Algebraic Geometry over the Additive Monoid of Natural Numbers

Artem Shevlyakov

Omsk Branch of Institute of Mathematics

19th December, 2008

The Philosophic Problem

The Philosophic Problem

L. Kronecker: The natural numbers were created by God. Another mathematics is due to humanity.

VS

The Philosophic Problem

L. Kronecker: The natural numbers were created by God. Another mathematics is due to humanity.

VS

V. Remeslennikov: The algebraic geometry over $\mathbb{N}$ is only an exercise before the geometry over free monoids.

Outline

1 Preliminaries

2 Coefficient-free equations

3 Systems with Coefficients

4 Generalizations

Languages

- Let $\mathcal{L} = \langle *^{(2)}, 1 \rangle$ be a standard language of monoid theory. Further we shall consider only commutative monoids and write $+, 0$ instead $*, 1$.

Languages

- Let $\mathcal{L} = \langle *^{(2)}, 1\rangle$ be a standard language of monoid theory. Further we shall consider only commutative monoids and write $+, 0$ instead $*, 1$.
- We add to $\mathcal{L}$ the set of constant symbols $\{c_a | a \in A\}$, where A is an arbitrary monoid. Denote the union $\mathcal{L} \cup \{c_a \ |a \in A\}$ by $\mathcal{L}_A$.

Languages

- Let $\mathcal{L} = \langle *^{(2)}, 1\rangle$ be a standard language of monoid theory. Further we shall consider only commutative monoids and write $+, 0$ instead $*, 1$.
- We add to $\mathcal{L}$ the set of constant symbols $\{c_a | a \in A\}$, where A is an arbitrary monoid. Denote the union $\mathcal{L} \cup \{c_a \ | a \in A\}$ by $\mathcal{L}_A$.
  We write the axioms of the theory of commutative monoids:
  1. $\forall x \forall y \forall z \ (x + y) + z = x + (y + z)$;
  2. $\forall x \ x + 0 = 0 + x = x$;
  3. $\forall x \forall y \ x + y = y + x$;

  and the obvious axioms with constant symbols:
  1. $c_{a_i} \neq c_{a_j}, \ i \neq j$;
  2. $c_{a_i} + c_{a_j} = c_{a_i + a_j}$;
  3. $c_a = 0 \Leftrightarrow a = 0$.

Languages

- Let $\mathcal{L} = \langle *^{(2)}, 1\rangle$ be a standard language of monoid theory. Further we shall consider only commutative monoids and write $+, 0$ instead $*, 1$.
- We add to $\mathcal{L}$ the set of constant symbols $\{c_a | a \in A\}$, where A is an arbitrary monoid. Denote the union $\mathcal{L} \cup \{c_a \ | a \in A\}$ by $\mathcal{L}_A$.
  We write the axioms of the theory of commutative monoids:
  1. $\forall x \forall y \forall z \ (x + y) + z = x + (y + z)$;
  2. $\forall x \ x + 0 = 0 + x = x$;
  3. $\forall x \forall y \ x + y = y + x$;

  and the obvious axioms with constant symbols:
  1. $c_{a_i} \neq c_{a_j}, \ i \neq j$;
  2. $c_{a_i} + c_{a_j} = c_{a_i + a_j}$;
  3. $c_a = 0 \Leftrightarrow a = 0$.
- An $\mathcal{L}_A$-structure (a model of the language $\mathcal{L}_A$) M is said to be an A-monoid if M satisfies all formulas above. In other words, A-monoid is a monoid with a fixed submonoid isomorphic to A.

Systems of Equations

- An atomic $\mathcal{L}_A$-formula $t(\bar{x}) = s(\bar{x})$ is called an *equation* over A (A-equation for short). An A-equation is said to be coefficients-free if it does not contain constant symbols. Remind that all 0-equations ($A = \{0\}$) are coefficient-free.
- A system of equations $\mathcal{S}$ over A is an arbitrary set of A-equations.

Systems of Equations

- An atomic $\mathcal{L}_A$-formula $t(\bar{x}) = s(\bar{x})$ is called an *equation* over A (A-equation for short). An A-equation is said to be coefficients-free if it does not contain constant symbols. Remind that all 0-equations $(A = \{0\})$ are coefficient-free.
- A system of equations $\mathcal{S}$ over A is an arbitrary set of A-equations.
- Clearly, each A-equation has an equivalent form

$$\sum_{i \in I} \gamma_i x_i + a = \sum_{j \in J} \gamma_j x_j + a',$$

where $a, a' \in A$.

We can seek a solution of $\mathcal{S}$ in every commutative monoid B.

We can seek a solution of $\mathcal{S}$ in every commutative monoid B.

The solutions of $\mathcal{S}$ over B is denoted by $V_B(\mathcal{S})$.

We can seek a solution of $\mathcal{S}$ in every commutative monoid B.

The solutions of $\mathcal{S}$ over B is denoted by $V_B(\mathcal{S})$.

We chose the most famous commutative monoid $\mathbb{N}$ (the additive monoid of natural numbers) and studied its algebraic geometry (further $B = \mathbb{N}$). Below we generalize our results for $\mathbb{N}$ to a wide class of commutative monoids.

- A set $Y \subseteq \mathbb{N}^n$ is called *algebraic over* $\mathbb{N}$ if there exists a system of equations with $Y = \mathrm{V}_{\mathbb{N}}(\mathcal{S})$.

- A set $Y \subseteq \mathbb{N}^n$ is called *algebraic over* $\mathbb{N}$ if there exists a system of equations with $Y = \mathrm{V}_{\mathbb{N}}(\mathcal{S})$.
- An algebraic set is said to be *irreducible* if it is not represented as a nontrivial union of algebraic sets.

- A set $Y \subseteq \mathbb{N}^n$ is called *algebraic over* $\mathbb{N}$ if there exists a system of equations with $Y = \mathrm{V}_{\mathbb{N}}(\mathcal{S})$.
- An algebraic set is said to be *irreducible* if it is not represented as a nontrivial union of algebraic sets.
- The *radical* of a set Y (or a system $\mathcal{S}$) contains all A-equations satisfied by all points from Y (by all solutions of $\mathcal{S}$).

- A set $Y \subseteq \mathbb{N}^n$ is called *algebraic over* $\mathbb{N}$ if there exists a system of equations with $Y = \mathrm{V}_{\mathbb{N}}(\mathcal{S})$.
- An algebraic set is said to be *irreducible* if it is not represented as a nontrivial union of algebraic sets.
- The *radical* of a set Y (or a system $\mathcal{S}$) contains all A-equations satisfied by all points from Y (by all solutions of $\mathcal{S}$).
- The radical of Y divides the set of $\mathcal{L}_A$-terms into equivalence classes. Indeed, two $\mathcal{L}_A$-terms $t(\bar{x}), s(\bar{x})$ are equivalent iff $t(\bar{y}) = s(\bar{y})$ for all $y \in Y$. It is easy to prove that equivalence relation preserves the operation $+$, thus $\mathrm{Rad}_{\mathbb{N}}(Y)$ defines the congruence θ_{Rad}.

- A set $Y \subseteq \mathbb{N}^n$ is called *algebraic over* $\mathbb{N}$ if there exists a system of equations with $Y = \mathrm{V}_{\mathbb{N}}(\mathcal{S})$.
- An algebraic set is said to be *irreducible* if it is not represented as a nontrivial union of algebraic sets.
- The *radical* of a set Y (or a system $\mathcal{S}$) contains all A-equations satisfied by all points from Y (by all solutions of $\mathcal{S}$).
- The radical of Y divides the set of $\mathcal{L}_A$-terms into equivalence classes. Indeed, two $\mathcal{L}_A$-terms $t(\bar{x}), s(\bar{x})$ are equivalent iff $t(\bar{y}) = s(\bar{y})$ for all $y \in Y$. It is easy to prove that equivalence relation preserves the operation $+$, thus $\mathrm{Rad}_{\mathbb{N}}(Y)$ defines the congruence θ_{Rad}.

The quotient monoid $\Gamma_A(Y) = \mathrm{T}_{\mathcal{L}_{\mathbb{N}}}(X)/\theta_{\mathrm{Rad}_B(Y)}$, where $\mathrm{T}_{\mathcal{L}_{\mathbb{N}}}(X)$ is a set of all $\mathcal{L}_A$-terms, is called the *coordinate A-monoid* of Y. The operation $+$ over $\Gamma_A(Y)$ is defined by

$$[t(\bar{x})] + [s(\bar{x})] = [s(\bar{x}) + t(\bar{x})],$$

where $[t(\bar{x})]$ is the equivalence class of $t(\bar{x})$.

Main Aims of Algebraic Geometry

The main goal of algebraic geometry can be considered as a classification of

1. algebraic sets;
2. radicals;
3. coordinate monoids;

Fact. The monoid $\mathbb{N}$ is A-equationally Noetherian, i.e. for each infinite system of A-equations $\mathcal{S}$ which depends on a finite set of variables $x_1, \ldots, x_n$ there exists a finite subsystem $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $V_{\mathbb{N}}(S) = V_{\mathbb{N}}(S_0)$.

Definitions for Unification Theorems

- An A-*homomorphism* φ is a homomorphism of A-monoid M and φ is an identity on A ($\varphi(a) = a$, $a \in A$).

Definitions for Unification Theorems

- An A-*homomorphism* φ is a homomorphism of A-monoid M and φ is an identity on A ($\varphi(a) = a$, $a \in A$).
- A *universal* formula has a form $\forall \bar{x} \Phi(\bar{x})$, where Φ is a quantifier-free formula.

Definitions for Unification Theorems

- An A-*homomorphism* φ is a homomorphism of A-monoid M and φ is an identity on A ($\varphi(a) = a$, $a \in A$).
- A *universal* formula has a form $\forall \bar{x} \Phi(\bar{x})$, where Φ is a quantifier-free formula.
- A *quasi-identity* is a universal formula, where $\Phi(\bar{x}) = (t_1(\bar{x}) = s_1(\bar{x})) \wedge \ldots \wedge (t_m(\bar{x}) = s_m(\bar{x})) \to (t(\bar{x}) = s(\bar{x}))$.

Definitions for Unification Theorems

- An A-*homomorphism* φ is a homomorphism of A-monoid M and φ is an identity on A ($\varphi(a) = a$, $a \in A$).
- A *universal* formula has a form $\forall \bar{x} \Phi(\bar{x})$, where Φ is a quantifier-free formula.
- A *quasi-identity* is a universal formula, where $\Phi(\bar{x}) = (t_1(\bar{x}) = s_1(\bar{x})) \wedge \ldots \wedge (t_m(\bar{x}) = s_m(\bar{x})) \to (t(\bar{x}) = s(\bar{x}))$.

We shall use standard denotations of algebraic geometry:

Definitions for Unification Theorems

- An A-*homomorphism* φ is a homomorphism of A-monoid M and φ is an identity on A ($\varphi(a) = a$, $a \in A$).
- A *universal* formula has a form $\forall \bar{x} \Phi(\bar{x})$, where Φ is a quantifier-free formula.
- A *quasi-identity* is a universal formula, where $\Phi(\bar{x}) = (t_1(\bar{x}) = s_1(\bar{x})) \wedge \ldots \wedge (t_m(\bar{x}) = s_m(\bar{x})) \to (t(\bar{x}) = s(\bar{x}))$.

We shall use standard denotations of algebraic geometry:

- the letter 'D' means 'E. Daniyarova';
- the letter 'M' means 'A. Myasnikov';
- the letter 'R' means 'V. Remeslennikov'.

The First Unification Theorem

Theorem (DMR)

Suppose C is an A-monoid and C is finitely generated over A. Then the following conditions are equivalent:
1) C is a coordinate monoid of an algebraic set over $\mathbb{N}$, and this set is defined by a system of A-equations.
2) C is A-separated by $\mathbb{N}$. In other words, for an arbitrary elements c_1, c_2, $c_1 \neq c_2$ there exists a A-homomorphism $\varphi \colon C \to \mathbb{N}$ such that $\varphi(c_1) \neq \varphi(c_2)$.
3) $C \in \mathrm{qvar}_A(\mathbb{N})$, i.e. each $\mathcal{L}_A$-quasi-identity which is true in $\mathbb{N}$ holds in C.

The Second Unification Theorem

Theorem (DMR)

Suppose C is an A-monoid and C is finitely generated over A. Then the following conditions are equivalent:
1) C is a irreducible coordinate monoid of an algebraic set over $\mathbb{N}$, and this set is defined by a system of A-equations.
2) C is A-discriminated by $\mathbb{N}$. In other words, for an arbitrary elements $c_1, \ldots, c_k$, $c_i \neq c_j$ there exists a A-homomorphism $\varphi\colon C \to \mathbb{N}$ such that $\varphi(c_i) \neq \varphi(c_j)$.
3) $C \in \mathrm{ucl}_A(\mathbb{N})$, i.e. each $\mathcal{L}_A$-universal formula which is true in $\mathbb{N}$ holds in C.

Comparing with the Group $\mathbb{Z}$

Theorem (a corollary from MR)

All coordinate groups over $\mathbb{Z}$ are the direct products $\mathbb{Z}^n$ and irreducible.

Coefficient-free equations

Positive property

In this subsection $A = \{0\}$.
A commutative monoid is called *positive*, if the following quasi-identity

$$\forall x \forall y \ (x + y = 0) \to (x = 0).$$

holds. In other words, the sum of two nonzero elements of positive monoid is not a zero.
Obviously, M is positive iff the set $M \setminus \{0\}$ is a semigroup.

Classification of Coordinate Monoids in Coefficient-Free Case

Theorem

A finitely generated monoid M is a coordinate monoid of an algebraic set Y over $\mathbb{N}$, where Y is defined by coefficient-free equations, iff M is commutative positive and with cancellation property $(\forall x \forall y \forall z\ (x+z=y+z) \to (x=y))$.

Theorem

All algebraic sets over $\mathbb{N}$ defined by coefficient-free systems are irreducible.

Model-Theoretic Corollary

The classes $\mathrm{qvar}_0(\mathbb{N}), \mathrm{ucl}_0(\mathbb{N})$ are equal and axiomatizable by the following $\mathcal{L}$-formulas

1. $\forall x \forall y \forall z\ (x+y)+z = x+(y+z)$;
2. $\forall x\ x+0 = 0+x = x$;
3. $\forall x \forall y\ x+y = y+x$;
4. $\forall x \forall y \forall z\ x+z = y+z \to x = y$ (cancellation property);
5. $\forall x \forall y\ x+y = 0 \to x = 0$ (positive property).

Geometrical Equivalence

1. Monoids M_1, M_2 are called *geometrical equivalent* if $\mathrm{Rad}_{M_1}(\mathcal{S}) = \mathrm{Rad}_{M_2}(\mathcal{S})$ for every system $\mathcal{S}$.
2. By definition, the geometrical equivalent monoids have the same set of coordinate monoids. Therefore, the obtained results for $\mathbb{N}$ can be applied to the wide class of commutative monoids.

Geometrical Equivalence

1. Monoids M_1, M_2 are called *geometrical equivalent* if $\mathrm{Rad}_{M_1}(\mathcal{S}) = \mathrm{Rad}_{M_2}(\mathcal{S})$ for every system $\mathcal{S}$.
2. By definition, the geometrical equivalent monoids have the same set of coordinate monoids. Therefore, the obtained results for $\mathbb{N}$ can be applied to the wide class of commutative monoids.

Theorem

Each nontrivial commutative positive monoid with cancellation property M is geometrical equivalent to $\mathbb{N}$. Moreover, all algebraic sets over M are irreducible, thus M is universal equivalent to $\mathbb{N}$.

Systems with Coefficients

Irreducible Coordinate Monoids

We can consider only the case $A = \mathbb{N}$, because each $\mathbb{N}$-equation can be transformed to an A-equation for every monoid $A \subseteq \mathbb{N}$.

Irreducible Coordinate Monoids

We can consider only the case $A = \mathbb{N}$, because each $\mathbb{N}$-equation can be transformed to an A-equation for every monoid $A \subseteq \mathbb{N}$.

An $\mathbb{N}$-monoid M is called $\mathbb{N}$-*positive* if for all pairs of nonzero elements $m_1, m_2 \notin \mathbb{N}$ their sum does not belong to $\mathbb{N}$ ($m_1 + m_2 \notin \mathbb{N}$).

Irreducible Coordinate Monoids

We can consider only the case $A = \mathbb{N}$, because each $\mathbb{N}$-equation can be transformed to an A-equation for every monoid $A \subseteq \mathbb{N}$.

An $\mathbb{N}$-monoid M is called $\mathbb{N}$-*positive* if for all pairs of nonzero elements $m_1, m_2 \notin \mathbb{N}$ their sum does not belong to $\mathbb{N}$ ($m_1 + m_2 \notin \mathbb{N}$).

The $\mathbb{N}$-positive property for $\mathbb{N}$ is written by the series of $\mathbb{N}$-formulas ($\alpha \in \mathbb{N}$)

$$\varphi_\alpha = \forall x \forall y \ (x + y = \alpha) \to ((x = 0) \vee (x = 1) \vee (x = 2) \vee \ldots \vee (x = \alpha)).$$

$\mathbb{N}$-monoid M is $\mathbb{N}$-positive iff the set $M \backslash \mathbb{N}$ is a semigroup.

$\mathbb{N}$ is obviously $\mathbb{N}$-positive.

Theorem

Suppose $\mathbb{N}$-monoid M is coordinate monoid of an algebraic set over $\mathbb{N}$. Then M is irreducible iff M is $\mathbb{N}$-positive.

Reducible Sets. They really exist.

There are reducible sets over $\mathbb{N}$ defined by systems with coefficient. For example, the solution of $x+y=1$ is represented by the union $(0,1)\cup(1,0)$.

Reducible Sets. They really exist.

There are reducible sets over $\mathbb{N}$ defined by systems with coefficient. For example, the solution of $x+y=1$ is represented by the union $(0,1)\cup(1,0)$.

Below we find necessary and sufficient conditions for an $\mathbb{N}$-monoid M to be coordinate over $\mathbb{N}$. First Unification Theorem made us to seek the set of quasi-identities $\mathcal{Q}$ such that

1. if an $\mathbb{N}$-monoid $M\models\mathcal{Q}$ then the set of $\mathbb{N}$-homomorphisms $\mathrm{Hom}_{\mathbb{N}}(M,\mathbb{N})$ is not empty;
2. if an $\mathbb{N}$-monoid $M\models\mathcal{Q}$ then $\mathbb{N}$ $\mathbb{N}$-separates M.

Congruent Closure

Let S be a set of atomic $\mathcal{L}_A$-formulas. The congruent closure $[S] \supseteq S$ is a minimal set with the properties

- If $t(\bar{x}) = s(\bar{x}) \in S$, then $t(\bar{x}) = t(\bar{x}) \in [S]$ and $s(\bar{x}) = s(\bar{x}) \in [S]$.
- If $t(\bar{x}) = s(\bar{x}) \in S$, then $s(\bar{x}) = t(\bar{x}) \in [S]$.
- If $t(\bar{x}) = s(\bar{x}), s(\bar{x}) = u(\bar{x}) \in S$, then $s(\bar{x}) = u(\bar{x}) \in [S]$.
- If $t_1(\bar{x}) = s_1(\bar{x}), t_2(\bar{x}) = s_2(\bar{x}) \in S$, then $t_1(\bar{x}) + t_2(\bar{x}) = s_1(\bar{x}) + s_2(\bar{x}) \in [S]$.

Congruent Closure

Let S be a set of atomic $\mathcal{L}_A$-formulas. The congruent closure $[S] \supseteq S$ is a minimal set with the properties

- If $t(\bar{x}) = s(\bar{x}) \in S$, then $t(\bar{x}) = t(\bar{x}) \in [S]$ and $s(\bar{x}) = s(\bar{x}) \in [S]$.
- If $t(\bar{x}) = s(\bar{x}) \in S$, then $s(\bar{x}) = t(\bar{x}) \in [S]$.
- If $t(\bar{x}) = s(\bar{x}), s(\bar{x}) = u(\bar{x}) \in S$, then $s(\bar{x}) = u(\bar{x}) \in [S]$.
- If $t_1(\bar{x}) = s_1(\bar{x}), t_2(\bar{x}) = s_2(\bar{x}) \in S$, then $t_1(\bar{x}) + t_2(\bar{x}) = s_1(\bar{x}) + s_2(\bar{x}) \in [S]$.

The congruent closure of a system of equations contains only elementary corollaries of this system. By definition, $[S] \subseteq \mathrm{Rad}_{\mathbb{N}}(S)$.

Radicals and Congruent Closures of $\mathbb{N}$-equations

If an $\mathbb{N}$-equation $t(\bar{x}) = s(\bar{x})$ has not a form $t'(\bar{x}) = n$ the radical $\mathrm{Rad}_{\mathbb{N}}(t(\bar{x}) = s(\bar{x}))$ is equal to the congruent closure. The radical of an equation $t(\bar{x}) = n$ often strictly contains the congruent closure of this equation.

Radicals and Congruent Closures of $\mathbb{N}$-equations

If an $\mathbb{N}$-equation $t(\bar{x}) = s(\bar{x})$ has not a form $t'(\bar{x}) = n$ the radical $\mathrm{Rad}_{\mathbb{N}}(t(\bar{x}) = s(\bar{x}))$ is equal to the congruent closure. The radical of an equation $t(\bar{x}) = n$ often strictly contains the congruent closure of this equation.

For example, consider the equation $4x + 3y + 7z = 7$ which has only two solutions $(0, 0, 1), (1, 1, 0)$. The radical of this equation is generated by $4x + 3y + 7z = 7$ and $x = y$ and therefore it is not equal to congruent closure of $4x + 3y + 7z = 7$.

In other words, the equation $4x + 3y + 7z = 7$ implies $x = y$, thus the quasi-identity

$$\forall x \forall y \forall z (4x + 3y + 7z = 7) \to (x = y)$$

must be true in every coordinate $\mathbb{N}$-monoid over $\mathbb{N}$.

Quasi-identities $\mathcal{Q}$

- Suppose an $\mathbb{N}$-equation $t(\bar{x}) = s(\bar{x})$ is unsolvable over $\mathbb{Z}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = s(\bar{x})) \to (0 = 1)$.

Quasi-identities $\mathcal{Q}$

- Suppose an $\mathbb{N}$-equation $t(\bar{x}) = s(\bar{x})$ is unsolvable over $\mathbb{Z}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = s(\bar{x})) \to (0 = 1)$.
- Suppose an $\mathbb{N}$-equation of a form $t(\bar{x}) = n$, and it is unsolvable over $\mathbb{N}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = n) \to (0 = 1)$.

Quasi-identities $\mathcal{Q}$

- Suppose an $\mathbb{N}$-equation $t(\bar{x}) = s(\bar{x})$ is unsolvable over $\mathbb{Z}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = s(\bar{x})) \to (0 = 1)$.
- Suppose an $\mathbb{N}$-equation of a form $t(\bar{x}) = n$, and it is unsolvable over $\mathbb{N}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = n) \to (0 = 1)$.
- Suppose the equations $eq_1, \ldots, eq_l$ generate the radical of an equation $t(\bar{x}) = n$. Then we write the quasi-identities
  - $\forall \bar{x}\ (t(\bar{x}) = n) \to eq_1$,
  - $\ldots$
  - $\forall \bar{x}\ (t(\bar{x}) = n) \to eq_l$,

Quasi-identities $\mathcal{Q}$

- Suppose an $\mathbb{N}$-equation $t(\bar{x}) = s(\bar{x})$ is unsolvable over $\mathbb{Z}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = s(\bar{x})) \to (0 = 1)$.
- Suppose an $\mathbb{N}$-equation of a form $t(\bar{x}) = n$, and it is unsolvable over $\mathbb{N}$. Then we write a quasi-identity $\forall \bar{x}\ (t(\bar{x}) = n) \to (0 = 1)$.
- Suppose the equations $eq_1, \ldots, eq_l$ generate the radical of an equation $t(\bar{x}) = n$. Then we write the quasi-identities
  - $\forall \bar{x}\ (t(\bar{x}) = n) \to eq_1$,
  - $\ldots$
  - $\forall \bar{x}\ (t(\bar{x}) = n) \to eq_l$,

Theorem

A commutative $\mathbb{N}$-monoid with cancellation property M is a coordinate monoid of a nonempty algebraic set over $\mathbb{N}$ iff all quasi-identities $\mathcal{Q}$ hold in M.

Unions of Algebraic Sets

Suppose $Y_1, \ldots, Y_n$ are algebraic irreducible sets and each Y_i does not contain in the union of $\bigcup_{j \neq i} Y_j$.

Further we find a criterion of the set $Y_1 \cup \ldots \cup Y_n$ to be algebraic.

Theorem

Suppose $Y_1 \cup \ldots \cup Y_n$ is algebraic. Then $Y_1, \ldots, Y_n$ can be obtained as a parallel shift of the set Y_0 via vectors with natural coordinates, where Y_0 is algebraic and defined by a system of coefficient-free equations. (Necessary condition)

A variable x of a system $\mathcal{S}$ is called *fixed* if $x = n \in \mathrm{Rad}_{\mathbb{N}}(\mathcal{S})$.

Theorem

Suppose systems $\mathcal{S}_1, \ldots, \mathcal{S}_n$ depend on variables $x_1, \ldots, x_m$, and let the union of the solutions of $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be algebraic. Then all systems have the same nonempty set of fixed coordinates. (Necessary condition)

Criterion

Theorem

The union of algebraic sets $Y_1, \dots, Y_n$ is an algebraic set iff

1. *there exist systems of a form*

$$\mathcal{S}_1 = \begin{cases} x_1 = \alpha_{11}, \\ \dots\dots \\ x_l = \alpha_{1l}, \\ t_1(\bar{y}) + \beta_{11} = s_1(\bar{y}), \\ \dots\dots \\ t_m(\bar{y}) + \beta_{1m} = s_m(\bar{y}), \end{cases} \quad \dots \; \mathcal{S}_n = \begin{cases} x_1 = \alpha_{n1}, \\ \dots\dots \\ x_l = \alpha_{nl}, \\ t_1(\bar{y}) + \beta_{n1} = s_1(\bar{y}), \\ \dots\dots \\ t_m(\bar{y}) + \beta_{nm} = s_m(\bar{y}) \end{cases}$$

   such that $V_{\mathbb{N}}(\mathcal{S}_i) = Y_i$

2. *The union of the solutions of the subsystems with variables x_j is an algebraic set.*

3. $rk(A|e|B) = rk(A|B)$ *(over the field $\mathbb{R}$), where $A = (\alpha_{ij})$ is a $n \times l$-matrix, $B = (\beta_{ij})$ is a $n \times m$-matrix, and e is a column of 1.*

Generalizations

Generalizations

Suppose A, B are commutative positive monoids and $A \subseteq B$.

Theorem

Let M be a commutative A-positive monoid with cancellation property. Suppose the set of A-homomorphisms $\mathrm{Hom}_A(M, B)$ is not empty. Then M is irreducible coordinate monoid of an algebraic set over B.

Corollary

Let A-positive monoid M be a coordinate monoid of an nonempty algebraic set over B. Then M is irreducible.

Generalizations

Suppose A, B are commutative positive monoids and $A \subseteq B$.

Theorem

Let M be a commutative A-positive monoid with cancellation property. Suppose the set of A-homomorphisms $\mathrm{Hom}_A(M, B)$ is not empty. Then M is irreducible coordinate monoid of an algebraic set over B.

Corollary

Let A-positive monoid M be a coordinate monoid of an nonempty algebraic set over B. Then M is irreducible.

Remind that the theorem and corollary above contain only the necessary condition. Indeed, if $B = A = \mathbb{R}^+$ and it is easy to prove that all algebraic sets over $\mathbb{R}^+$ are irreducible. Moreover, there is not a universal formula which expresses $\mathbb{R}^+$-positiveness property.