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# BOUNDARY OPERATOR ALGEBRAS FOR FREE UNIFORM TREE LATTICES

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Abstract. Let $X$ be a finite connected graph, each of whose vertices has degree at least three. The fundamental group $\Gamma$ of $X$ is a free group and acts on the universal covering tree $\Delta$ and on its boundary $\partial\Delta$, endowed with a natural topology and Borel measure. The crossed product $C^*$-algebra $C(\partial\Delta) \rtimes \Gamma$ depends only on the rank of $\Gamma$ and is a Cuntz-Krieger algebra whose structure is explicitly determined. The crossed product von Neumann algebra does not possess this rigidity. If $X$ is homogeneous of degree $q+1$ then the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_\lambda$ where $\lambda = 1/q^2$ if $X$ is bipartite, and $\lambda = 1/q$ otherwise.

## Introduction

Let $\Delta$ be a locally finite tree whose automorphism group $\mathrm{Aut}(\Delta)$ is equipped with the compact open topology. Let $\Gamma$ be a discrete subgroup of $\mathrm{Aut}(\Delta)$ which acts freely on $\Delta$. That is, no element $g \in \Gamma - \{1\}$ stabilizes any vertex or geometric edge of $\Delta$. Assume furthermore that $\Gamma$ acts cocompactly on $\Delta$, so that the quotient $\Gamma\backslash\Delta$ is a finite graph. Then $\Gamma$ is a finitely generated free group and is referred to as a free uniform tree lattice.

Conversely, if $X$ is a finite connected graph and $\Gamma$ is the fundamental group of $X$, then $\Gamma$ is a finitely generated free group and acts freely and cocompactly on the universal covering tree $\Delta$.

It is fruitful to think of the tree $\Delta$ as a combinatorial analogue of the Poincaré disc and $\Gamma$ as an analogue of a Fuchsian group. The group $\Gamma$ is the free group on $\gamma$

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generators, where $\gamma = 1-\chi(\Gamma\backslash\Delta)$ and $\chi(\Gamma\backslash\Delta)$ is the Euler-Poincaré characteristic of the quotient graph. Let $S$ be a free set of generators for $\Gamma$.

Define a $\{0,1\}$-matrix $A$ of order $2\gamma$, with entries indexed by elements of $S \cup S^{-1}$, by

$$A(x,y) = \begin{cases} 1 & \text{if } y \neq x^{-1}, \\ 0 & \text{if } y = x^{-1}. \end{cases} \tag{0.1}$$

Notice that the matrix $A$ depends only on the rank of the free group $\Gamma$.

The boundary $\partial\Delta$ of the tree $\Delta$ is the set of equivalence classes of infinite semi-geodesics in $\Delta$, where equivalent semi-geodesics contain a common sub-semi-geodesic. There is a natural compact totally disconnected topology on $\partial\Delta$ [S, I.2.2]. Denote by $C(\partial\Delta)$ the algebra of continuous complex valued functions on $\partial\Delta$. The full crossed product algebra $C(\partial\Delta) \rtimes \Gamma$ is the universal $C^*$-algebra generated by the commutative $C^*$-algebra $C(\partial\Delta)$ and the image of a unitary representation $\pi$ of $\Gamma$, satisfying the covariance relation

$$f(g^{-1}\omega) = \pi(g) \cdot f \cdot \pi(g)^{-1}(\omega)$$

for $f \in C(\partial\Delta)$, $g \in \Gamma$ and $\omega \in \partial\Delta$ [Ped, Chapter 7].

**Theorem 1.** *Let $\Delta$ be a locally finite tree whose vertices all have degree at least three. Let $\Gamma$ be a free uniform lattice in $\mathrm{Aut}(\Delta)$. Then the boundary $C^*$-algebra $\mathcal{A}(\Gamma) = C(\partial\Delta) \rtimes \Gamma$ depends only on the rank of $\Gamma$, and $\Gamma$ is itself determined by $K_0(\mathcal{A}(\Gamma))$. More precisely,*

(1) *$\mathcal{A}(\Gamma)$ is isomorphic to the simple Cuntz-Krieger algebra $\mathcal{O}_A$ associated with the matrix $A$;*
(2) *$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^\gamma \oplus \mathbb{Z}/(\gamma-1)\mathbb{Z}$ and the class of the identity $[\mathbf{1}]$ is the generator of the summand $\mathbb{Z}/(\gamma-1)\mathbb{Z}$. Moreover $K_1(\mathcal{A}(\Gamma)) = \mathbb{Z}^\gamma$.*

The algebra $\mathcal{A}(\Gamma)$ satisfies the hypotheses of the classification theorem of [K],[Ph]. Therefore the isomorphism class of the algebra $\mathcal{A}(\Gamma)$ is determined by its K-theory together with the class of the identity in $K_0$. The fact that the class $[\mathbf{1}]$ in $K_0$ has order equal to $-\chi(\Gamma\backslash\Delta)$ strengthens the result of [Rob, Section 1] and provides an exact analogy with the Fuchsian case [AD].

Theorem 1 will be proved in Lemmas 1.4 and 2.1 below. The key point in the proof is that the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined uniquely, up to isomorphism, by a finite number of generators and relations [CK], and it is possible to identify these explicitly in $\mathcal{A}(\Gamma)$. The original motivation for this result was the paper of J. Spielberg [Spi], which showed that if $\Gamma$ acts freely and transitively on the tree

$\Delta$ then $\mathcal{A}(\Gamma)$ is a Cuntz-Krieger algebra. Higher rank analogues were studied in [RS].

There is a natural Borel measure on $\partial\Delta$ and one may also consider the crossed product von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$. This is the von Neumann algebra arising from the classical group measure space construction of Murray and von Neumann [Su]. In contrast to Theorem 1, the structure of this algebra depends on the tree $\Delta$ and on the action of $\Gamma$. For simplicity, only the case where $\Delta$ is a homogeneous tree is considered.

**Theorem 2.** *Let $\Delta$ be a homogeneous tree of degree $q+1$, where $q \geq 1$, and let $\Gamma$ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$. *Then $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is the hyperfinite factor of type* $\mathrm{III}_\lambda$ *where*

$$\lambda = \begin{cases} 1/q^2 & \text{if the graph } \Gamma\backslash\Delta \text{ is bipartite}, \\ 1/q & \text{otherwise.} \end{cases}$$

Theorem 2 will be proved in Section 3. The result could equally well have been stated as a classification of the measure theoretic boundary actions up to orbit equivalence [HO]. The analogous result for a Fuchsian group $\Gamma$ acting on the circle is that $\mathrm{L}^\infty(S^1) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_1$ [Spa].

The special case of Theorem 2 where $\Gamma$ acts freely and transitively on the vertices of $\Delta$ was dealt with in [RR]. In that case $q$ is odd, $\Gamma$ is the free group of rank $\frac{q+1}{2}$, and $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_{1/q}$. We remark that R. Okayasu [Ok] constructs similar algebras in a different way, but does not explicitly compute the value of $\lambda$.

There is a type map $\tau$ defined on the vertices of $\Delta$ and taking values in $\mathbb{Z}/2\mathbb{Z}$, defined as follows. Fix a vertex $v_0 \in \Delta$ and let $\tau(v) = d(v_0, v) \pmod 2$, where $d(u,v)$ denotes the usual graph distance between vertices of the tree. The type map is independent of $v_0$, up to addition of 1 (mod 2). It therefore induces a canonical partition of the vertex set of $\Delta$ into two classes, so that two vertices are in the same class if and only if the distance between them is even. An automorphism $g \in \mathrm{Aut}(\Delta)$ is said to be *type preserving* if, for every vertex $v$, $\tau(gv) = \tau(v)$. The graph $\Gamma\backslash\Delta$ is bipartite if and only if the action of $\Gamma$ is type preserving.

Let $\mathbb{F}$ be a nonarchimedean local field with residue field of order $q$. The Bruhat-Tits building associated with $\mathrm{PGL}(2,\mathbb{F})$ is a regular tree $\Delta$ of degree $q+1$ whose boundary may be identified with the projective line $\mathbb{P}_1(\mathbb{F})$. If $\Gamma$ is a torsion free lattice in $\mathrm{PGL}(2,\mathbb{F})$ then $\Gamma$ is necessarily a free group of rank $\gamma \geq 2$, which acts freely and cocompactly on $\Delta$ [S, Chapitres I.3.3, II.1.5], and the results apply to the action of $\Gamma$ on $\mathbb{P}_1(\mathbb{F})$.

Let $\mathcal{O}$ denote the valuation ring of $\mathbb{F}$. Then $K = \mathrm{PGL}(2,\mathcal{O})$ is an open maximal compact subgroup of $\mathrm{PGL}(2,\mathbb{F})$ and the vertex set of $\Delta$ may be identified with the homogeneous space $\mathrm{PGL}(2,\mathbb{F})/K$. If the Haar measure $\mu$ on $\mathrm{PGL}(2,\mathbb{F})$ is normalized so that $\mu(K) = 1$, then the covolume $\mathrm{covol}(\Gamma)$ is equal to the number of vertices of $X = \Gamma\backslash\Delta$ and $\gamma - 1 = \frac{(q-1)}{2}\,\mathrm{covol}(\Gamma)$, (c.f. [S, Chapitre II.1.5]).

The action of $\Gamma$ on $\Delta$ is type preserving if and only if $\Gamma$ is a subgroup of $\mathrm{PSL}(2,\mathbb{F})$. Combining Theorem 1 and Theorem 2, in this special case, yields

**Corollary 1.** *Let* $\Gamma$ *be a torsion free lattice in* $\mathrm{PGL}(2,\mathbb{F})$. *Using the above notation, the boundary algebras are determined as follows.*

(1) *The* $C^*$*-algebra* $\mathcal{A}(\Gamma) = C(\mathbb{P}_1(\mathbb{F})) \rtimes \Gamma$ *is the unique Cuntz-Krieger algebra satisfying*

$$(K_0(\mathcal{A}(\Gamma)), [\mathbf{1}]) = (\mathbb{Z}^\gamma \oplus \mathbb{Z}/(\gamma-1)\mathbb{Z}, (0,0,\dots,0,1)).$$

(2) *The von Neumann algebra* $\mathrm{L}^\infty(\mathbb{P}_1(\mathbb{F})) \rtimes \Gamma$ *is the hyperfinite factor of type* $\mathrm{III}_\lambda$ *where*

$$\lambda = \begin{cases} 1/q^2 & \textit{if } \Gamma \subset \mathrm{PSL}(2,\mathbb{F}), \\ 1/q & \textit{otherwise.} \end{cases}$$

## 1. The Cuntz-Krieger algebra

Let $\Delta$ be a locally finite tree whose vertices all have degree at least three. The results and terminology of [S] will be used extensively. The edges of $\Delta$ are directed edges and each geometric edge of $\Delta$ corresponds to two directed edges $d$ and $\overline{d}$. Let $\Delta^0$ denote the set of vertices and $\Delta^1$ the set of directed edges of $\Delta$.

Suppose that $\Gamma$ is a torsion free discrete group acting freely on $\Delta$ : that is no element $g \in \Gamma - \{1\}$ stabilizes any vertex or geometric edge of $\Delta$. Then $\Gamma$ is a free group [S, I.3.3] and there is an orientation on the edges which is invariant under $\Gamma$ [S, I.3.1]. Choose such an orientation. This orientation consists of a partition $\Delta^1 = \Delta^1_+ \sqcup \overline{\Delta^1_+}$ and a bijective involution $d \mapsto \overline{d} : \Delta^1 \to \Delta^1$ which interchanges the two components of $\Delta^1$. Each directed edge $d$ has an origin $o(d) \in \Delta^0$ and a terminal vertex $t(d) \in \Delta^0$ such that $o(\overline{d}) = t(d)$.

Assume that $\Gamma$ acts cocompactly on $\Delta$. This means that the quotient $\Gamma\backslash\Delta$ is a finite connected graph with vertex set $V = \Gamma\backslash\Delta^0$ and directed edge set $E = E_+ \sqcup \overline{E_+} = \Gamma\backslash\Delta^1_+ \sqcup \Gamma\backslash\overline{\Delta^1_+}$. The Euler-Poincaré characteristic of the graph is $\chi(\Gamma\backslash\Delta) = n_0 - n_1$ where $n_0 = \#(V)$ and $n_1 = \#(E_+)$, and $\Gamma$ is the free group on $\gamma$ generators, where $\gamma = 1 - \chi(\Gamma\backslash\Delta)$.

Choose a tree $T$ of representatives of $\Delta \pmod \Gamma$; that is a lifting of a maximal tree of $\Gamma\backslash\Delta$. The tree $T$ is finite, since $\Gamma$ acts cocompactly on $\Delta$. Let $S$ be the set of elements $x \in \Gamma - \{1\}$ such that there exists an edge $e \in \Delta^1_+$ with $o(e) \in T$ and $t(e) \in xT$. Then $S$ is a free set of generators for the free group $\Gamma$ [S, I.3.3, Théorème 4′] and $\gamma = \#S$. It is clear that $S^{-1}$ is the set of elements $x \in \Gamma - \{1\}$ such that there exists an edge $e \in \Delta^1_-$ with $o(e) \in T$ and $t(e) \in xT$. The map $g \mapsto gT$ is a bijection from $\Gamma$ onto the set of $\Gamma$ translates of the tree $T$ in $\Delta$, and these translates are pairwise disjoint [S, I.3.3, Proof of Théorème 4′]. Moreover each vertex of $\Delta$ lies in precisely one of the sets $gT$.

The boundary $\partial\Delta$ of the tree $\Delta$ is the set of equivalence classes of infinite semi-geodesics in $\Delta$, where equivalent semi-geodesics agree except on finitely many edges. Also $\partial\Delta$ has a natural compact totally disconnected topology [S, I.2.2]. The group $\Gamma$ acts on $\partial\Delta$ and one can form the crossed product algebra $C(\partial\Delta) \rtimes \Gamma$. This is the universal $C^*$-algebra generated by the commutative $C^*$-algebra $C(\partial\Delta)$ and the image of a unitary representation $\pi$ of $\Gamma$, satisfying the covariance relation

$$f(g^{-1}\omega) = \pi(g) \cdot f \cdot \pi(g)^{-1}(\omega) \tag{1.1}$$

for $f \in C(\partial\Delta)$, $g \in \Gamma$ and $\omega \in \partial\Delta$ [Ped]. This covariance relation implies that for each clopen set $E \subset \partial\Delta$ we have

$$\chi_{gE} = \pi(g) \cdot \chi_E \cdot \pi(g)^{-1}. \tag{1.2}$$

In this equation, $\chi_E$ is a continuous function and is regarded as an element of the crossed product algebra via the embedding $C(\partial\Delta) \subset C(\partial\Delta) \rtimes \Gamma$. In the present setup the algebra $C(\partial\Delta) \rtimes \Gamma$ is seen *a posteriori* to be simple. Therefore $C(\partial\Delta) \rtimes \Gamma$ coincides with the reduced crossed product algebra [Ped, 7.7.4] and there is no need to distinguish between them notationally.

Fix a vertex $O \in \Delta$ with $O \in T$. Each $\omega \in \partial\Delta$ has a unique representative semi-geodesic $[O, \omega)$ with initial vertex $O$. A basic open neighbourhood of $\omega \in \partial\Delta$ consists of those $\omega' \in \partial\Delta$ such that $[O,\omega) \cap [O,\omega') \supset [O, v]$ for some fixed $v \in [O, \omega)$. If $g \in \Gamma - \{1\}$, let $\Pi_g$ denote the set of all $\omega \in \partial\Delta$ such that $[O,\omega)$ meets the tree $gT$. Note that $\Pi_g$ is clopen, since $T$ is finite. The characteristic function $p_g$ of the set $\Pi_g$ is continuous and so lies in $C(\partial\Delta) \subset C(\partial\Delta) \rtimes \Gamma$. The identity element $\mathbf{1}$ of $C(\partial\Delta) \rtimes \Gamma$ is the constant function defined by $\mathbf{1}(\omega) = 1$, $\omega \in \partial\Delta$.

**Lemma 1.1.** *If $x, y \in S \cup S^{-1}$ with $x \neq y^{-1}$ then*

(a) $\pi(x)p_{x^{-1}}\pi(x^{-1}) = \mathbf{1} - p_x$ ;
(b) $\pi(x)p_y\pi(x^{-1}) = p_{xy}$.

Proof. (a) By (1.2), the element $\pi(x)p_{x^{-1}}\pi(x^{-1})$ is the characteristic function of the set

$$\begin{aligned} F_x &= \{x\omega \ ; \ \omega \in \partial\Delta, x^{-1}T \cap [O,\omega) \neq \emptyset\} \\ &= \{x\omega \ ; \ \omega \in \partial\Delta, T \cap [xO, x\omega) \neq \emptyset\} \\ &= \{\omega \in \partial\Delta \ ; \ T \cap [xO,\omega) \neq \emptyset\}. \end{aligned}$$

Now there exists a unique edge $e \in \Delta^1$ such that $o(e) \in T$ and $t(e) \in xT$. If $x \in S$ then $e \in \Delta^1_+$ and if $x \in S^{-1}$ then $e \in \overline{\Delta^1_+}$. Therefore

$$\begin{aligned} \partial\Delta - F_x &= \{\omega \in \partial\Delta \ ; \ T \cap [xO,\omega) = \emptyset\} \\ &= \{\omega \in \partial\Delta \ ; \ xT \cap [O,\omega) \neq \emptyset\} \\ &= \Pi_x, \end{aligned}$$

and the characteristic function of this set is $p_x$. See Figure 1.

The proof of (b) is an easy consequence of (1.2). □

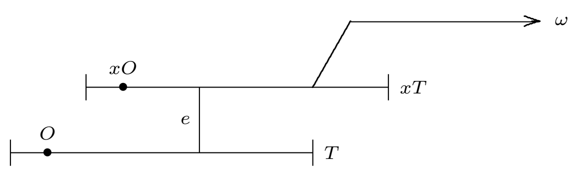


Figure 1. A boundary point $\omega \in \Pi_x$.

**Lemma 1.2.** *The family of projections $P = \{p_g \ ; \ g \in \Gamma - \{1\}\}$ generates $C(\partial\Delta)$ as a $C^*$-algebra.*

Proof. We show that $P$ separates points of $\partial\Delta$. Let $\omega_1, \omega_2 \in \partial\Delta$ with $\omega_1 \neq \omega_2$. Let $[O,\omega_1) \cap [O,\omega_2) = [O,v]$, and choose $u \in [v,\omega_1)$ such that $d(v,u)$ is greater than the diameter of $T$. See Figure 2.

Let $g \in \Gamma$ be the unique element such that $u \in gT$. Then $v \notin gT$ and so $gT \cap [O,\omega_2) = \emptyset$. Therefore $p_g(\omega_1) = 1$ and $p_g(\omega_2) = 0$. □

**Lemma 1.3.** *The sets of the form $\Pi_x$, $x \in S \cup S^{-1}$, are pairwise disjoint and their union is $\partial\Delta$.*

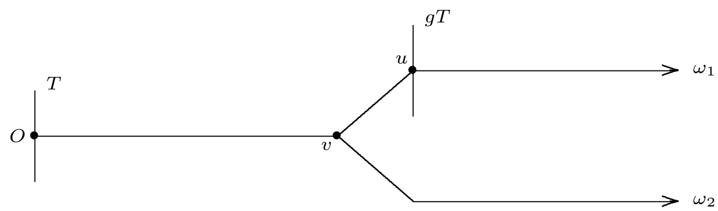


FIGURE 2. Separation of boundary points.

PROOF. Given $\omega \in \partial\Delta$, let $v$ be the unique vertex of $\Delta$ such that $[O, \omega) \cap T = [O, v]$. Let $v'$ be the vertex of $[O, \omega)$ such that $d(O, v') = d(O, v) + 1$. Then let $x$ be the unique element of $S \cup S^{-1}$ such that $v' \in xT$. See Figure 3. Then $\omega \in \Pi_x$. The sets $\Pi_x$, $x \in S \cup S^{-1}$, are pairwise disjoint since the sets $xT$, $x \in S \cup S^{-1}$, are pairwise disjoint. $\square$

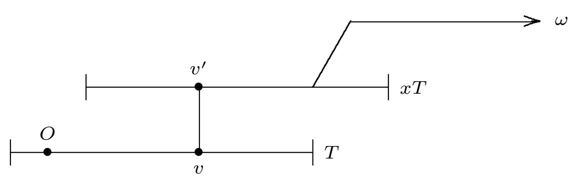


FIGURE 3. Definition of the set $\Pi_x$ containing $\omega$.

For $x \in S \cup S^{-1}$ define a partial isometry

$$s_x = \pi(x)(\mathbf{1} - p_{x^{-1}}) \in C(\partial\Delta) \rtimes \Gamma.$$

Then, by Lemma 1.1,

$$s_x s_x^* = \pi(x)(\mathbf{1} - p_{x^{-1}})\pi(x^{-1}) = \mathbf{1} - \pi(x)p_{x^{-1}}\pi(x^{-1}) = p_x,$$

and

$$s_x^* s_x = \mathbf{1} - p_{x^{-1}}.$$

Therefore the elements $s_x$ satisfy the relations

$$s_x^* s_x = \sum_{\substack{y \in S \cup S^{-1} \\ y \neq x^{-1}}} s_y s_y^*. \tag{1.3}$$

Also, it follows from Lemma 1.3 that

$$\mathbf{1} = \sum_{x \in S \cup S^{-1}} p_x = \sum_{x \in S \cup S^{-1}} s_x s_x^*. \tag{1.4}$$

The relations (1.3),(1.4) are precisely the Cuntz-Krieger relations [CK] corresponding to the $\{0,1\}$-matrix $A$, with entries indexed by elements of $S \cup S^{-1}$, defined by

$$A(x,y) = \begin{cases} 1 & \text{if } y \neq x^{-1}, \\ 0 & \text{if } y = x^{-1}. \end{cases} \tag{1.5}$$

The matrix $A$ depends only on the rank of the free group $\Gamma$. Also $A$ is irreducible and not a permutation matrix. It follows that the $C^*$-subalgebra $\mathcal{A}$ of $C(\partial\Delta) \rtimes \Gamma$ generated by $\{s_x \,;\, x \in S \cup S^{-1}\}$ is isomorphic to the simple Cuntz-Krieger algebra $\mathcal{O}_A$ [CK]. It remains to show that $\mathcal{A}$ is the whole of $C(\partial\Delta) \rtimes \Gamma$.

**Lemma 1.4.** *Under the above hypotheses, $C(\partial\Delta) \rtimes \Gamma = \mathcal{A}$.*

Proof. By the discussion above, it is enough to show that

$$\mathcal{A} \supseteq C(\partial\Delta) \rtimes \Gamma.$$

First of all we show that $\mathcal{A} \supseteq \pi(\Gamma)$. It suffices to show that $\pi(x) \in \mathcal{A}$ for each $x \in S \cup S^{-1}$. Now

$$s_{x^{-1}}^* = (\mathbf{1} - p_x)\pi(x) = \pi(x)p_{x^{-1}},$$

by Lemma 1.1. Therefore

$$s_x + s_{x^{-1}}^* = \pi(x)(\mathbf{1} - p_{x^{-1}}) + \pi(x)p_{x^{-1}} = \pi(x). \tag{1.6}$$

It follows that $\pi(x) \in \mathcal{A}$, as required.

Finally, we must show that $\mathcal{A} \supseteq C(\partial\Delta)$. Since $s_x s_x^* = p_x$, it is certainly true that $p_x \in \mathcal{A}$ for all $x \in S \cup S^{-1}$. It follows by induction from Lemma 1.1(b), that $p_g \in \mathcal{A}$ for all $g \in \Gamma$. Lemma 1.2 now implies that $\mathcal{A} \supseteq C(\partial\Delta)$. □

**Example 1.5.** Consider the graphs $X$, $Y$ in Figure 4. Each of them has as universal covering space the 3-homogeneous tree $\Delta$. Each has fundamental group the free group $\Gamma$ on two generators. Consequently, each gives rise to an action of $\Gamma$ on $\Delta$. These two actions cannot be conjugate via an element of $\mathrm{Aut}(\Delta)$ because their quotients are not isomorphic as graphs.

Copies of the free group on two generators, acting with these actions on the corresponding Bruhat-Tits tree $\Delta$, can be found inside $\mathrm{PGL}(2,\mathbb{Q}_2)$, and also inside $\mathrm{PGL}(2,\mathbb{F})$ for any local field $\mathbb{F}$ with residue field of order 2. To see this, note that

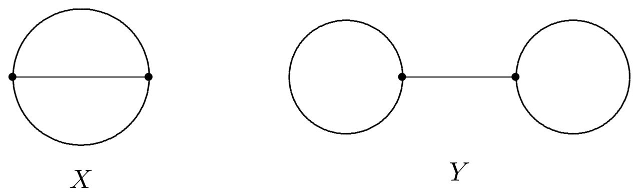


FIGURE 4. The graphs $X$, $Y$.

by [FTN, Appendix, Proposition 5.5] $\mathrm{PGL}(2, \mathbb{Q}_2)$ contains cocompact lattices $\Gamma_1$, $\Gamma_2$ which act freely and transitively on the vertex set $\Delta^0$ with

$$\Gamma_1 = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$$
$$\Gamma_2 = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z}) = \langle x, d \mid d^2 = 1 \rangle.$$

The subgroups $\Gamma_X = \langle ab, ac \rangle$ and $\Gamma_Y = \langle x, dxd \rangle$ are both isomorphic to the free group on two generators. Moreover $\Gamma_X \backslash \Delta = X$ and $\Gamma_Y \backslash \Delta = Y$.

## 2. K-THEORY

Using the results of [C1], it is now easy to determine the K-theory of $\mathcal{A}(\Gamma)$. For each $x \in S \cup S^{-1}$, the element $p_x$ is a projection in $\mathcal{A}(\Gamma)$ and therefore defines an equivalence class $[p_x]$ in $K_0(\mathcal{A}(\Gamma))$. It is shown in [C1] that the classes $[p_x]$ generate $K_0(\mathcal{A}(\Gamma))$. Indeed, let $L$ denote the abelian group with generating set $S \cup S^{-1}$ and relations

$$x = \sum_{\substack{y \in S \cup S^{-1} \\ y \neq x^{-1}}} y \qquad \text{for } x \in S \cup S^{-1}. \tag{2.1}$$

The map $x \mapsto [p_x]$ extends to an isomorphism $\theta$ from $L$ onto $K_0(\mathcal{A}(\Gamma))$ [C1]. Moreover $\theta(\varepsilon) = [\mathbf{1}]$, where $\varepsilon = \sum_{x \in S \cup S^{-1}} x$. Now it follows from (2.1) that, for each $x \in S$,

$$\varepsilon = x + x^{-1}. \tag{2.2}$$

Also

$$\varepsilon = \sum_{x \in S} (x + x^{-1}) = \sum_{x \in S} \varepsilon = \gamma\varepsilon.$$

Thus

$$(\gamma - 1)\varepsilon = 0. \tag{2.3}$$

The group $L$ is therefore generated by $S \cup \{\varepsilon\}$, and the relation (2.3) is satisfied.

On the other hand, starting with an abstract abelian group with generating set $S \cup \{\varepsilon\}$ and the relations (2.2), one can make the formal definition $x^{-1} = \varepsilon - x$, for each $x \in S$, and recover the relations (2.1) via

$$\sum_{x \in S}(x + x^{-1}) = \gamma\varepsilon = \varepsilon = x + x^{-1} \qquad \text{for } x \in S.$$

This discussion proves

**Lemma 2.1.** $K_0(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^\gamma \oplus \mathbb{Z}/(\gamma - 1)\mathbb{Z}$ *via an isomorphism which sends* $[\mathbf{1}]$ *to the generator of* $\mathbb{Z}/(\gamma - 1)\mathbb{Z}$.

It is known that the $C^*$-algebra $\mathcal{A}(\Gamma)$ is purely infinite, simple, unital and nuclear [CK, C1, C2]. The classification theorem of [K] therefore shows that $\mathcal{A}(\Gamma)$ is determined by its K-theory.

**Remark 2.2.** If $\Gamma$ is a torsion free cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$, so that $\Gamma$ is the fundamental group of a Riemann surface of genus $g$, then it is known, [AD, Proposition 2.9], [HN], that $\mathcal{A}(\Gamma) = C(\mathbb{P}_1(\mathbb{R})) \rtimes \Gamma$ is the unique p.i.s.u.n. $C^*$-algebra whose K-theory is specified by

$$\begin{aligned}(K_0(\mathcal{A}(\Gamma)), [\mathbf{1}]) &= (\mathbb{Z}^{2g+1} \oplus \mathbb{Z}/(2g-2)\mathbb{Z}, (0, 0, \ldots, 0, 1)),\\ K_1(\mathcal{A}(\Gamma)) &= \mathbb{Z}^{2g+1}.\end{aligned}$$

The proof of this result in [AD] makes use of the Thom Isomorphism Theorem of A. Connes (which has no $p$-adic analogue) to identify $K_*(\mathcal{A}(\Gamma))$ with the topological K-theory $K^*(\Gamma \backslash \mathrm{PSL}(2, \mathbb{R}))$. It follows from the classification theorem of [K, Ph] that $\mathcal{A}(\Gamma)$ is a Cuntz-Krieger algebra. However there is no apparent dynamical reason for this fact. In contrast, the Cuntz-Krieger algebras of the present article appear naturally and explicitly.

## 3. The measure theoretic result

The purpose of this section is to prove Theorem 2 of the Introduction. From now on $\Delta$ is a homogeneous tree of degree $q + 1$, where $q \geq 1$, and $\Gamma$ is a free uniform lattice in $\mathrm{Aut}(\Delta)$. A similar Theorem could be stated for non-homogeneous trees, and proved by the same methods. The boundary $\partial\Delta$ is endowed with a natural Borel measure. In contrast to the topological result, measure theoretic rigidity for the boundary action fails: the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$

depends on the tree $\Delta$ and on the action of $\Gamma$. Before proceeding with the proof here are some examples.

**Example 3.1.** Let $\Gamma$ be the free group on two generators. Then $\Gamma$ is the fundamental group of each of the graphs $X$, $Y$ of Figure 4. The 3-homogeneous tree $\Delta_3$ is the universal covering of both these graphs and there are two corresponding (free, cocompact) actions of $\Gamma$ on $\Delta_3$. It follows from Theorem 2 that the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta_3) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_{1/4}$ in the first case, since $X$ is bipartite, and type $\mathrm{III}_{1/2}$ in the second case, since $Y$ is not bipartite.

The group $\Gamma$ is also the fundamental group of a bouquet of two circles and the corresponding action of $\Gamma$ on the 4-homogeneous tree $\Delta_4$ produces the hyperfinite factor of type $\mathrm{III}_{1/3}$. These three actions are the only free and cocompact actions of the free group on two generators on a tree $\Delta$ with no vertices of degree $\leq 2$.

**Remark 3.2.** For each $\gamma \geq 2$, it is easy to construct bipartite and non-bipartite 3-homogeneous graphs with fundamental group the free group on $\gamma$ generators. The corresponding boundary actions are of types $\mathrm{III}_{1/4}$ and $\mathrm{III}_{1/2}$ respectively.

We now proceed with the proof of Theorem 2. As before, fix a vertex $O \in \Delta$. If $u, v$ are vertices in $\Delta^0$, let $[u, v]$ be the directed geodesic path between them, with origin $u$. The graph distance $d(u, v)$ between $u$ and $v$ is the length of $[u, v]$, where each edge is assigned unit length. If $v \in \Delta^0$ let $\Omega_v$ be the clopen set consisting of all $\omega \in \partial\Delta$ such that $v \in [O, \omega)$.

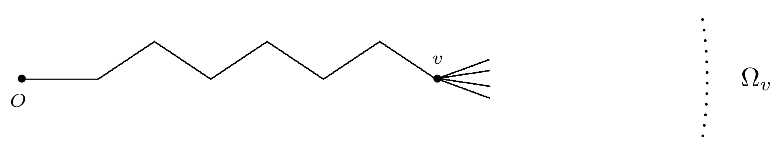


FIGURE 5. A subset $\Omega_v$ of the boundary.

There is a natural Borel measure $\mu$ on $\partial\Delta$ defined by $\mu(\Omega_v) = q^{(1-n)}$, where $n = d(O, v)$. The consistency of this definition is easily established, using the fact that there are precisely $q$ vertices $w$ adjacent to $v$ and not lying on the path $[O, v]$. The set $\Omega_v$ is the disjoint union of the corresponding sets $\Omega_w$, each of which has measure $q^{-n}$. Note that the normalization of this measure is different from that in [FTN]. This is immaterial for the result, but makes the formulae simpler. The measure $\mu$ clearly depends on the choice of the vertex $O$ in $\Delta$, but its measure class does not.

**Lemma 3.3.** *The action of* $\Gamma$ *on* $\partial\Delta$ *is measure-theoretically free, i.e.*

$$\mu\left(\{\omega \in \partial\Delta : g\omega = \omega\}\right) = 0$$

*for all elements* $g \in \Gamma - \{e\}$.

PROOF. Let $g \in \Gamma - \{e\}$. Since the action of $\Gamma$ on $\Delta$ is free, $g$ is hyperbolic; that is $g$ fixes no point of $\Delta$. It follows that the set $\{\omega \in \partial\Delta : g\omega = \omega\}$ contains exactly two elements and so certainly has measure zero. □

It is well known (and it is an easy consequence of Lemma 3.13 below) that the action of $\Gamma$ on $\partial\Delta$ is also ergodic. Therefore the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is a factor. A convenient reference for this fact and for the classification of von Neumann algebras is [Su]. Most of this section will be devoted to establishing that this factor is of type $\mathrm{III}_\lambda$, for an appropriate value of $\lambda$. This will be done by determining the ratio set of W. Krieger.

**Definition 3.4.** Let $G$ be a countable group of automorphisms of a measure space $(\Omega, \mu)$. Define the **ratio set** $r(G)$ to be the subset of $[0, \infty)$ such that if $\lambda \geq 0$ then $\lambda \in r(G)$ if and only if for every $\epsilon > 0$ and measurable set $A$ with $\mu(A) > 0$, there exists $g \in G$ and a measurable set $B$ such that $\mu(B) > 0$, $B \cup gB \subseteq A$ and

$$\left| \frac{d\mu\circ g}{d\mu}(\omega) - \lambda \right| < \epsilon$$

for all $\omega \in B$.

**Remark 3.5.** The ratio set $r(\Gamma)$ depends only on the quasi-equivalence class of the measure $\mu$. If the action of $\Gamma$ is ergodic then $r(\Gamma) - \{0\}$ is a subgroup of the multiplicative group of positive real numbers [HO, §I-3, Lemma 14].

In order to compute $r(\Gamma)$, for the action of $\Gamma$ on $\partial\Delta$, the first step is to find the possible values of the Radon-Nikodym derivatives $\frac{d\mu\circ g}{d\mu}(\omega)$, for $g \in \Gamma$ and $\omega \in \partial\Delta$.

Fix $g \in \Gamma$ and $\omega \in \partial\Delta$. Choose an open set of the form $\Omega_v$ with $v \in [O, \omega)$ and $d(O, v) > d(O, gO)$. Such sets $\Omega_v$ form a neighbourhood base of $\omega$. Then $v \notin [O, gO]$ (Figure 6), and $g^{-1}\Omega_v = \Omega_{g^{-1}v}$. Since $d(O, g^{-1}v) = d(gO, v)$, we have $\mu(g^{-1}\Omega_v) = q^{-d(gO,v)}$.

It follows that

$$\frac{d\mu\circ g}{d\mu}(\omega) = \frac{\mu(g^{-1}\Omega_v)}{\mu(\Omega_v)} = \frac{q^{-d(gO,v)}}{q^{-d(O,v)}} = q^{\delta(g,\omega)} \tag{3.1}$$

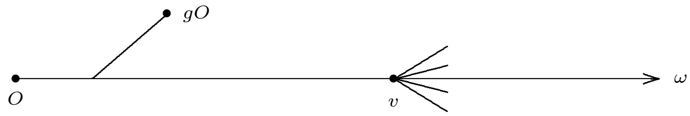


FIGURE 6

where $\delta(g, \omega) = d(O, v) - d(gO, v)$. It is clear that $\delta(g, \omega)$ depends only on $g$ and $\omega$, not on the choice of $v$. In the language of [GH, Chapter 8], $\delta(g, \omega)$ is the Busemann function $\beta_\omega(O, gO)$ relating the horocycles centered at $\omega$ containing $O$, $gO$ respectively. For a fixed vertex $v$ with $d(O, v) > d(O, gO)$, the formula (3.1) remains true for all $\omega \in \Omega_v$.

We have therefore proved

**Lemma 3.6.** *The values of the Radon-Nikodym derivatives* $\frac{d\mu\circ g}{d\mu}(\omega)$, *for* $g \in \Gamma$ *and* $\omega \in \partial\Delta$, *are given by*

$$\frac{d\mu\circ g}{d\mu}(\omega) = q^{\delta(g,\omega)}$$

*Moreover, for each* $g \in \Gamma$, *each of these values is attained on a nonempty open subset of* $\partial\Delta$.

These considerations show that

$$r(\Gamma) \subseteq \{q^{\delta(g,\omega)}\,;\, g \in \Gamma, \omega \in \partial\Delta\} \cup \{0\}. \tag{3.2}$$

Since the action of $\Gamma$ is ergodic, $r(\Gamma) - \{0\}$ is a multiplicative group of positive real numbers [HO, Lemma 14]. What must be done now is to show that the inclusion in (3.2) is in fact an equality. Clearly $r(\Gamma) \neq [0, \infty)$. Therefore if we can show that $r(\Gamma)$ contains a number in the open interval $(0, 1)$ then, by [HO, Lemma 15], it must equal $\{\lambda^n\,;\, n \in \mathbb{Z}\} \cup \{0\}$, for some $\lambda \in (0, 1)$. By definition, this will show that the action of $\Gamma$, and hence the associated von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma)$, is of type $\mathrm{III}_\lambda$.

Before proceeding, it is useful to interpret the situation in terms of the quotient graph $X = \Gamma\backslash\Delta$. In a connected graph $X$ a *proper* path is a path which has no backtracking. That is, no edge $[a, b]$ in the path is immediately followed by its inverse $[b, a]$. A *cycle* is a closed path, which is said to be based at its initial vertex (= final vertex). Note that a proper cycle can have a tail beginning at its base vertex, but that it can have no other tail (Figure 7). Every proper cycle determines a unique tail-less cycle which is obtained by removing the tail. A circuit is a cycle which does not pass more than once through any vertex. There is clearly an upper bound for the possible length of a circuit in $X$.

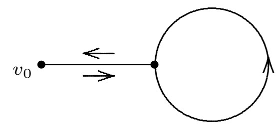


FIGURE 7. A proper cycle with tail, based at $v_0$.

If $g \in \Gamma$, then the geodesic path $[O, gO]$ in $\Delta$ projects to a proper cycle $C$ in the quotient graph $X = \Gamma\backslash\Delta$ based at $v_0 = \Gamma O$. Moreover $d(O, gO)$ is equal to the length $\ell(C)$ of that cycle.

Conversely if $C$ is a proper cycle based at $v_0$ in the graph $X$ then the homotopy class of $C$ is an element $g$ of the fundamental group $\Gamma$ of $X$. The cycle $C$ lifts to a unique proper path in $\Delta$ with initial vertex $O$, namely $[O, gO]$, and $\ell(C) = d(O, gO)$.

In order to prove equality in (3.2) we need the auxiliary concept of the *full group*.

**Definition 3.7.** Given a group $\Gamma$ acting on a measure space $(\Omega, \mu)$, we define the *full group*, $[\Gamma]$, of $\Gamma$ by

$$[\Gamma] = \{T \in \mathrm{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega\}.$$

**Remark 3.8.** The ratio set $r(\Gamma)$ of a countable subgroup $\Gamma$ of $\mathrm{Aut}(\Omega)$ depends only on the full group in the sense that $r(\Gamma_1) = r(\Gamma_2)$ whenever $[\Gamma_1] = [\Gamma_2]$.

The basis for the proof of equality in (3.2) is the following well known result. It is stated without proof in [HO, I.3].

**Lemma 3.9.** *Let $\Gamma$ be a countable group acting ergodically on a measure space $\Omega$. Suppose that the full group $[\Gamma]$ contains an ergodic measure preserving subgroup $H$.*

*If $r \in (0, \infty)$, $g \in \Gamma$ and the set $D = \{\omega \in \Omega\,;\, \frac{d\mu\circ g}{d\mu}(\omega) = r\}$ has positive measure, then $r \in r(\Gamma)$.*

PROOF. Let $A$ be a measurable subset of $\Omega$ with $\mu(A) > 0$. By the ergodicity of $H$, there exist $h_1, h_2 \in H$ such that the set $B = \{\omega \in A : h_1\omega \in D \text{ and } h_2 g h_1\omega \in A\}$ has positive measure.

Let $\Gamma'$ denote the group generated by $h_1, h_2$ and $\Gamma$. By Remark 3.8, $r(\Gamma') = r(\Gamma)$.

Let $t = h_2 g h_1 \in \Gamma'$. By construction, $B \cup tB \subseteq A$. Moreover, since $H$ is measure-preserving,

$$\frac{d\mu\circ t}{d\mu}(\omega) = \frac{d\mu\circ g}{d\mu}(h_1\omega) = r \text{ for all } \omega \in B,$$

since $h_1\omega \in D$. This proves that $r \in r(\Gamma') = r(\Gamma)$, as required. □

In view of Lemma 3.6, all that is now needed in the present setup is the construction of a subgroup $H$. This will require the following result.

**Lemma 3.10.** *Let $H$ be subgroup of* $\mathrm{Aut}(\Delta)$. *Suppose that the induced action of $H$ on $\partial\Delta$ is measure preserving and that, for each positive integer $n$, $H$ acts transitively on the collection of sets*

$$\{\Omega_v : v \in \Delta,\, d(O,v) = n\}.$$

*Then $H$ acts ergodically on $\partial\Delta$.*

Proof. Suppose that $S_0 \subseteq \partial\Delta$ is a Borel set which is invariant under $H$ and such that $\mu(S_0) > 0$. We show that this implies $\mu(\partial\Delta - S_0) = 0$, thereby establishing the ergodicity of the action.

Define a new measure $\lambda$ on $\partial\Delta$ by $\lambda(S) = \mu(S \cap S_0)$, for each Borel set $S \subseteq \partial\Delta$. Now, for each $k \in H$,

$$\begin{aligned}
\lambda(kS) &= \mu(kS \cap S_0) = \mu(S \cap k^{-1}S_0) \\
&\leq \mu(S \cap S_0) + \mu(S \cap (k^{-1}S_0 - S_0)) \\
&= \mu(S \cap S_0) \\
&= \lambda(S),
\end{aligned}$$

and therefore $\lambda$ is $H$-invariant.

Fix a positive integer $n$. The transitivity hypothesis on the action of $H$ implies that

$$\lambda(\Omega_v) = \lambda(\Omega_w)$$

whenever $v, w \in \Delta$, $d(O,v) = d(O,w) = n$. Since $\partial\Delta$ is the union of $q^{(n-1)}(q+1)$ disjoint sets $\{\Omega_v;\, d(O,v) = n\}$, each of which has equal measure with respect to $\lambda$, we deduce that, if $d(O,v) = n$,

$$\lambda(\Omega_v) = \frac{\lambda(\partial\Delta)}{q^{(n-1)}(q+1)} = \frac{\mu(S_0)}{q^{(n-1)}(q+1)}.$$

Thus $\lambda(\Omega_v) = c\mu(\Omega_v)$ for every $v \in \Delta$, where $c = \frac{\mu(S_0)}{(q+1)} > 0$. Since the sets $\Omega_v$, $v \in \Delta$, generate the Borel $\sigma$-algebra, we deduce that $\lambda(S) = c\mu(S)$ for each Borel set $S$. Therefore

$$\mu(\partial\Delta - S_0) = c^{-1}\lambda(\partial\Delta - S_0) = c^{-1}\mu((\partial\Delta - S_0) \cap S_0) = 0,$$

thus proving ergodicity. □

It is now convenient to introduce some new terminology.

**Definition 3.11.** Let $X$ be a finite connected graph. Let $v_0$ be a vertex of $X$ and let $K \geq 0$. Say that $(X, v_0)$ has property $\mathfrak{L}(K)$ if for any two proper paths $P_1, P_2$ having the same length $n$ and the same initial vertex $v_0$, there exists $k \geq 0$, with $k \leq K$, and proper cycles $C_1, C_2$ based at $v_0$ such that

(a) The initial segment of $C_i$ is $P_i$, $i = 1, 2$;
(b) the cycles $C_i$ have the same length $n + k$, $i = 1, 2$.

Property $\mathfrak{L}(K)$ says that any two proper paths of the same length starting at $v_0$ can be completed to proper cycles of the same length, with a uniform bound on how much must be added to each path.

**Lemma 3.12.** *Let $X$ be a finite connected graph whose vertices all have degree at least three and let $v_0$ be a vertex of $X$. Then $(X, v_0)$ has property $\mathfrak{L}(K)$ for some $K \geq 0$.*

The proof of this technical result is deferred to Section 4. We can now prove that the action of $\Gamma$ on $\partial\Delta$ satisfies the hypotheses of Lemma 3.9.

**Lemma 3.13.** *Let $\Delta$ be a homogeneous tree of degree $q+1$, where $q \geq 1$ and let $\Gamma$ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$. *Then, relative to the action of $\Gamma$ on $\partial\Delta$, the full group $[\Gamma]$ contains an ergodic measure preserving subgroup $H$.*

Proof. By Lemma 3.10, it suffices to prove the following assertion for any $u, v \in \Delta^0$, with $d(O, u) = d(O, v) = n$.

($\star$) There exists a measure preserving automorphism $\phi \in [\Gamma]$ such that $\phi$ is almost everywhere a bijection from $\Omega_u$ onto $\Omega_v$.

The geodesic paths $[O, u], [O, v]$ in $\Delta$ project to proper paths $P_u, P_v$ in $X$ with initial vertex $v_0 = \Gamma O$ and length $n$. By hypothesis, the graph $X$ has property $\mathfrak{L}(K)$ for some constant $K \geq 0$, relative to $v_0$. Therefore there exists an integer $k \leq K$ and proper cycles $C_u, C_v$ based at $v_0$ which have initial segments $P_u, P_v$ respectively and $\ell(P_u) = \ell(P_v) = n + k$.

The cycles $C_u, C_v$ lift to unique geodesic paths $[O, u^*], [O, v^*]$ in $\Delta$ with initial segments $[O, u]$, $[O, v]$ respectively and $\Gamma u^* = \Gamma v^* = \Gamma O = v_0$. Since the vertices of $X$ all have degree at least three, we can choose an edge $e$ with $o(e) = v_0$ in $X$ such that $e$ meets the terminal edges of $C_u$ and $C_v$ only at $v_0$. There are unique vertices $u_1, v_1 \in \Delta^1$ such that $e = \Gamma[u^*, u_1] = \Gamma[v^*, v_1]$. Therefore there exists an element $g \in \Gamma$ such that $g[u^*, u_1] = [v^*, v_1]$. The restriction of the action of $g$ to $\Omega_{u_1} = \{\omega \in \partial\Delta;\ u_1 \in [u^*, \omega)\}$ defines a measure preserving bijection from $\Omega_{u_1}$ onto $\Omega_{v_1}$. Define $\phi(\omega) = g(\omega)$ for $\omega \in \Omega_{u_1}$.

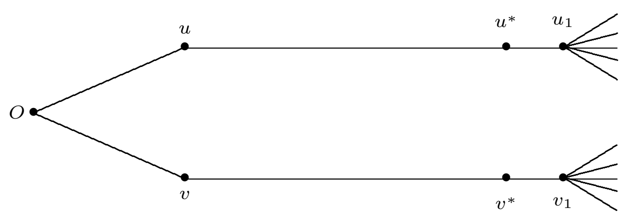


FIGURE 8. Definition of $\phi$ on $\Omega_{u_1}$.

The set $\Omega_u$ is a disjoint union of $q^{k+1}$ sets of the form $\Omega_w$ where $d(O, w) = n + k + 1$. Each such set therefore has measure $\mu(\Omega_w) = q^{-k-1}\mu(\Omega_u)$. The map $\phi$ has been defined only on the set $\Omega_w$ with $w = u_1$. Therefore $\phi$ has not yet been defined on a proportion $(1 - q^{-k-1})$ of the set $\Omega_u$. Since $k \le K$, the measure of the subset of $\Omega_u$ for which $\phi$ has not yet been defined is at most $(1 - q^{-K-1})\mu(\Omega_u)$.

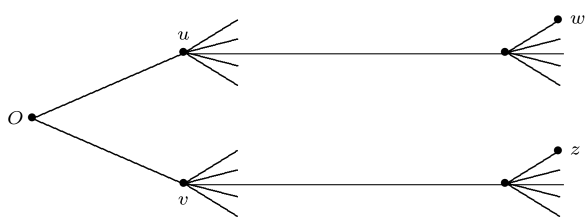


FIGURE 9. Second step in the definition of $\phi$.

Now repeat the process on each of the $q^{k+1}$ sets $\Omega_w \subset \Omega_u$, with $d(O, w) = n + k + 1$, on which $\phi$ has not yet been defined. In the preceding argument, replace $\Omega_u$ by $\Omega_w$ and $\Omega_v$ by an appropriate subset $\Omega_z$ of $\Omega_v$ disjoint from $\Omega_{v_1}$. Note that there is a large amount of arbitrariness in the choice of which $\Omega_z$ is to be paired with a particular $\Omega_w$. In each such $\Omega_w$, $\phi$ is then defined on a subset whose complement in $\Omega_w$ has measure at most $(1-q^{-K-1})\mu(\Omega_w)$.

Thus after two steps, $\phi$ has been defined except on a set of measure at most $(1-q^{-K-1})^2\mu(\Omega_u)$. Continue in this way. After $j$ steps, $\phi$ has been defined except on a set of measure at most $(1-q^{-K-1})^j\mu(\Omega_u)$.

Since $(1-q^{-K-1})^j \to 0$ as $j \to \infty$, the measure preserving map $\phi$ is defined almost everywhere on $\Omega_u$, with $\phi(\omega) \in \Gamma\omega$ for almost all $\omega \in \Omega_u$. Finally define $\phi$ to be the inverse of the map already constructed on $\Omega_v$ and the identity map on $\partial\Delta - (\Omega_u \cup \Omega_v)$. The proof of $(\star)$ is complete. $\square$

It follows from Lemmas 3.6, 3.9, and 3.13 that we have equality in (3.2). That is

$$r(\Gamma) = \{q^{\delta(g,\omega)}\,;\, g \in \Gamma, \omega \in \partial\Delta\} \cup \{0\}. \tag{3.3}$$

The final step is to identify this set more precisely. Recall that there is a canonical bipartition of the vertex set of $\Delta$, such that two vertices have the same type if and only if the distance between them is even. The graph $X = \Gamma\backslash\Delta$ is bipartite if and only if the action of $\Gamma$ is type preserving. Recall also that $\delta(g,\omega) = d(O,v) - d(gO,v)$, for any vertex $v \in [O,\omega) \cap [gO,\omega)$.

**Lemma 3.14.** *Let $\Delta$ be a locally finite tree whose vertices all have degree at least three. Let $\Gamma$ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$ *and let $X = \Gamma\backslash\Delta$. Then*

$$\{\delta(g,\omega)\,;\, g \in \Gamma, \omega \in \partial\Delta\} = \begin{cases} 2\mathbb{Z} & \textit{if } X \textit{ is bipartite},\\ \mathbb{Z} & \textit{otherwise}. \end{cases}$$

Proof. Suppose first of all that $X$ is not bipartite. Then $X$ contains a circuit of odd length. Connecting this circuit to $v_0$ by a minimal path and going around the circuit an appropriate number of times shows that $X$ contains proper cycles based at $v_0$ of arbitrarily large even and odd lengths.

It follows that we may choose $g \in \Gamma$ such that $d(O,gO) = 2n$, for arbitrarily large $n$. If $k \in \mathbb{Z}$,with $-n \le k \le n$, let $a \in [O,gO]$ with $d(O,a) = n+k$, $d(gO,a) = n-k$. Choose $\omega \in \partial\Delta$ with $[O,\omega) \cap [gO,\omega) = [a,\omega)$. This is possible since the vertex $a$ has degree at least three (Figure 10). Then $\delta(g,\omega) = n+k-(n-k) = 2k$.

We may also choose $g \in \Gamma$ such that $d(O,gO) = 2n+1$ for arbitrarily large $n$. If $k \in \mathbb{Z}$,with $-n \le k \le n$, choose $a \in [O,gO]$ with $d(O,a) = n+k$, $d(gO,a) = n+1-k$. Choose $\omega \in \partial\Delta$ with $[O,\omega) \cap [gO,\omega) = [a,\omega)$. Then $\delta(g,\omega) = 2k-1$.

It follows that the range of the function $\delta(g,\omega)$ is $\mathbb{Z}$.

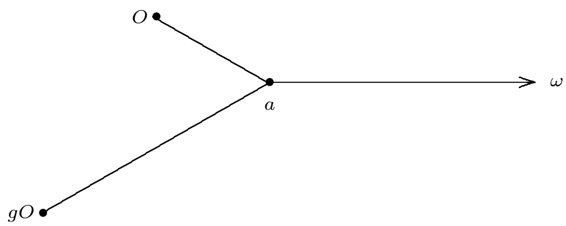


FIGURE 10

Now suppose that $X$ is bipartite. Then the graph $X$ contains proper cycles of arbitrarily large even length only. The preceding argument shows that the range of the function $\delta(g, \omega)$ is $2\mathbb{Z}$. □

Assume the hypotheses of Theorem 2. Then by (3.3) and Lemma 3.14, the action of $\Gamma$ on $\partial\Delta$ is of type $\mathrm{III}_{1/q^2}$, if $X$ is bipartite and of type $\mathrm{III}_{1/q}$ otherwise.

In order to complete the proof of Theorem 2, it only remains to prove that the factor $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is hyperfinite. By [Z1], this follows from the next result.

**Proposition 3.15.** *The action of* $\Gamma$ *on* $\partial\Delta$ *is amenable.*

PROOF. The group $G = \mathrm{Aut}(\Delta)$ acts transitively on $\partial\Delta$ [FTN, Chapter I.8]. Fix an element $\omega \in \partial\Delta$, and let $G_\omega = \{g \in G : g\omega = \omega\}$. Then $\partial\Delta \cong G/G_\omega$, and $\mu$ corresponds to a measure in the unique quasi-invariant measure class of $G/G_\omega$. The group $G_\omega$ is amenable by [FTN, Theorem 8.3]. It follows from [Z2, Corollary 4.3.7] that the action of $\Gamma$ on $G/G_\omega$ is amenable. □

## 4. APPENDIX: PROOF OF A TECHNICAL LEMMA

This section contains a proof of the technical result, Lemma 3.12. During the course of the proof it will be necessary to concatenate paths. A difficulty arises because two proper paths cannot necessarily be concatenated to produce a proper path. The product path may backtrack at the initial edge of the second path. This problem is overcome by introducing a detour around a proper cycle attached at the initial vertex of the second path. The following auxiliary Lemma will be used to do this.

**Lemma 4.1.** (ATTACHING A LOOP TO AN EDGE.) *Let $X$ be a finite connected graph whose vertices all have degree at least three. Let $e$ be an edge of $X$. Then there is a proper cycle $L$ based at the terminal vertex $t(e)$, not passing through $e$ and having length $\ell(L) \leq \delta + \lambda$, where $\delta$ is the diameter of $X$ and $\lambda$ is the maximum length of a circuit in $X$.*

PROOF. The edge $e$ is contained in a maximal tree $T$ in $X$. Every vertex of $X$ is a vertex of $T$. Let $P$ be a maximal proper (geodesic) path in $T$ with initial vertex $o(e)$ and initial edge $e$. Let $v$ be the terminal vertex of $P$ and $f$ the terminal edge. Then $v$ is an endpoint of $T$. The vertex $v$ has degree at least three. It follows that there are two edges in $X$ other than $\overline{f}$ with initial vertex $v$. These two edges may both have terminal vertex $v$ (in fact one may be the opposite of the other) or else one or both of them may end at a vertex other than $v$. However in all cases we may use one or both of these edges together with edges in $T$ to construct a circuit $L_0$ based at $v$ and not passing through $e$. The required proper cycle $L$ can the be constructed from $P \cup L_0$. $\square$

PROOF OF LEMMA 3.12. Let $\delta$ be the diameter of $X$ and $\lambda$ the maximum length of a circuit in $X$. We show that property $\mathfrak{L}(K)$ is satisfied with $K = 10+10\delta+6\lambda$.

Let $P_1, P_2$ be proper paths in $X$ having the same length $n$ and the same initial vertex $v_0$. Let $p_1, p_2$ be the terminal vertices of $P_1, P_2$ respectively. We must construct proper cycles $C_1, C_2$ based at $v_0$ satisfying the conditions of Definition 3.11.

Choose once and for all a path $[p_1, p_2]$ of shortest length between $p_1$ and $p_2$. There are two separate cases to consider.

CASE 1. The length of $[p_1, p_2]$ is even. Denote this length by $2s$ where $s \geq 0$ and let $p_0$ be the midpoint of $[p_1, p_2]$. If $s = 0$ then a simpler argument will apply, and produce a smaller bound for the lengths of $C_1, C_2$, so we assume that $s > 0$.

Choose a path $R$ of minimal length from $p_0$ to $v_0$. The cycles $C_1, C_2$ will be constructed from portions the paths $P_1, P_2, R, [p_1, p_2]$, with loops attached to avoid backtracking. Refer to Figure 11.

Choose an edge $e_1$ with $o(e_1) = p_1$ such that $e_1$ is not the initial edge of $[p_1, p_2]$ and $\overline{e}_1$ is not the final edge of $P_1$. Choose an edge $e_2$ with $o(e_2) = p_2$ such that $e_2$ is not the initial edge of $[p_2, p_1]$ and $\overline{e}_2$ is not the final edge of $P_2$. For $i = 1, 2$, attach a proper cycle $L_i$ at $t(e_i)$, as in Lemma 4.1.

Assume that the initial edge of $R$ does not meet either of the edges of $[p_1, p_2]$ which contain $p_0$. Let $C_1$ be the proper cycle based at $v_0$ obtained by passing through the following sequence of paths and edges in the order indicated.

$$P_1 \to e_1 \to L_1 \to \overline{e}_1 \to [p_1, p_2] \to e_2 \to L_2 \to \overline{e}_2 \to [p_2, p_0] \to R$$

Similarly, let $C_2$ be obtained from

$$P_2 \to e_2 \to L_2 \to \overline{e}_2 \to [p_2, p_1] \to e_1 \to L_1 \to \overline{e}_1 \to [p_1, p_0] \to R$$

The proper cycles $C_1, C_2$ have initial segments $P_1, P_2$ respectively and have the same length $n+k$, where $k = 4+\ell(L_1)+\ell(L_2)+3s+\ell(R) \leq 4+2(\delta+\lambda)+\frac{3}{2}\delta+\delta < 4+5\delta+2\lambda$.

Now assume that the initial edge of $R$ meets an edge of $[p_1, p_2]$ which contains $p_0$. This is precisely the situation illustrated in Figure 11. The cycles $C_1, C_2$ described above will no longer both be proper, since there will be a backtrack for one of them at the first edge of $R$. In order to avoid this, choose an edge $e_0$ with $o(e_0) = p_0$ such that $e_0$ does not meet either of the edges of $[p_1, p_2]$ containing $p_0$. Attach a proper cycle $L_0$ at $t(e_0)$, as in Lemma 4.1. Modify the cycles $C_1, C_2$ above so that the final part of each becomes

$$\ldots, p_0] \to e_0 \to L_0 \to \overline{e}_0 \to R$$

The proper cycles $C_1, C_2$ now have the same length $n+k$, where $k < 6+6\delta+3\lambda$.

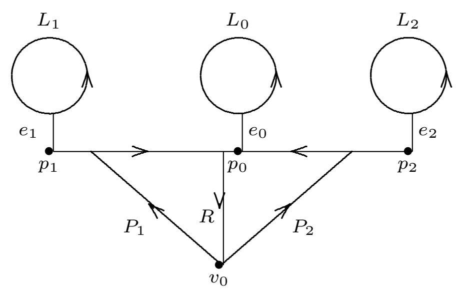


FIGURE 11. Constructing proper cycles of the same length.

CASE 2. The length of $[p_1, p_2]$ is odd. Denote this length by $2s+1$ where $s \geq 0$ and let $p_0$ be the vertex of $[p_1, p_2]$ with $d(p_0, p_1) = s, d(p_0, p_2) = s+1$. The argument that follows will be slightly different, but simpler, if $s = 0$. We therefore again assume that $s > 0$.

Exactly the same argument as in Case 1 shows that there are proper cycles $C_1, C_2$ based at $v_0$ and with initial segments $P_1, P_2$ respectively. The only difference is that $\ell(C_1) = n+k+1$, $\ell(C_2) = n+k$, where $k < 6+6\delta+3\lambda$.

The cycles will be modified to have the same length by adding to the end of each an appropriate proper cycle based at $v_0$. The (possibly improper) cycle

$$P_1 \to [p_1, p_2] \to \overline{P}_2$$

has odd length. Deleting appropriate parts of this cycle shows that $X$ contains a circuit $C_0$ of odd length $2t+1$. (In other words, the graph $X$ is not bipartite.)

Choose a path $S_1$ of minimal length from $v_0$ to $C_0$. Let $v_1$ be the terminal vertex of $S_1$. The circuit $C_0$ is the union of two proper paths $C_0^+$, $C_0^-$ with lengths $t+1$, $t$ respectively and initial vertex $v_1$. Let $v_2$ be the terminal vertex of the paths $C_0^+$, $C_0^-$. Choose a path $S_2$ of minimal length from $v_2$ to $v_1$. Add to the end of each of the cycles $C_1, C_2$ a cycle based at $v_0$, as indicated below

$$C_1 \to S_1 \to C_0^- \to S_2$$

$$C_2 \to S_1 \to C_0^+ \to S_2$$

The resulting cycles have the same length, namely $n+k+1+t+\ell(S_1)+\ell(S_2) = n+k'$, where $k' \le k+\lambda+2\delta < 6+8\delta+4\lambda$. Either or both of these cycles may have backtracking at $v_0$ or at $v_2$ (but not at $v_1$). If this happens add an edge (and its reverse) and adjoin a loop to both cycles at the relevant vertex as in Lemma 4.1. The resulting cycles are proper (i.e. have no backtracking) and have the same length $n+k''$, where $k'' \le 10+10\delta+6\lambda$. $\square$

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