 \rangle$, $F_2\phi^3 = \langle a, b^8 \rangle$, …. So, $F_2\phi^\infty = \langle a \rangle \leqslant_{\text{ff}} F_2$.

The stable image

Definition

Let $\phi \in End(F_n)$. The *stable image* of ϕ is $F_n\phi^\infty = \cap_{i=1}^\infty F_n\phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi\colon F_n \to F_n$,

i) $F_n\phi^\infty$ *is ϕ-invariant,*

ii) *the restriction $\phi\colon F_n\phi^\infty \to F_n\phi^\infty$ is an isomorphism,*

iii) $F_n\phi^\infty$ *is a retract.*

Example: For $\phi\colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2 \rangle$, $F_2\phi^2 = \langle a, b^4 \rangle$, $F_2\phi^3 = \langle a, b^8 \rangle$, …. So, $F_2\phi^\infty = \langle a \rangle \leqslant_{\mathrm{ff}} F_2$.

The stable image

Definition

Let $\phi \in End(F_n)$. The *stable image* of ϕ is $F_n \phi^\infty = \cap_{i=1}^{\infty} F_n \phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi \colon F_n \to F_n$,

  i) $F_n \phi^\infty$ *is* ϕ-*invariant,*

  ii) *the restriction* $\phi \colon F_n \phi^\infty \to F_n \phi^\infty$ *is an isomorphism,*

  iii) $F_n \phi^\infty$ *is a retract.*

Example: For $\phi \colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2 \phi = \langle a, b^2 \rangle$, $F_2 \phi^2 = \langle a, b^4 \rangle$, $F_2 \phi^3 = \langle a, b^8 \rangle$, ... So, $F_2 \phi^\infty = \langle a \rangle \leqslant_{\text{ff}} F_2$.

Stallings' graphs and intersections

Theorem (Stallings, 83)

For any free group $F_n = F(A)$, there is an effectively computable bijection

$$\{\text{f.g. subgroups of } F_n\} \longleftrightarrow \{\text{finite } A\text{-labeled core graphs}\}$$

Theorem

Given sets of generators for $H, K \leqslant_{\text{fg}} F_n$, one can algorithmically compute a basis for $H \cap K$.

Stallings' graphs and intersections

Theorem (Stallings, 83)

For any free group $F_n = F(A)$, there is an effectively computable bijection

$$\{f.g. \text{ subgroups of } F_n\} \longleftrightarrow \{\text{finite A-labeled core graphs}\}$$

Theorem

Given sets of generators for $H, K \leqslant_{\mathrm{fg}} F_n$, one can algorithmically compute a basis for $H \cap K$.

Algebraic extensions

Definition

An extension of subgroups $H \leqslant K \leqslant F_n$ is called algebraic, denoted $H \leqslant_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. Write

$$\mathcal{AE}(H) = \{K \leqslant F_n \mid H \leqslant_{\mathrm{alg}} K\}.$$

Theorem (Takahasi, 51)

If $H \leqslant_{\mathrm{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

Algebraic extensions

Definition

An extension of subgroups $H \leqslant K \leqslant F_n$ is called algebraic, denoted $H \leqslant_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. Write

$$\mathcal{AE}(H) = \{K \leqslant F_n \mid H \leqslant_{\mathrm{alg}} K\}.$$

Theorem (Takahasi, 51)

If $H \leqslant_{\mathrm{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

Outline

The automorphism case

Theorem (McCool)

Let $H \leqslant_{\mathrm{fg}} F_n$. Then $Aut_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-Cl(H) is finitely generated and algorithmically computable.

Proof. $a\text{-}Cl(H)$ $=$ $\mathrm{Fix}\,(\mathrm{Aut}_H(F_n))$
$=$ $\mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle)$
$=$ $\mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m).$ □

The automorphism case

Theorem (McCool)

Let $H \leqslant_{\mathrm{fg}} F_n$. Then $\mathrm{Aut}_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, $a\text{-}Cl(H)$ is finitely generated and algorithmically computable.

Proof. $\quad a\text{-}Cl(H) \quad = \quad \mathrm{Fix}\,(\mathrm{Aut}_H(F_n))$
$$= \quad \mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle)$$
$$= \quad \mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m). \quad \square$$

The automorphism case

Theorem (McCool)

Let $H \leqslant_{\mathrm{fg}} F_n$. Then $Aut_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-Cl$(H)$ is finitely generated and algorithmically computable.

Proof. $\begin{aligned} a\text{-}Cl(H) \quad &= \quad \mathrm{Fix}\,(\mathrm{Aut}_H(F_n)) \\ &= \quad \mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle) \\ &= \quad \mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m). \quad \square \end{aligned}$

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, $e\text{-}Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.

- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.

- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.

- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.

- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

1.Some history
oooooooooo

Algorithmic results
oo

Needed tools
oooooo

The proof
oo●ooo

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-Cl(H) is finitely generated and algorithmically computable.

Proof. Given *H* (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of *H* as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\text{fg}} F_n$, e-Cl(H) is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ ($1 \leqslant r \leqslant q$).
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, $e\text{-}Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

1.Some history
○○○○○○○○○○

Algorithmic results
○○

Needed tools
○○○○○○

The proof
○●○○○○

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-Cl$(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ ($1 \leqslant r \leqslant q$).
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-Cl(H) is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ ($1 \leqslant r \leqslant q$).
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-Cl(H) is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \,\in\, \mathsf{Aut}\,(H_i) \\ H \,\leqslant\, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \quad \bigcap_{\substack{\beta \,\in\, \mathsf{End}\,(F_n) \\ H \,\leqslant\, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $\beta \in \mathsf{End}\,(F_n)$ with $H \leqslant \mathsf{Fix}\,(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant F$.
- Now, β restricts to an automorphism $\alpha\colon H_i \to H_i$.
- And, clearly, $\mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta) \geqslant H$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H).$

Proof. Let us see that

$$
\bigcap_{i=1}^{r} \ \bigcap_{\substack{\alpha \in \mathsf{Aut}(H_i) \\ H \leqslant \mathsf{Fix}(\alpha)}} \mathsf{Fix}(\alpha) \quad = \quad \bigcap_{\substack{\beta \in \mathsf{End}(F_n) \\ H \leqslant \mathsf{Fix}(\beta)}} \mathsf{Fix}(\beta).
$$

- Take $\beta \in \mathsf{End}(F_n)$ with $H \leqslant \mathsf{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^\infty \leqslant F$.
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $\mathsf{Fix}(\alpha) = \mathsf{Fix}(\beta) \geqslant H$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \,\in\, \mathsf{Aut}\,(H_i) \\ H \,\leqslant\, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \quad \bigcap_{\substack{\beta \,\in\, \mathsf{End}\,(F_n) \\ H \,\leqslant\, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $\beta \in \mathsf{End}\,(F_n)$ with $H \leqslant \mathsf{Fix}\,(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant F$.
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $\mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta) \geqslant H$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \,\in\, \mathsf{Aut}\,(H_i) \\ H \,\leqslant\, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \quad \bigcap_{\substack{\beta \,\in\, \mathsf{End}\,(F_n) \\ H \,\leqslant\, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $\beta \in \mathsf{End}\,(F_n)$ with $H \leqslant \mathsf{Fix}\,(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant F$.
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $\mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta) \geqslant H$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathsf{Aut}\,(H_i) \\ H \leqslant \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \bigcap_{\substack{\beta \in \mathsf{End}\,(F_n) \\ H \leqslant \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $\beta \in \mathsf{End}\,(F_n)$ with $H \leqslant \mathsf{Fix}\,(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant F$.
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $\mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta) \geqslant H$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \, \in \, \mathsf{Aut}\,(H_i) \\ H \, \leqslant \, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \quad \bigcap_{\substack{\beta \, \in \, \mathsf{End}\,(F_n) \\ H \, \leqslant \, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $\beta \in \mathsf{End}\,(F_n)$ with $H \leqslant \mathsf{Fix}\,(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant F$.
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $\mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta) \geqslant H$.
- Hence, we have "$\leqslant$".

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \,\in\, \mathsf{Aut}\,(H_i) \\ H \,\leqslant\, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \;=\; \bigcap_{\substack{\beta \,\in\, \mathsf{End}\,(F_n) \\ H \,\leqslant\, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathsf{Aut}\,(H_i)$ with $H \leqslant \mathsf{Fix}\,(\alpha)$.
- Let $\rho\colon F \to H_i$ be a retraction, and consider the endomorphism,
  $\beta\colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$
\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \, \in \, \mathsf{Aut}\,(H_i) \\ H \, \leqslant \, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \bigcap_{\substack{\beta \, \in \, \mathsf{End}\,(F_n) \\ H \, \leqslant \, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).
$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathsf{Aut}\,(H_i)$ with $H \leqslant \mathsf{Fix}\,(\alpha)$.

- Let $\rho \colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta \colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.

- Clearly, $H \leqslant \mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta)$.

- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \,\in\, \mathsf{Aut}\,(H_i) \\ H \,\leqslant\, \mathsf{Fix}\,(\alpha)}} \mathsf{Fix}\,(\alpha) \quad = \bigcap_{\substack{\beta \,\in\, \mathsf{End}\,(F_n) \\ H \,\leqslant\, \mathsf{Fix}\,(\beta)}} \mathsf{Fix}\,(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathsf{Aut}\,(H_i)$ with $H \leqslant \mathsf{Fix}\,(\alpha)$.
- Let $\rho \colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta \colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \mathsf{Fix}\,(\alpha) = \mathsf{Fix}\,(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \, \in \, \text{Aut}\,(H_i) \\ H \, \leqslant \, \text{Fix}\,(\alpha)}} \text{Fix}\,(\alpha) \quad = \quad \bigcap_{\substack{\beta \, \in \, \text{End}\,(F_n) \\ H \, \leqslant \, \text{Fix}\,(\beta)}} \text{Fix}\,(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \text{Aut}\,(H_i)$ with $H \leqslant \text{Fix}\,(\alpha)$.
- Let $\rho \colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta \colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \text{Fix}\,(\alpha) = \text{Fix}\,(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \, \in \, \text{Aut}\,(H_i) \\ H \, \leqslant \, \text{Fix}\,(\alpha)}} \text{Fix}\,(\alpha) \quad = \quad \bigcap_{\substack{\beta \, \in \, \text{End}\,(F_n) \\ H \, \leqslant \, \text{Fix}\,(\beta)}} \text{Fix}\,(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \text{Aut}\,(H_i)$ with $H \leqslant \text{Fix}\,(\alpha)$.
- Let $\rho \colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta \colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \text{Fix}\,(\alpha) = \text{Fix}\,(\beta)$.
- Hence, we have "$\geqslant$". $\square$

THANKS