STRONG SINGULARITY FOR SUBALGEBRAS OF FINITE FACTORS

Guyan Robertson[1]
Department of Mathematics
University of Newcastle
Callaghan, NSW 2308
AUSTRALIA
e-mail: guyan@frey.newcastle.edu.au

Allan M. Sinclair
Department of Mathematics
University of Edinburgh
Edinburgh EH9 3JZ
SCOTLAND
allan@maths.ed.ac.uk

Roger R. Smith[2]
Department of Mathematics
Texas A&M University
College Station, TX 77843
USA
rsmith@math.tamu.edu

Abstract

In this paper we develop the theory of strongly singular subalgebras of von Neumann algebras, begun in earlier work. We mainly examine the situation of type II_1 factors arising from countable discrete groups. We give simple criteria for strong singularity, and use them to construct strongly singular subalgebras. We particularly focus on groups which act on geometric objects, where the underlying geometry leads to strong singularity.

2000 Mathematics Subject Classification Numbers: 46L10, 22D25

---

[1] Partially supported by the Australian Research Council

[2] Partially supported by the National Science Foundation.

1 Introduction

Let $\mathcal{A}$ be a maximal abelian self–adjoint subalgebra (masa) in a type II_1 factor $\mathcal{M}$ with trace tr. In [6], Dixmier identified various classes of masas based on the structure of the normalizer

$$N(\mathcal{A}) = \{u \in \mathcal{M} : u\mathcal{A}u^* = \mathcal{A}, u \text{ unitary}\}. \tag{1.1}$$

In particular, $\mathcal{A}$ is said to be singular if $N(\mathcal{A}) \subseteq \mathcal{A}$, so that the only normalizing unitaries already belong to $\mathcal{A}$. He also provided some examples of singular masas inside factors arising from discrete groups. However, it is a difficult problem to decide whether a given masa is singular, and this prompted the second and third authors to introduce the concept of strong singularity in [20]. The trace induces a norm $\|\cdot\|_2$ on $\mathcal{M}$ by $\|x\|_2 = (tr(x^*x))^{1/2}$, and a norm $\|\cdot\|_{\infty,2}$ may then be defined for a map $\phi : \mathcal{M} \to \mathcal{M}$ by

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2 : \|x\| \leq 1\}. \tag{1.2}$$

Letting $\mathbb{E}_{\mathcal{N}}$ denote the unique trace preserving conditional expectation onto any von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$, strong singularity of a masa $\mathcal{A}$ (or of a general von Neumann subalgebra) is then defined by requiring the inequality

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \geq \|(I - \mathbb{E}_{\mathcal{A}})(u)\|_2 \tag{1.3}$$

to hold for all unitaries $u \in \mathcal{M}$. Singularity of any such masa is an immediate consequence of (1.3). The objective in introducing this concept was to have an easily verifiable criterion for singularity. For example, the masa in $VN(\mathbb{F}_2)$ generated by one of the generators of $\mathbb{F}_2$ satisfies (1.3), showing singularity (which was, of course, known to Dixmier, [6]). However, in [20], the problem of exhibiting large classes of strongly singular masas was not addressed beyond examples arising from hyperbolic groups. The purpose of this paper is to examine several general contexts in which strongly singular masas and subalgebras appear naturally and in profusion. Since type II_1 factors are closely connected to discrete groups, much of the work (but not exclusively) will be in this area. We now give a brief outline of the paper.

The second section gives a criterion for determining when a von Neumann subalgebra is strongly singular (Lemma 2.1). This is a minor modification of a result from [20], but the new version is slightly more flexible and thus more widely applicable. We use it to generate classes of strongly singular masas based on semi–direct products of groups, a construction which translates to crossed products of type II_1 factors by groups acting as automorphisms. We also demonstrate that the hyperfinite type II_1 factor can possess both strongly singular masas and subfactors (see Corollary 2.3, Example 2.5 and also Section 6). In the third section we investigate the crossed product of a von Neumann algebra $\mathcal{N}$ by an action of $\mathbb{Z}$, with particular reference to the abelian subalgebra $\mathcal{B}$ generated by the unitary which implements the action. When the action is ergodic and the automorphisms are trace preserving, it is well known that the resulting crossed product is a type II_1 factor, [11]. Under the additional hypothesis that the action is either strongly or weakly mixing (Lemmas 3.1 and 3.2), we obtain strong singularity of $\mathcal{B}$, showing that such masas arise naturally from classical ergodic theory.

In the remaining three sections we examine the situation of a discrete I.C.C. group Γ with an abelian subgroup Γ_0, and we consider $VN(\Gamma_0)$ as an abelian subalgebra of $VN(\Gamma)$. Lemma 4.1 gives a group–theoretic criterion for strong singularity, which we then verify in various contexts. The unifying theme is to let Γ act on a space X of nonpositive curvature, and to exploit the geometry to show that the hypothesis of this lemma is satisfied. The classical example of $VN(\mathbb{F}_2)$, alluded to earlier, fits into this pattern, by considering the action of $\mathbb{F}_2$ on its Cayley graph X: a homogeneous tree of degree four. Now the group $\mathbb{F}_2$ is a torsion free lattice in the rank one p-adic semisimple group $\mathrm{SL}_2(\mathbb{Q}_p)$. Conversely any torsion free lattice in $\mathrm{SL}_2(\mathbb{Q}_p)$ is a free group [19]. Any semisimple Lie group acts in a natural way upon a space of nonpositive curvature. In the case of a real group this space is a (Riemannian) symmetric space, whereas in the p-adic case it is a euclidean building (a tree, if the group has rank one). These cases are dealt with in Sections 4 and 5 respectively. In Section 4, Γ is the fundamental group of a compact locally symmetric space of nonpositive curvature, while in Section 5, Γ acts by isometries on locally finite euclidean buildings. The motivating examples here come from [17], which was our starting point in constructing strongly singular masas. The building examples are of particular interest because many of the groups constructed in [5] do not embed naturally into linear groups. In the final section of the paper, we give another class of examples based on, but extending, those of Dixmier, [6]. As before, the geometry of the spaces on which our groups act is the crucial ingredient.

In [16], Popa was able to construct singular masas in any type II_1 factor. At this time, we do not know if this is also true for strongly singular masas, or indeed whether all singular masas must also be strongly singular.

The third author gratefully acknowledges the hospitality and support of the Department of Mathematics, University of Newcastle, where part of this work was completed.

2 Group von Neumann algebras

Let $\mathcal{A} \subseteq \mathcal{M}$ be a von Neumann subalgebra of a type II_1 factor $\mathcal{M}$. For the case of an abelian subalgebra, *strong singularity* of $\mathcal{A}$ was defined, in [20], by requiring the inequality

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \geq \|(I - \mathbb{E}_{\mathcal{A}})(u)\|_2 \tag{2.1}$$

to hold for all unitaries $u \in \mathcal{M}$. There is no need for commutativity of $\mathcal{A}$, and so the definition extends without change to all von Neumann subalgebras.

For masas $\mathcal{A} \subseteq \mathcal{M}$, the concept of an asymptotic homomorphism (with respect to a unitary $v \in \mathcal{A}$) was introduced in [20]. We say that $\mathbb{E}_{\mathcal{A}}$ is an asymptotic homomorphism, with respect to $v \in \mathcal{A}$, if

$$\lim_{|n|\to\infty} \|\mathbb{E}_{\mathcal{A}}(xv^n y) - \mathbb{E}_{\mathcal{A}}(x)v^n\mathbb{E}_{\mathcal{A}}(y)\|_2 = 0 \tag{2.2}$$

for all $x, y \in \mathcal{M}$. Strong singularity is a consequence of having an expectation which is an asymptotic homomorphism, [20], and (2.2) gives a criterion for strong singularity which can be easily checked in specific cases. Our first result is a minor variant of this. We weaken the requirement of (2.2) slightly, but obtain the same conclusion. The resulting criterion is then more flexible. The proof is essentially that given in [20], but we include it for completeness. It will become apparent later why we state condition (2.3) in a stronger form than is necessary for the proof of the result.

Lemma 2.1. *Let $\mathcal{A} \subseteq \mathcal{M}$ be a von Neumann subalgebra of a type II_1 factor $\mathcal{M}$ and suppose that, given $\varepsilon > 0$ and $\{x_1, \ldots, x_n; y_1, \ldots, y_m\} \in \mathcal{M}$, there exists a unitary $v \in \mathcal{A}$, such that*

$$\|\mathbb{E}_{\mathcal{A}}(x_i v y_j) - \mathbb{E}_{\mathcal{A}}(x_i)v\mathbb{E}_{\mathcal{A}}(y_j)\|_2 < \varepsilon. \tag{2.3}$$

Then $\mathcal{A}$ is strongly singular in $\mathcal{M}$.

Proof. We will make use of the simple relationship

$$\|h\|^2 = \|Ph\|^2 + \|(I-P)h\|^2 \tag{2.4}$$

for any element h in a Hilbert space H and for any projection $P \in B(H)$. Fix a unitary $u \in \mathcal{M}$ and $\varepsilon > 0$. Apply the hypothesis to the set $\{u^*; u\}$ to obtain a unitary $v \in \mathcal{A}$ such that

$$\|\mathbb{E}_{\mathcal{A}}(u^* vu) - \mathbb{E}_{\mathcal{A}}(u^*)v\mathbb{E}_{\mathcal{A}}(u)\|_2 < \varepsilon. \tag{2.5}$$

Using this inequality, we see that

$$
\begin{aligned}
\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}^2 &\geq \|v - \mathbb{E}_{u\mathcal{A}u^*}(v)\|_2^2 \\
&= \|v - u\mathbb{E}_{\mathcal{A}}(u^*vu)u^*\|_2^2 \\
&= \|u^*vu - \mathbb{E}_{\mathcal{A}}(u^*vu)\|_2^2 \\
&= 1 - \|\mathbb{E}_{\mathcal{A}}(u^*vu)\|_2^2 \\
&\geq 1 - (\|\mathbb{E}_{\mathcal{A}}(u^*)v\mathbb{E}_{\mathcal{A}}(u)\|_2 + \varepsilon)^2 \\
&\geq 1 - (\|\mathbb{E}_{\mathcal{A}}(u)\|_2 + \varepsilon)^2 \\
&= \|(I - \mathbb{E}_{\mathcal{A}})(u)\|_2^2 - \varepsilon^2 - 2\varepsilon\|\mathbb{E}_{\mathcal{A}}(u)\|_2. \qquad (2.6)
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

The two basic ways to obtain type II_1 factors are to consider the von Neumann algebras arising from discrete groups, and to form crossed products by group actions. Such actions on group von Neumann algebras can take place at the level of groups, as we now describe. Let K and H be discrete groups with an action $\alpha\colon\ K \to \mathrm{Aut}(H)$. Then the semi-direct product $\Gamma = H \rtimes_\alpha K$ is the set of formal products $\{hk\colon\ h \in H,\ k \in K\}$ with multiplication

$$(hk)(h'k') = (h\alpha_k(h'))(kk'). \qquad (2.7)$$

The action α lifts from H to $VN(H)$, and $VN(\Gamma) = VN(H) \rtimes_\alpha K$, [21]. We assume this notation in the next result. Identity elements of groups are denoted e_H or e_K, and the abbreviation I.C.C. means *infinite conjugacy class.*

Theorem 2.2. *Let H and K be infinite discrete groups, let $\alpha\colon\ K \to Aut(H)$ be an action, and let $\Gamma = H \rtimes_\alpha K$. Consider the following statements.*

(i) *For each $k \in K\backslash\{e_K\}$, the only fixed point of α_k is e_H;*

(ii) Γ *is* I.C.C. *and, given finite subsets $F_1, F_2 \subseteq H\backslash\{e_H\}$, there exists $k \in K$ such that $\alpha_k(F_1) \cap F_2 = \emptyset$;*

(iii) Γ *is* I.C.C. *and, given a finite subset $F \subseteq H\backslash\{e_H\}$, there exists $k \in K$ such that $\alpha_k(F) \cap F = \emptyset$;*

(iv) *$VN(\Gamma)$ is a type* II_1 *factor and $VN(K)$ is a strongly singular von Neumann subalgebra.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

Proof. The equivalence of (ii) and (iii) is clear. In one direction, given F_1 and F_2, take F to be $F_1 \cup F_2$; in the other, given F, take $F_1 = F_2 = F$. We now show that (i) implies (iii). Suppose that (i) holds, but that there exists a finite set $F \subseteq H\backslash\{e_H\}$ so that $\alpha_k(F) \cap F \neq \emptyset$ for all $k \in K$. Then let

$$S_f = \{k \in K\colon\ \alpha_k(f) \in F\}, \qquad f \in F, \tag{2.8}$$

and note that $K = \bigcup\limits_{f\in F} S_f$, so that at least one S_f is infinite. For this f, there must exist distinct $k_1, k_2 \in K$ such that $\alpha_{k_1}(f) = \alpha_{k_2}(f)$, since F is finite. But then $\alpha_{k_1k_2^{-1}}$ has a fixed point, contradicting (i). It remains to show that Γ is I.C.C.

Consider first $k_1 \in K\backslash\{e_K\}$. Then

$$h^{-1}k_1h = h^{-1}\alpha_{k_1}(h)k_1, \qquad h \in H, \tag{2.9}$$

and so there are infinitely many distinct conjugates of k_1 unless α_{k_1} has fixed points, contrary to hypothesis. If $h_1 \neq e_H$, consider conjugates of $h_1k_1 \in \Gamma$ by elements $k \in K$. Then

$$kh_1k_1k^{-1} = \alpha_k(h_1)kk_1k^{-1}, \qquad k \in K. \tag{2.10}$$

The set $\{\alpha_k(h_1)\colon\ k \in K\}$ consists of an infinite number of distinct elements, otherwise h_1 is a fixed point of some α_k, and so Γ is I.C.C.

We now assume (iii). Then the I.C.C. condition on Γ ensures that $VN(\Gamma)$ is a type II_1 factor. We will verify that (2.3) is satisfied, and then obtain the result from Lemma 2.1. A simple approximation argument shows that we may take the x_i's and y_j's to be group elements in (2.3). Moreover, by expanding the set of such elements, we may assume that the inequality to be verified takes the form

$$\|\mathbb{E}_{\mathcal{A}}(x_ivx_j) - \mathbb{E}_{\mathcal{A}}(x_i)v\mathbb{E}_{\mathcal{A}}(x_j)\|_2 < \varepsilon \tag{2.11}$$

for a given set $\{x_1, \ldots, x_r\} \subseteq \Gamma$. The module map property of $\mathbb{E}_{\mathcal{A}}$ shows that (2.11) is true for any $x_i \in K$, so we may assume that each x_i has the form h_ik_i with $k_i \in K$ and $h_i \in H\backslash\{e_H\}$. Then $\mathbb{E}_{\mathcal{A}}(x_i) = 0$, $1 \leq i \leq r$, so (2.11) will be satisfied by a group element $k \in K$, chosen so that

$$h_ik_ikh_jk_j \notin K, \qquad 1 \leq i, j \leq r. \tag{2.12}$$

This condition is equivalent to

$$h_i\alpha_{k_ik}(h_j)k_ikk_j \notin K, \qquad 1 \leq i, j \leq r, \tag{2.13}$$

which will be true if

$$h_i \alpha_{k_i k}(h_j) \neq e_H, \qquad 1 \leq i, j \leq r. \tag{2.14}$$

This last condition may be reformulated as

$$\alpha_k(h_j) \neq \alpha_{k_i^{-1}}(h_i^{-1}), \qquad 1 \leq i, j \leq r. \tag{2.15}$$

Let $F = \{h_1, \ldots, h_r, \alpha_{k_1}^{-1}(h_1^{-1}), \ldots, \alpha_{k_r^{-1}}(h_r^{-1})\} \subseteq H \backslash \{e_H\}$. By hypothesis, there exists $k \in K$ such that $\alpha_k(F) \cap F = \emptyset$. In particular (2.15) is satisfied for this choice of k, completing the proof. □

It is now easy to produce examples of strongly singular subalgebras by constructing groups which satisfy Theorem 2.2 (i).

Corollary 2.3. *The hyperfinite type* II_1 *factor contains a strongly singular hyperfinite subfactor.*

Proof. Let K be a countable amenable I.C.C. group with no elements of finite order except the identity. An example of such a group is given below. Then let H be the countable abelian group, under pointwise multiplication, of functions $f\colon\ K \to \{\pm 1\}$ which are identically 1 off a finite set. Then define an action $\alpha\colon\ K \to \mathrm{Aut}(H)$ by

$$\alpha_k(f)(x) = f(k^{-1}x), \qquad x \in K, \tag{2.16}$$

for each $k \in K$ and $f \in H$. Consider a fixed $k \in K \backslash \{e_K\}$, and suppose that $f \in H \backslash \{e_H\}$ is a fixed point of α_k, and is thus a fixed point for all powers of α_k. There exists $x_0 \in K$ such that $f(x_0) = -1$, and it then follows that

$$f(k^{-n}x_0) = -1, \qquad n \in \mathbb{Z}. \tag{2.17}$$

The definition of H shows that $\{k^{-n}x_0\colon\ n \in \mathbb{Z}\}$ is a finite set, contradicting the assumption that k has infinite order.

Let $\Gamma = H \rtimes_\alpha K$. Then $VN(\Gamma) = VN(H) \rtimes_\alpha K$ and so is hyperfinite. The hypothesis of Theorem 2.2 (i) is satisfied, and so $VN(\Gamma)$ is the hyperfinite type II_1 factor, while $VN(K)$ is a strongly singular hyperfinite type II_1 subfactor, by (i) $\Rightarrow$ (iv) of this theorem. □

Example 2.4. Take the group $\mathbb{Z}$, and let H be the group, under pointwise addition, of functions $f\colon\ \mathbb{Z} \to \mathbb{Z}$ which are identically 0 off a finite set. Define an action $\alpha\colon\ \mathbb{Z} \to \mathrm{Aut}(H)$ by

$$\alpha_k(f)(n) = f(n+k), \qquad n \in \mathbb{Z}, \tag{2.18}$$

for $k \in \mathbb{Z}$ and $f \in H$. For $k \neq 0$, α_k has no fixed points other than the identity of H, and so (i) $\Rightarrow$ (ii) of Theorem 2.2 shows that $K = H \rtimes_\alpha \mathbb{Z}$ is I.C.C. Since H and $\mathbb{Z}$ are abelian it follows that the semi-direct product is amenable. It is easy to verify that elements of finite order ($\neq e$) in semi-direct products can exist only when they exist in at least one of the constituent subgroups, so the group K defined above is an example of an amenable I.C.C. group with no elements of finite order except the identity. Another possibility is to let the multiplicative group $\mathbb{Q}^+$ act on the additive group $\mathbb{Q}$ by $\alpha_q(p) = qp$. The resulting semi–direct product has exactly the same properties. $\square$

Example 2.5. With the notation of the previous example, Theorem 2.2 shows that $VN(\mathbb{Z})$ is a strongly singular masa inside the hyperfinite type II_1 factor $VN(K)$. $\square$

Example 2.6. Let $\mathbb{F}_\infty$ be the free group on countably many generators g_i, indexed by $i \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, the map $\alpha_n\colon\ g_i \mapsto g_{i+n}$ on generators lifts to an automorphism of $\mathbb{F}_\infty$ with no fixed points except the identity when $n \neq 0$. Theorem 2.2 (i) then gives a strongly singular masa $VN(\mathbb{Z})$ inside the non-hyperfinite factor $VN(\mathbb{F}_\infty) \rtimes_\alpha \mathbb{Z}$. $\square$

3 Ergodic actions

In this section we consider a von Neumann algebra $\mathcal{N}$ (denoted by $\mathcal{A}$ when abelian) with a faithful normal bounded trace tr, together with a trace preserving automorphism θ. We assume that $\mathcal{N}$ is represented on $L^2(\mathcal{N}, tr)$, and we define an action of $\mathbb{Z}$ on $\mathcal{N}$ by $\alpha_n = \theta^n$. The resulting crossed product $\mathcal{N} \rtimes_\alpha \mathbb{Z}$ is represented on $L^2(\mathcal{N}, tr) \otimes \ell^2(\mathbb{Z})$. There is a representation π of $\mathcal{N}$ on this Hilbert space and a unitary operator u so that

$$\pi(\theta^n(x)) = u^n x u^{-n}, \qquad x \in \mathcal{N}, \quad n \in \mathbb{Z}, \tag{3.1}$$

and elements of $\mathcal{N} \rtimes_\alpha \mathbb{Z}$ have unique representations as $\sum\limits_{n\in\mathbb{Z}} \pi(x_n)u^n$, where such sums converge ultraweakly. Since θ is trace preserving, there is a faithful normal trace on $\mathcal{N} \rtimes_\alpha \mathbb{Z}$ given by

$$tr\left(\sum_{n\in\mathbb{Z}} \pi(x_n)u^n\right) = tr(x_0). \tag{3.2}$$

This is standard theory which may be found in [11]. When $\mathcal{N}$ is an abelian von Neumann algebra $\mathcal{A}$, its image in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ has a normalizer which generates the crossed product. Thus $\mathcal{A}$ is Cartan whenever the action is such that $\mathcal{A}$ is maximal abelian in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$. The unitary u which implements θ always generates a canonical abelian von Neumann subalgbebra $\mathcal{B}$ of $\mathcal{A} \rtimes_\alpha \mathbb{Z}$, which we now investigate.

Recall that an action α of $\mathbb{Z}$ on $\mathcal{N}$ is *ergodic* if each α_n $(n \neq 0)$ has only multiples of the identity as its fixed points. If the automorphism group $\{\theta^n\}_{n\in\mathbb{Z}}$ is both trace preserving and ergodic then it is well known (see [11, p. 546]) that $\mathcal{N} \rtimes_\alpha \mathbb{Z}$ is a type II_1 factor with $\mathcal{B}$ as a masa. The automorphism θ is *strongly mixing* (called *mixing* in classical abelian ergodic theory) if

$$\lim_{n\to\infty} tr(x\theta^n(y)) = tr(x)tr(y), \qquad x, y \in \mathcal{N}. \tag{3.3}$$

When θ is trace preserving, we may use the limit as $|n| \to \infty$ in (3.3), and we also note that ergodicity is an easy consequence of (3.3). We say that θ is *weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |tr(x\theta^k(y)) - tr(x)tr(y)| = 0 \tag{3.4}$$

for all $x, y \in \mathcal{A}$. Ergodicity also follows from this weaker definition.

Lemma 3.1. *Let $\mathcal{N}$ be a von Neumann algebra with a faithful normal trace, let θ be a trace preserving ergodic automorphism of $\mathcal{N}$ defining an action $\alpha_n = \theta^n$ of $\mathbb{Z}$ on $\mathcal{N}$,* *let u be the unitary in $\mathcal{M} = \mathcal{N} \rtimes_\alpha \mathbb{Z}$ which implements θ, and let $\mathcal{B}$ be the abelian von Neumann algebra generated by u. Then $\mathbb{E}_{\mathcal{B}}$ is an asymptotic homomorphism with respect to u if and only if θ is strongly mixing. In particular, $\mathcal{B}$ is a strongly singular masa in $\mathcal{M}$ when θ is strongly mixing.*

Proof. The set $\{u^n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathcal{B}, tr)$, and so $\mathbb{E}_{\mathcal{B}}$ may be expressed by

$$\mathbb{E}_{\mathcal{B}}(x) = \sum_{n\in\mathbb{Z}} tr(xu^{-n})u^n, \qquad x \in \mathcal{M}. \tag{3.5}$$

In particular

$$\mathbb{E}_{\mathcal{B}}(x) = tr(x)1, \qquad x \in \mathcal{N}. \tag{3.6}$$

Elements of the form xu^n, $x \in \mathcal{N}$, $n \in \mathbb{Z}$, generate $\mathcal{M}$, so it is sufficient to check the asymptotic homomorphism condition for such operators. If $x, y \in \mathcal{N}$ and $k, r \in \mathbb{Z}$, then

$$\begin{aligned}\mathbb{E}_{\mathcal{B}}(xu^k u^n y u^r) &= \mathbb{E}_{\mathcal{B}}(x\theta^{n+k}(y)u^{n+k+r}) \\ &= \mathbb{E}_{\mathcal{B}}(x\theta^{n+k}(y))u^{n+k+r} \\ &= tr(x\theta^{n+k}(y))u^{n+k+r},\end{aligned} \tag{3.7}$$

from (3.6). On the other hand,

$$\mathbb{E}_{\mathcal{B}}(xu^k)u^n\mathbb{E}_{\mathcal{B}}(yu^r) = tr(x)tr(y)u^{n+k+r}, \tag{3.8}$$

using (3.5), and so

$$\begin{aligned}\|\mathbb{E}_{\mathcal{B}}(xu^k u^n y u^r) - \mathbb{E}_{\mathcal{B}}(xu^k)u^n\mathbb{E}_{\mathcal{B}}(yu^r)\|_2 &= |tr(x\theta^n(\theta^k(y))) - tr(x)tr(y)| \\ &= |tr(x\theta^n(\theta^k(y))) - tr(x)tr(\theta^k(y))|,\end{aligned} \tag{3.9}$$

where the last equality uses trace preservation by θ. Letting $|n| \to \infty$ in (3.9) immediately gives the conclusion that $\mathbb{E}_{\mathcal{B}}$ is an asymptotic homomorphism for u if and only if θ is strongly mixing. The last statement of the lemma then follows from [20]. □

We now consider a weakly mixing automorphism θ, and we maintain the notation of the previous lemma.

Lemma 3.2. *Let θ be a trace preserving weakly mixing automorphism of $\mathcal{N}$. Then $\mathcal{B}$ is a strongly singular masa in $\mathcal{M} = \mathcal{N} \rtimes_\alpha \mathbb{Z}$.*

Proof. We will verify (2.3) in Lemma 2.1, from which the result will follow. It suffices to consider a finite set of generators so, to obtain a contradiction, we may assume that there exist $\varepsilon > 0$ and elements $x_j u^k \in \mathcal{M}$, $0 \leq j$, $|k| \leq J$, so that (2.3) fails for all unitaries $v \in \mathcal{B}$. In particular

$$\max\{\|\mathbb{E}_{\mathcal{B}}(x_j u^{n+k} x_r u^s) - \mathbb{E}_{\mathcal{B}}(x_j u^k) u^n \mathbb{E}_{\mathcal{B}}(x_r u^s)\|_2 \colon\ 0 \leq j, r, |k|, |s| \leq J\} \geq \varepsilon \quad (3.10)$$

for all $n \in \mathbb{Z}$. Using (3.7), this condition becomes

$$\max\{|tr(x_j \theta^{n+k}(x_r)) - tr(x_j) tr(\theta^k(x_r))| \colon\ 0 \leq j, r, |k|, \leq J\} \geq \varepsilon \quad (3.11)$$

for all $n \in \mathbb{Z}$. Let $y_{k,r}$ denote the element $\theta^k(x_r)$. If we sum in (3.11) and average from 0 to $n-1$, then we obtain

$$\frac{1}{n} \sum_{i=0}^{n-1} \sum_{0 \leq j, r, |k| \leq J} |tr(x_j \theta^i(y_{k,r}) - tr(x_j) tr(y_{k,r})| \geq \varepsilon \quad (3.12)$$

for all $n \geq 1$, and this violates the defining inequality (3.4) of weakly mixing. This completes the proof . □

Remark 3.3. Classical ergodic theory (see [15]) provides many examples of strongly mixing transformations of measure spaces, as well as examples which are weakly but not strongly mixing. The two previous lemmas then give examples of strongly singular masas $\mathcal{B}$, some of which do not arise from asymptotic homomorphisms for u. □

4 Groups acting on symmetric spaces

Let Γ be an I.C.C. group with an abelian subgroup Γ_0. Then $VN(\Gamma_0)$ is an abelian subalgebra of the type II_1 factor $VN(\Gamma)$, and in this section we investigate when it is a strongly singular masa. For the case of group von Neumann algebras, Lemma 2.1 takes the following form.

Lemma 4.1. *Let* Γ *be a discrete* I.C.C. *group with an abelian subgroup* Γ_0. *The following condition implies that* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$*:*
If $x_1, \ldots, x_m \in \Gamma$ *and*

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,, \tag{4.1}$$

then $x_i \in \Gamma_0$ *for some* i.

Proof. The condition in question is equivalent to the following:
If $x_1, \ldots, x_m, y_1, \ldots, y_n \in \Gamma \backslash \Gamma_0$, then there exists $\gamma_0 \in \Gamma_0$ such that

$$x_i \gamma_0 y_j \notin \Gamma_0, \quad 1 \le i \le m, \ 1 \le j \le n. \tag{4.2}$$

To see this replace each of the sets $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ by their union and replace x_i by x_i^{-1}, and y_j by y_j^{-1}. Now apply Lemma 2.1, with each operator approximated by a finite linear combination of group elements. □

The aim now is to apply this lemma to construct strongly singular masas of $VN(\Gamma)$, for certain geometrically defined groups Γ, acting on spaces of nonpositive curvature.

In order to establish a connection with geometry, consider the following general setup. Let (X, d) be a metric space and let Γ be a group of isometries of X. If P, Q are subsets of X, and $\delta > 0$, then use the notation $P \underset{\delta}{\subset} Q$ to mean that $d(p, Q) \le \delta$, for all $p \in P$.

Let Γ_0 be an abelian subgroup of Γ and let A be a Γ_0-invariant subset of X. Consider the conditions:

(C1) There exists a compact subset K of A such that $\Gamma_0 K = A$.

(C2) If $A \underset{\delta}{\subset} x_1 A \cup x_2 A \cup \cdots \cup x_m A$, for some $x_1, \ldots, x_m \in \Gamma$, and $\delta > 0$, then $x_j \in \Gamma_0$, for some j.

Proposition 4.2. *If* **(C1)** *and* **(C2)** *hold then* $VN(\Gamma_0)$ *is a strongly singular masa of* $VN(\Gamma)$.

Proof. Suppose that $x_1, \dots, x_m \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,. \tag{4.3}$$

Let $\delta = \max\{d(x_j k, k); 1 \le j \le n,\ k \in K\}$. For $1 \le j \le n$, this implies that $x_j K \underset{\delta}{\subset} K$ and so

$$\Gamma_0 x_j K \underset{\delta}{\subset} \Gamma_0 K = A \,. \tag{4.4}$$

Hence, for each i, j, we have $x_i \Gamma_0 x_j K \underset{\delta}{\subset} x_i A$. It follows from **(C1)** and (4.3) that

$$A = \Gamma_0 K \underset{\delta}{\subset} x_1 A \cup x_2 A \cup \cdots \cup x_m A \,. \tag{4.5}$$

Applying condition **(C2)**, we see that $x_j \in \Gamma_0$ for some j, contrary to hypothesis. The result now follows from Lemma 4.1. □

In the first class of examples, Γ is the fundamental group of a compact locally symmetric space of nonpositive curvature. The classic book [13] is a convenient reference for the background and necessary results. There is a clear introduction to the theory of symmetric spaces in [2, Chapter II.10].

Let X be a symmetric space of noncompact type, by which we mean that X is a quotient G/K of a semisimple Lie group by a maximal compact subgroup K.

Example 4.3. $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$, $n \ge 2$. If $n = 2$ one obtains the hyperbolic plane. □

The *rank* r of X is the dimension of a maximal *flat* in X. That is, the maximal dimension of an isometrically embedded euclidean space in X. If $r = 1$, then the flats are geodesics. If $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$, then $r = n - 1$. Call a flat of maximal dimension r an *r-flat* (or *maximal flat*). A geodesic L in X is called *regular* if it lies in only one r-flat; it is called *singular* if it is not regular.

Let F be an r-flat in X and let $x \in F$. Let S_x denote the union of all the singular geodesics through x. A connected component of $F - S_x$ is called a *Weyl chamber* with origin x.

Example 4.4. If $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, the hyperbolic plane, then $r = 1$. The 1-flats are geodesics. If x is a point on a geodesic L then the two Weyl chambers in L with origin x are two semi-geodesics. □

Example 4.5. If $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$, then $r = 2$ and there are six Weyl chambers in any 2-flat F with a given origin $x \in F$, as illustrated in Figure 1.

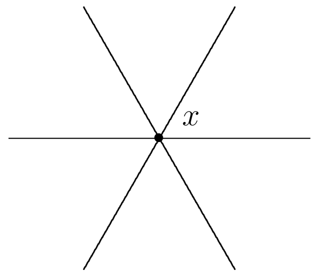

Figure 1: Weyl chambers with origin x

If A, B are subsets of X, define the Hausdorff distance between them by

$$\mathrm{hd}(A, B) = \inf\{\delta \le \infty; A \underset{\delta}{\subset} B \text{ and } B \underset{\delta}{\subset} A\}\,. \tag{4.6}$$

Let $\mathcal{W}$ denote the set of all Weyl chambers in X and define an equivalence relation $\sim$ on $\mathcal{W}$ by

$$W_1 \sim W_2 \Longleftrightarrow \mathrm{hd}(W_1, W_2) < \infty\,. \tag{4.7}$$

The *boundary* Ω of X is defined to be the quotient space $\mathcal{W}/\sim$. It is well known [13, Lemma 4.1] that Ω may be identified with the topological homogeneous space G/P, where P is a Borel subgroup of G. The action of a discrete subgroup Γ of G on the boundary Ω will play an important role in our argument, just as it did in Mostow's proof of rigidity [13].

If F is an r-flat in X, then the restriction of the equivalence relation $\sim$ to F allows one to define the boundary of F, which is a finite set. There is a natural embedding of the boundary of F into the boundary of X and it is convenient to identify each boundary point of F with the corresponding boundary point of X.

Suppose that Γ is a cocompact lattice in a semisimple Lie group G. It is well known [2, Proposition II.6.10], [13, §11] that each element $\gamma \in \Gamma$ is *semi-simple*. Geometrically this means that the displacement function defined on X by $\xi \mapsto d(\xi, \gamma\xi)$ attains its minimum at some point $\xi_0 \in X$. If the group Γ acts freely on X then the minimum value $d(\xi_0, \gamma\xi_0)$ is strictly positive if $\gamma \neq 1$ (γ is *hyperbolic*). This implies [2, Proposition II.6.8] that there is a geodesic line (an *axis* of γ) upon which γ acts by translation.

We now have enough background information to begin the main result of this section. Let G be a semisimple Lie group with no centre and no compact factors. Let

Γ be a torsion free cocompact lattice in G. Then Γ acts freely on the symmetric space $X = G/K$ and the quotient manifold $M = \Gamma\backslash X$ has universal covering space X. Thus M is a compact locally symmetric space of nonpositive curvature, with fundamental group $\pi(M) = \Gamma$. Moreover every compact locally symmetric space M arises in this way.

Let $T^r \subset M$ be a totally geodesic embedding of a flat r-torus in M. By the easy part of the Flat Torus Theorem [12, Theorem 1] the inclusion $i : T^r \to M$ induces an injective homomorphism $i_* : \pi(T^r) \to \pi(M)$. Thus $\Gamma_0 = i_*\pi(T^r) \cong \mathbb{Z}^r$. Conversely, if Γ_0 is any free abelian subgroup of rank r in Γ, then by [12, Theorem 1], [2, Theorem II.7.1], there exists an r-flat F_0 in X such that $\Gamma_0 F_0 = F_0$, Γ_0 acts upon F_0 by translations, and $\Gamma_0\backslash F_0 = T^r$.

Let $\sigma = \sigma(M)$ denote the length of a shortest closed geodesic in M. The aim of the rest of this section is to prove strong singularity of $VN(\Gamma_0)$ in this setting. We accomplish this through the following series of lemmas. Note that the group Γ is I.C.C. by [9, Lemma 3.3.1], so VN(Γ) is a II_1 factor. We shall verify conditions **(C1)**, **(C2)** for the action of Γ on the symmetric space X, taking the subset A of X to be the r-flat $F_0 \subset X$, upon which the abelian subgroup $\Gamma_0 \cong \mathbb{Z}^r$ acts. The result is then a consequence of Proposition 4.2. Verification of **(C1)** is easy.

Lemma 4.6. *The action of Γ_0 on F_0 satisfies* **(C1)**.

Proof. This is immediate since $\Gamma_0\backslash F_0$ is compact. Let K be the closure of a bounded fundamental domain for the action of Γ_0 on F_0. $\square$

Verification of **(C2)** requires some preparation.

Lemma 4.7. *If $F, F_1, \dots, F_m$ are r-flats in X and $F \underset{k}{\subset} F_1 \cup \cdots \cup F_m$, for some $k > 0$, then each boundary point of F is a boundary point of some F_j, $1 \le j \le m$.*

Proof. Let W be a Weyl chamber in F. Write each flat F_j $(1 \le j \le m)$, as a finite union of Weyl chambers W_{jl}. Then $W \subset \bigcup_{j,l} W_{jl}$. It follows from [13, Lemma 15.1] that $\mathrm{hd}(W, W_{rs}) < \infty$ for some r, s. That is, W and W_{rs} represent the same boundary point of X. $\square$

Lemma 4.8. *Assume that T^r has diameter $< \sigma$. If $x \in \Gamma$ and the r-flats F_0 and xF_0 have a common boundary point, then $x \in \Gamma_0$.*

Proof. This depends crucially upon the fact that the embedded torus $T^r = \Gamma_0 \backslash F_0$ has diameter $< \sigma$, where σ is the minimum length of a nontrivial closed geodesic in X. This means that for any two points a, $b \in F_0$, there exists $\gamma \in \Gamma_0$ such that

$$d(a, \gamma b) \leq \mathrm{diam}(T^r) < \sigma\,, \tag{4.8}$$

where d is the canonical G-invariant metric on X.

The hypothesis that the r-flats F_0, xF_0 have a common boundary point implies that there exist Weyl chambers W, W' in F_0, xF_0 respectively such that $\mathrm{hd}(W, W') < \infty$. Let v be the origin of W. The Weyl chamber $x^{-1}W'$ in F_0 is equivalent to a Weyl chamber W_0 in F_0 with origin v. Thus the Weyl chamber xW_0 in xF_0 is equivalent to W' and hence to W.

Choose $\gamma_0 \in \Gamma_0$ such that $\gamma_0 v \in W_0$. This is possible, since Γ_0 acts freely on F_0 by translations and $\Gamma_0 \backslash F_0 = T^r$. Let $[v, \gamma_0 v]$ denote the geodesic segment in F_0 from v to $\gamma_0 v$. Then $L = \bigcup_{n \geq 0} \gamma_0^n [v, \gamma_0 v]$ is a regular geodesic ray contained in W_0 and γ_0 acts on L by translation.

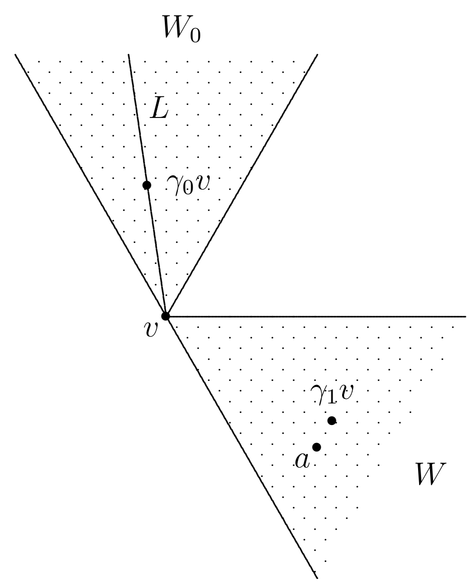

Figure 2: Weyl chambers with origin v in the flat F_0

It follows that xL is a regular geodesic ray lying in the Weyl chamber xW_0, and $x\gamma_0 x^{-1}$ acts by translation on xL. Thus $xL = \bigcup_{n \geq 0} [x\gamma_0^n v, x\gamma_0^{n+1} v]$.

Since $\mathrm{hd}(xW_0, W) < \infty$, we have $xL \underset{t}{\subset} W$ for some $t > 0$.

It follows that xL is actually asymptotic to W, meaning that $d(xL, W) = 0$ [13, Lemma 7.3(iii)]. Now xL approaches W monotonically at infinity [13, Lemma 4.2] and so $d(x\gamma_0^n v, W) \to 0$ as $n \to \infty$.

Choose $k \geq 1$ and $a \in W$ such that $d(x\gamma_0^k v, a) < \sigma - \mathrm{diam}(T^r)$.

Using (4.8), choose $\gamma_1 \in \Gamma_0$ such that $d(a, \gamma_1 v) \leq \mathrm{diam}(T^r)$. Then $d(x\gamma_0^k v, \gamma_1 v) < \sigma$. Equivalently, $d(\gamma_1^{-1} x\gamma_0^k v, v) < \sigma$.

Now this implies that $\gamma_1^{-1} x\gamma_0^k v = v$. For otherwise the geodesic segment from v to $\gamma_1^{-1} x\gamma_0^k v$ in X projects to a nontrivial closed geodesic in M of length $< \sigma$, contradicting the definition of σ. Since Γ acts freely on X, we deduce that $\gamma_1^{-1} x\gamma_0^k = 1$. Therefore $x = \gamma_1\gamma_0^{-k} \in \Gamma_0$. □

Theorem 4.9. *Let T^r be a totally geodesic flat torus in a compact locally symmetric space M of nonpositive curvature and rank r. Let $\Gamma_0 \cong \mathbb{Z}^r$ be the image of the fundamental group $\pi(T^r)$ under the natural monomorphism from $\pi(T^r)$ into $\Gamma = \pi(M)$. Assume that* $\mathrm{diam}(T^r) < \sigma(M)$. *Then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

Proof. Condition **(C1)** is satisfied by Lemma 4.6. It remains to verify condition **(C2)**. Suppose therefore that $x_1, \dots, x_m \in \Gamma$ and $\delta > 0$ satisfy $F_0 \underset{\delta}{\subset} x_1F_0 \cup x_2F_0 \cup \dots \cup x_mF_0$. Choose a boundary point ω of F_0. By Lemma 4.7, ω is also a boundary point of x_jF for some j. It follows from Lemma 4.8 that $x_j \in \Gamma_0$. Therefore condition **(C2)** is satisfied.

Finally, $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$, by Proposition 4.2. □

If Γ is a torsion free cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$ then the result of Theorem 4.9 becomes particularly simple.

Corollary 4.10. *Let Γ be the fundamental group of a compact Riemann surface M of genus $g \geq 2$. Let $\gamma_0 \in \Gamma$ be the class of a closed geodesic of minimal length in M, and let $\Gamma_0 \cong \mathbb{Z}$ be the subgroup of Γ generated by γ_0. Then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

Proof. If C is a closed geodesic of minimal length σ in the class of γ_0, then $\mathrm{diam}(C) = \frac{1}{2}\sigma < \sigma$. The result follows directly from Theorem 4.9. □

Remark 4.11. The usual presentation of the fundamental group of the compact Riemann surface M is as the one-relator group

$$\Gamma = \left\langle a_1, \dots, a_g, b_1, \dots, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

With this presentation of Γ, the generator γ_0 of Γ_0 can be any one of the generators $a_i^{\pm 1}$, $b_j^{\pm 1}$. To see this recall that, by a theorem of Poincaré [10, VB], there exist hyperbolic isometries a_i, $b_j \in \mathrm{PSL}_2(\mathbb{R})$, $(1 \leq i, j \leq g)$, which generate Γ inside $\mathrm{PSL}_2(\mathbb{R})$. Moreover one can ensure that a fundamental domain for the action of Γ on the hyperbolic plane X is a regular hyperbolic $(4g)$-gon P_g, and the isometries a_i, b_j map $2g$ of the edges of P_g to the other $2g$ edges in an appropriate way. To see that we can choose $\gamma_0 = a_1$, for example, choose v to be the mid-point of the edge of P_g such that $a_1 v$ is also the mid-point of an edge (Figure 3). Then the geodesic segment $[v, a_1 v]$ in P_g projects to a closed geodesic of minimal length in $M = \Gamma \backslash X$.

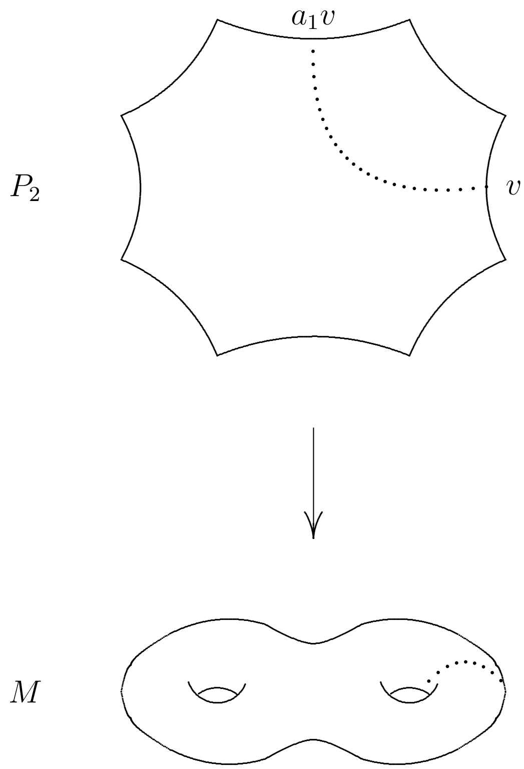

Figure 3: The genus 2 case.

Remark 4.12. Corollary 4.10 also follows immediately from [20, Corollary 6.3], since γ_0 is a prime element of the non-elementary hyperbolic group Γ. $\square$

5 Groups acting on euclidean buildings

In a second class of examples, the group Γ acts cocompactly by isometries on a locally finite euclidean building Δ of rank r. The building Δ is the combinatorial counterpart of a symmetric space X. The analogy becomes particularly evident if one considers groups of p-adic type. Specifically, let G be a connected semisimple group defined over a nonarchimedean local field. Then G acts on its Bruhat-Tits building Δ [4], and the vertex set of Δ may be identified with G/K, where K is a maximal compact subgroup.

We refer to [18] for the general theory of buildings. It is worth making a few remarks about the structure of euclidean buildings.

A building Δ is an r-dimensional simplicial complex whose maximal simplices are called *chambers*. All chambers have the same dimension r and adjacent chambers have a common face of dimension $r-1$.

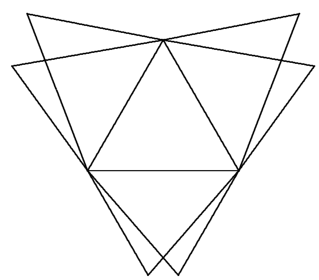

Figure 4: Six chambers adjacent to a chamber in an $\tilde{A}_2$ building.

Any two chambers can be connected by a sequence of adjacent chambers (called a *gallery*). An *apartment* in Δ is a subcomplex which is isomorphic to a Coxeter complex. All the apartments are isomorphic and any two simplices in Δ lie in a common apartment. If the apartments are infinite then Δ is contractible as a topological space. The apartments are then euclidean Coxeter complexes isometric to $\mathbb{R}^n$ and Δ is said to be a *euclidean building*. A euclidean building has a canonical piecewise smooth metric which is consistent with the euclidean structure on the apartments [3, VI.3]. It is convenient to normalise the distance on Δ so that any point of Δ is at distance < 1 from some vertex. The simplest examples of euclidean buildings are the homogeneous trees. In such a tree, a chamber is an edge and an apartment is an infinite geodesic.

The boundary of Δ is defined in terms of equivalence classes of sectors [18, Chp. 9.3]. A *sector* is a simplicial cone of dimension r, with a *special* base vertex, lying in some apartment of Δ. A sector in a euclidean building plays the role of a Weyl chamber in a symmetric space.

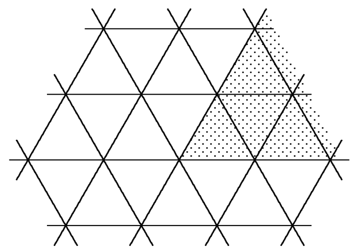

Figure 5: Part of an apartment and a sector in an $\tilde{A}_2$ building.

Two sectors are said to be *equivalent* if the Hausdorff distance between them is finite. This is considerably stronger than the notion of equivalence of Weyl chambers in a symmetric space, since two sectors are equivalent if and only if they contain a common subsector. Equivalent Weyl chambers, by contrast, usually have no points in common. As for symmetric spaces, the *boundary* Ω of Δ is the quotient space, whose points are equivalence classes of sectors.

For the rest of this section we fix a group Γ of automorphisms of Δ with the following properties.

(B1) Γ acts freely on the vertex set Δ^0, with finitely many vertex orbits (i.e. cocompactly).

(B2) There is an apartment F_0 in Δ and an abelian subgroup Γ_0 of Γ such that $\Gamma_0 \backslash F_0^0$ is finite, where F_0^0 is the vertex set of F_0.

(B3) The natural mapping $\Gamma_0 \backslash F_0^0 \to \Gamma \backslash \Delta^0$ is injective.

Remark 5.1.

(a) These conditions have been stated in a form applicable to a large class of examples. The assumption **(B1)** does not imply that Γ acts freely on (the geometric realization of) Δ. In fact, Γ acts freely on Δ if and only if Γ is torsion free [8, Proof of Theorem 4.1].

If we assume that Γ is torsion free then the setup is a precise combinatorial analogue of that in Section 4. For then $\Gamma\backslash\Delta$ is a finite cell complex of nonpositive curvature with universal covering space Δ and fundamental group Γ. Moreover $\Gamma_0\backslash F_0$ is homeomorphic to the r-torus, and **(B3)** implies that the natural mapping $\Gamma_0\backslash F_0 \to \Gamma\backslash\Delta$ is an embedding of the r-torus into $\Gamma\backslash\Delta$.

(b) The sole reason for assuming that Γ_0 is abelian in condition **(B2)** is to obtain an abelian von Neumann algebra $\mathrm{VN}(\Gamma_0)$. Everything else works equally well without this assumption.

(c) Let Γ be a group of automorphisms of Δ which acts properly discontinuously and cocompactly on Δ. (This is the case if condition **(B2)** is satisfied.) An apartment F in Δ is called *periodic* (or Γ-*closed*) if the stabilizer Γ_F of F acts cocompactly on F. The existence of an abundance of periodic apartments follows from [1, Theorem 8.9]. If F is a periodic apartment then the group Γ_F, being a Bieberbach group, necessarily contains a finite index subgroup $\Gamma_0 \cong \mathbb{Z}^r$, which also acts cocompactly on F. Condition **(B2)** is therefore satisfied for many apartments F_0 and subgroups $\Gamma_0 < \Gamma$.

The assumption **(B3)** has a simple interpretation in terms of the action of Γ on Δ.

Lemma 5.2. *Condition* **(B3)** *is equivalent to the following statement.*

(I) *If $\gamma \in \Gamma$, $a \in F_0^0$ and $\gamma a \in F_0^0$, then $\gamma \in \Gamma_0$.*

Proof. Assuming **(B3)**, let $\gamma \in \Gamma$, $a \in F_0^0$ and $\gamma a \in F_0^0$. Then $\Gamma a = \Gamma\gamma a$, and so $\Gamma_0 a = \Gamma_0\gamma a$, by injectivity. Thus $\gamma a = \gamma_0 a$ for some $\gamma_0 \in \Gamma_0$. However, Γ acts freely on Δ^0. Therefore $\gamma = \gamma_0 \in \Gamma_0$.

Conversely, if **(I)** holds, suppose that a, $a' \in F_0^0$ and $\Gamma a = \Gamma a'$. Then $\gamma a = a'$, for some $\gamma \in \Gamma$. In particular, $\gamma a \in F_0^0$. Therefore $\gamma \in \Gamma_0$, and so $\Gamma_0 a = \Gamma_0 a'$. □

Our next aim is to give a combinatorial analogue of Theorem 4.9. We begin with some preliminary results.

Lemma 5.3. *[17, Lemma 2.2] Let $C > 0$ and let S, S' be sectors in Δ. Then either S and S' contain a common subsector or S contains a subsector all of whose points are at a distance greater than C from S'.*

Proof. Choose subsectors S_1 and S_1' of S and S' respectively which lie in a common apartment [18, Chapter 9, Proposition (9.5)]. If S_1 and S_1' point in the same direction, then they have a common subsector, which is also a common subsector of S and S'.

Otherwise, fix a finite $C_1 > 0$ so that $d(v, S_1') \leq C_1$ for any $v \in S'$ [18, Chapter 9, Lemma (9.2)]. Choose a subsector S_2 of S_1 all of whose points are at a distance greater than $C + C_1$ from S_1'. Then those points are all at a distance greater than C from S'. $\square$

Lemma 5.4. *[c.f. Lemma 4.7] Let F, $F_1, \ldots, F_m$ be apartments in Δ such that, for some $\delta > 0$, $F^0 \underset{\delta}{\subset} F_1^0 \cup \cdots \cup F_m^0$. If S is a sector in F then there exists a subsector $S^* \subset S$ such that $S^* \subset F_j$, for some j.*

Proof. For $1 \leq j \leq m$, express F_j as a finite union of sectors. Let $\{S_\alpha : \alpha \in I\}$ denote the set of all such sectors.

Suppose that the sector S does *not* contain a subsector in common with any S_α. By Lemma 5.3, for each $\alpha \in I$ there exists a subsector S_α^* of S, all of whose points are at distance $> \delta$ from S_α.

Now $T = \cap_{\alpha \in I} S_\alpha^*$ is a (nonempty) subsector of S. Choose a vertex $t \in T^0$. Then $d(t, F_j) > \delta$ for each j. This contradicts the assumption that $F^0 \underset{\delta}{\subset} F_1^0 \cup \cdots \cup F_m^0$. $\square$

Corollary 5.5. *If F_1, F_2 are apartments in Δ and $F_1 \underset{\delta}{\subset} F_2$, for some $\delta > 0$, then $F_1 = F_2$.*

Proof. Express F_1 as a finite union of sectors $\{S_\alpha : \alpha \in I\}$, based at a vertex $v \in \Delta$. By Lemma 5.4, each S_α contains a subsector $S_\alpha^* \subset F_2$. In particular $F_1 \cap F_2 \neq \emptyset$ and we may assume from the start that $v \in F_1 \cap F_2$.

Now for each $\alpha \in I$, F_2 contains v and S_α^*, and hence also S_α, which is the convex hull of v and S_α^*. Thus $F_2 \supseteq F_1$. However F_1, F_2 are isomorphic Coxeter complexes in Δ. Therefore $F_1 = F_2$. $\square$

Before proceeding, recall that $VN(\Gamma)$ is a II_1 factor if and only if the group Γ is I.C.C.. If Γ were a lattice in a p–adic Lie group then the argument of [9, Lemma 3.3.1] (which uses the Borel density theorem) could be modified to prove that Γ is I.C.C.. However not all the groups considered in this section are embedded in a natural way as subgroups of p-adic linear groups. We therefore use a geometric argument to verify the I.C.C. property of Γ.

Lemma 5.6. *Let Δ be a euclidean building. Let Γ be a group of automorphisms of Δ which acts cocompactly on Δ. Then Γ is I.C.C.*

Proof. We have $\Gamma\mathcal{K} = \Delta$, where $\mathcal{K} \subset \Delta$ is compact. Let $x \in \Gamma - \{e\}$, and suppose that $C = \{y^{-1}xy : y \in \Gamma\}$ is finite. Let

$$\delta = \max\{d(\kappa, y^{-1}xy\kappa) : \kappa \in \mathcal{K}, y \in \Gamma\}. \tag{5.1}$$

Then

$$d(y\kappa, xy\kappa) = d(\kappa, y^{-1}xy\kappa) \le \delta, \tag{5.2}$$

for all $y \in \Gamma, \kappa \in \mathcal{K}$. Therefore, for all $\xi \in \Delta$,

$$d(\xi, x\xi) \le \delta. \tag{5.3}$$

Choose $\eta \in \Delta$ such that $x\eta \ne \eta$ and choose an apartment F in Δ with $\eta \in F$, $x\eta \notin F$. Now by (5.3), $F \underset{\delta}{\subset} xF$. Corollary 5.5 therefore implies that $F = xF$. In particular $x\eta \in F$, a contradiction. □

Remark 5.7. The proof of Lemma 5.6 also applies to a cocompact group Γ of isometries of a symmetric space. (The analogue of Corollary 5.5 is [13, Lemma 5.4].) In particular one obtains a proof of [9, Lemma 3.3.1] in the cocompact case which avoids the use of Borel's density theorem.

Theorem 5.8. *Let Γ be a group of automorphisms of a locally finite euclidean building Δ. Assume that* **(B1), (B2), (B3)** *hold. Then $VN(\Gamma_0)$ is a strongly singular masa of the* II_1 *factor $VN(\Gamma)$.*

Proof. In view of Lemma 4.6, it suffices to verify condition **(C2)**. Suppose that $x_1, \dots, x_m \in \Gamma$ and $\delta > 0$ satisfy $F_0^0 \underset{\delta}{\subset} x_1F_0^0 \cup x_2F_0^0 \cup \dots \cup x_mF_0^0$. Let S be a sector in F_0. By Lemma 5.4, there exists a subsector in $S^* \subset S$ such that $S^* \subset x_jF_0$, for some j. Now $S^* \subset F_0 \cap x_jF_0$. Choose a vertex s of S^*. Then $s = x_ja$ for some vertex $a \in F_0^0$. In particular $a \in F_0^0$ and $x_ja \in F_0^0$. It follows from Lemma 5.2 that $x_j \in \Gamma_0$. Therefore condition **(C2)** is satisfied. □

Example 5.9. [*Groups acting on buildings of type $\tilde{A}_2$*] Suppose that the building Δ has the property that there is a group Γ of automorphisms of Δ which acts freely and transitively on the vertex set Δ^0. For buildings of type $\tilde{A}_2$, groups with this property have been intensively studied in [5, I,II]. Suppose in addition that Γ has an abelian subgroup Γ_0 which acts transitively on the vertex set of an apartment in Δ. Then **(B1), (B2), (B3)** hold and so, by Theorem 5.8, $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$. Of the groups acting on $\tilde{A}_2$ buildings which are enumerated in [5, II], those labeled (4.1), (5.1), (6.1), (9.2), (13.1), (28.1) in that article contain such a subgroup Γ_0. The groups (4.1), (5.1), (6.1) are lattices in $\mathrm{PGL}_3(\mathbb{Q}_3)$, but the groups (9.2), (13.1), (28.1) do not have a natural embedding into a linear group. These groups all have 3-torsion, but act freely on the vertex set of Δ.

Example 5.10. [*A strongly singular but not ultrasingular masa in a* II_1 *factor with property (T)*] Let Γ be the group denoted by (4.1) in [5, II]. Then Γ is a lattice subgroup of $\mathrm{PGL}_3(\mathbb{Q}_3)$, and acts freely and transitively on the vertex set of a building Δ of type $\tilde{A}_2$. The presentation of Γ given in [5, II] has 13 generators x_i, $(0 \le i \le 12)$, and 20 relations, among which are

$$\begin{aligned}
x_2x_4x_7 &= 1 = x_2x_7x_4\,,\\
x_2x_3x_9 &= 1 = x_2x_9x_3\,,\\
x_7x_{11}x_{12} &= 1 = x_7x_{12}x_{11}\,,\\
x_6x_8x_9 &= 1 = x_6x_9x_8\,.
\end{aligned}$$

The abelian subgroups $\Gamma_1 = \langle x_2, x_4, x_7\rangle$, $\Gamma_2 = \langle x_2, x_3, x_9\rangle$, $\Gamma_3 = \langle x_7, x_{11}x_{12}\rangle$, $\Gamma_0 = \langle x_6, x_8, x_9\rangle$ are all free abelian of rank 2 and each acts transitively on the vertex set of an apartment. Thus each $\mathrm{VN}(\Gamma_i)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$. (Of course this implies that each Γ_i is a maximal abelian subgroup of Γ.)

According to [5, II, §5], the group Γ has an automorphism $f : x_i \mapsto x_{\pi(i)}$, where π is the permutation $(0\ 1\ 10\ 5)(6\ 11\ 8\ 12)(3\ 4)(7\ 9)$. The action of f on Γ interchanges Γ_1 and Γ_2, and $f^2|_{\Gamma_1} = id$, $f^2|_{\Gamma_2} = id$. Moreover f interchanges Γ_3 and Γ_0 and $f^2|_{\Gamma_0}$ exchanges the generators x_6 and x_8.

We claim that f^2 is an outer automorphism of Γ. For suppose that $f^2x = gxg^{-1}$ where $g \in \Gamma$. Since $f^2|_{\Gamma_1} = id$, and Γ_1 is a maximal abelian subgroup of Γ, we must have $g \in \Gamma_1$. Similarly $g \in \Gamma_2$, since $f^2|_{\Gamma_2} = id$. Thus $g \in \Gamma_1 \cap \Gamma_2 = \{1\}$.

It follows that f^2 induces an outer automorphism α of $\mathrm{VN}(\Gamma)$, under which $\mathrm{VN}(\Gamma_0)$ is invariant. In particular $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$ which is *not* ultrasingular in the sense of [16]. This answers in the negative a question raised in [17, Remark 2.10].

Note that as a lattice in a higher rank group, Γ has Kazhdan's property (T), so that $\mathrm{VN}(\Gamma)$ does contain ultrasingular masas by [16, Corollary 4.5].

In the examples above the group Γ has torsion. It is worth examining some cases where Γ is torsion free.

Example 5.11. [*A torsion free lattice in* PGL_3] Let Γ be the Regular $\tilde{A}_2$ group, which is a lattice subgroup of $\mathrm{PGL}(3,\mathbb{K})$, where $\mathbb{K}$ is the Laurent series field $\mathbf{F}_4((X))$ over the field $\mathbf{F}_4$ with four elements. This group is described in [5, I, Section 4] and the embedding of Γ in $\mathrm{PGL}(3,\mathbf{F}_4((X)))$ is essentially unique, by the Strong Rigidity Theorem of Margulis. The group Γ is torsion free and has 21 generators $x_i, 0 \le i \le 20$, and relations (written modulo 21):

$$\begin{cases} x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & 0 \le j \le 6, \\ x_j x_{j+3} x_{j-6} = 1 & 0 \le j \le 20. \end{cases}$$

This group Γ acts freely and transitively on the vertex set of its building Δ.

It follows from the first seven pairs of relations above that, for each j with $0 \le j \le 6$, the generators x_j, x_{j+7}, x_{j+14} pairwise commute and generate a free abelian subgroup of rank two inside Γ, satisfying the hypotheses of Theorem 5.8.

Example 5.12. [*Groups acting on products of trees*]

Consider some specific examples studied in [14]. In [14, Section 3], there is constructed a lattice subgroup Γ of $G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$, where $p, l \equiv 1 \pmod 4$ are two distinct primes. This restriction is made because -1 has a square root in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod 4$. The building Δ of G is a product of two homogeneous trees T_1, T_2 of degrees $(p+1)$ and $(l+1)$ respectively (that is, a euclidean

building Δ of type $\tilde{A}_1 \times \tilde{A}_1$) and G is a subgroup of $\mathrm{Aut}(\Delta)$. The group Γ is a torsion free group which acts freely and transitively on the vertex set Δ^0, but which is *not* a product of free groups. In fact it is an irreducible lattice in G.

Here is how Γ is constructed [14]. Let $\mathbb{H}(\mathbb{Z}) = \{\alpha = a_0 + a_1 i + a_2 j + a_3 k; a_j \in \mathbb{Z}\}$, the ring of integer quaternions. Let i_p be a square root of -1 in $\mathbb{Q}_p$ and define

$$\psi : \mathbb{H}(\mathbb{Z}) \to PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

by

$$\psi(a_0 + a_1 i + a_2 j + a_3 k) = \left(\begin{bmatrix} a_0 + a_1 i_p & a_2 + a_3 i_p \\ -a_2 + a_3 i_p & a_0 - a_1 i_p \end{bmatrix}, \begin{bmatrix} a_0 + a_1 i_l & a_2 + a_3 i_l \\ -a_2 + a_3 i_l & a_0 - a_1 i_l \end{bmatrix} \right)$$

Let $\tilde{\Gamma} = \{\alpha = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}(\mathbb{Z}); a_0 \equiv 1 \pmod 2, a_j \equiv 0 \pmod 2, j = 1, 2, 3, |\alpha|^2 = p^r l^s\}$. Then $\Gamma = \psi(\tilde{\Gamma})$ is a torsion free cocompact lattice in G. Let

$A = \{a = a_0 + a_1 i + a_2 j + a_3 k \in \tilde{\Gamma}; a_0 > 0, |a|^2 = p\}$,

$B = \{b = b_0 + b_1 i + b_2 j + b_3 k \in \tilde{\Gamma}; b_0 > 0, |b|^2 = l\}$.

Then A contains $p+1$ elements and B contains $l+1$ elements. The images $\underline{A}$, $\underline{B}$ of A, B in Γ generate free groups Γ_p, Γ_l of orders $p+1$, $l+1$ respectively and Γ itself is generated by $\underline{A} \cup \underline{B}$. The 1-skeleton of Δ is the Cayley graph of Γ relative to this set of generators.

By abuse of notation, identify a quaternion in $\tilde{\Gamma}$ with its image in Γ. (If one quaternion is a rational multiple of the other then they have the same image in Γ.)

It is now easy to exhibit copies of $\Gamma_0 \cong \mathbb{Z}^2$ inside Γ, with Γ_0 acting freely and transitively on the vertex set of an apartment in Δ, and therefore satisfying the hypotheses of the Theorem 5.8. There are integers a_0, a_1 (essentially unique) with a_0 odd, a_1 even and $a_0^2 + a_1^2 = p$. (The Two Square Theorem.) Similarly there are b_0, b_1 with $b_0^2 + b_1^2 = l$. Let $a = a_0 + a_1 i$, $b = b_0 + b_1 i$. Then we can take $\Gamma_0 = \langle a, b\rangle$. There is nothing special about the choice of i rather than j or k. Thus we get two other possible groups Γ_0. Specific Example: $p = 5, l = 13$, $a = 1 + 2i$, $b = 3 + 2i$.

6 Borel subgroups of linear algebraic groups

J. Dixmier [6] constructed examples of singular masas by considering groups of homographies. The purpose of this section is to extend his construction. A basic example is the following.

Example 6.1. Let Γ be the upper triangular subgroup of $\mathrm{PSL}_n(\mathbb{Q})$, $n \geq 2$, and let Γ_0 be the diagonal subgroup of Γ. Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of the II_1 factor $\mathrm{VN}(\Gamma)$. [Diximer deals with the case $n = 2$.]

We shall prove this result by using Proposition 4.2, and the methods of the previous section. In order to do this, we let Γ act on an appropriate euclidean building. Choose a prime p and let $G = \mathrm{PSL}_n(\mathbb{Q}_p)$. Then G acts upon its Bruhat-Tits building Δ, whose vertex set is G/K, where $K = \mathrm{PSL}_n(\mathbb{Z}_p)$. Here $\mathbb{Q}_p$ is the p-adic field and $\mathbb{Z}_p$ the p-adic integers. Details can be found in [3, VI.9F].

Choose the apartment F_0 of Δ whose vertices are all the cosets of the form

$$\begin{bmatrix} p^{j_1} & & & \\ & p^{j_2} & & \\ & & \ddots & \\ & & & p^{j_n} \end{bmatrix} \cdot K, \qquad j_k \in \mathbb{Z},\ 1 \leq k \leq n\,.$$

Then Γ_0 clearly acts transitively on the vertex set F_0^0.

The boundary Ω of Δ is the quotient space G/B, where B is the Borel subgroup of upper triangular matrices in G. It is important to note that since Γ is a subgroup of B, there is a boundary point ω_0 (the coset B) which is stabilized by Γ.

Consider now the following general setup. Let Δ be a euclidean building and let G be a strongly transitive type preserving subgroup of $\mathrm{Aut}(\Delta)$ [7, §17]. This means that G acts transitively on the set of pairs (C, F) where F is an apartment of Δ and C is a chamber contained in Δ. Fix an apartment $F_0 \subset \Delta$ and a sector $S_0 \subset F_0$. Then S_0 represents a boundary point ω_0 of Δ.

Consider the *Borel subgroup* $B = \{g \in G : g\omega_0 = \omega_0\}$. Let $N = \{g \in G : gF_0 = F_0\}$ and let $\mathfrak{A} = B \cap N$, the *Cartan subgroup*. Then by [7, Theorem 17.3],

$$\mathfrak{A} = \{g \in G; g \text{ acts on } F_0 \text{ by translations}\}\,.$$

Example 6.1 is a special case of this setup, with $G = \mathrm{PSL}_n(\mathbb{Q}_p)$ [7, §19].

Theorem 6.2. *Under the above assumptions, let Γ be an I.C.C. subgroup of B and let $\Gamma_0 = \Gamma \cap N$. Suppose also that $\Gamma_0 \backslash F_0^0$ is finite. Then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

Remark 6.3. The group B itself is I.C.C.. One way to see this is to note that B acts on Δ with finitely many vertex orbits [7, Theorem 17.6], and so B acts cocompactly on Δ. The I.C.C. property follows from Lemma 5.6.

Remark 6.4. Theorem 6.2 applies in particular to Example 6.1, with Γ the group of upper triangular matrices in $\mathrm{PSL}_n(\mathbb{Q})$. In Example 6.1, $\Gamma_0 \backslash F_0^0$ is a singleton, since Γ_0 acts transitively on F_0^0. The action of B on Δ is continuous and the vertex set of Δ is discrete. Therefore the dense subgroup Γ of B has finitely many vertex orbits, since B does. Thus Γ is I.C.C. by Lemma 5.6, since it acts cocompactly on Δ.

Proof. (Theorem 6.2.) We verify conditions **(C1)**, **(C2)** of Proposition 4.2, with $X = \Delta^0$ and $A = F_0^0$. The fact that $\Gamma_0 \backslash F_0^0$ is finite implies condition **(C1)**.

Turning to **(C2)**, let $x_1, \dots, x_m \in \Gamma$, $\delta > 0$ and $F_0^0 \underset{\delta}{\subset} x_1 F_0^0 \cup x_2 F_0^0 \cup \cdots \cup x_m F_0^0$. Choose a sector S in F_0 opposite S_0. By Lemma 5.4, there exists a subsector $S^* \subset S$ such that $S^* \subset x_j F_0$ for some j.

Since $x_j \omega_0 = \omega_0$, the two sectors $x_j S_0$ and S_0 have a common subsector S_0^*.

We now have $S_0^* \cup S^* \subset F_0$ and $S_0^* \cup S^* \subset x_j F_0$. However, opposite sectors in an apartment determine that apartment completely [3, VI.9, Lemma 2 and IV.5, Theorem 1]. Therefore $x_j F_0 = F_0$. In other words, $x_j \in \Gamma \cap N = \Gamma_0$. This establishes condition **(C2)**. □

Remark 6.5. (a) Generalizing Example 6.1, one can clearly let Γ be the upper triangular subgroup of $\mathrm{PSL}_n(\mathbb{K})$, where $\mathbb{K}$ is any subfield of $\mathbb{Q}_p$ for some prime p, with $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{Q}_p$. In fact $\mathbb{K}$ could be an appropriate subfield of any nonarchimedean local field. The abelian subgroup Γ_0 is again the diagonal subgroup of Γ. Note that Γ is amenable and so $\mathrm{VN}(\Gamma)$ is the hyperfinite II_1 factor.
(b) Other generalizations are possible. For example, one could replace $\mathbb{Q}_p$ by $\mathbb{R}$, and work with symmetric spaces. In the case $n = 2$, Γ would be the subgroup of upper triangular matrices in $\mathrm{PSL}_2(\mathbb{Q})$, and Γ_0 its diagonal subgroup. These groups act on the

hyperbolic plane X and the crucial point in the argument is the fact that a geodesic in X is uniquely determined by its two boundary points.

References

[1] W. Ballmann and M. Brin, Orbihedra of nonpositive curvature, *Publ. Math. I. H. E. S.* **82** (1995), 169–209.

[2] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[3] K. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[4] F. Bruhat and J. Tits, Groupes réductifs sur un corps local. I, *Données radicielles valuées*, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5–251.

[5] D. I. Cartwright, A. M. Mantero, T. Steger, and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, I & II, *Geom. Ded.* **47** (1993) 143–166 and 167–226.

[6] J. Diximer, Sous-anneaux abéliens maximaux dans les facteurs de type fini, *Ann. Math.* **59** (1954), 279–286

[7] P. Garrett, *Buildings and Classical Groups*, Chapman and Hall, London, 1997.

[8] S. M. Gersten and H. Short, Small cancellation theory and automatic groups: Part II, *Invent. Math.* **105** (1991), 641–662.

[9] F. Goodman, P. de la Harpe and V. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications **14**, Springer-Verlag, New York, 1989.

[10] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago, 2000.

[11] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras, Vol. II*, Academic Press, Orlando, 1986.

[12] H. B. Lawson and S. T. Yau, Compact manifolds of nonpositive curvature, *J. Differential Geometry* **7** (1972), 211–228.

[13] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Princeton, New Jersey, 1973.

[14] S. Mozes, Actions of Cartan Subgroups, Israel J. Math. **90** (1995), pp. 253–294.

[15] K. E. Petersen, *Ergodic Theory*, Cambridge University Press, Cambridge, 1983.

[16] S. Popa, Singular maximal abelian $*$-subalgebras in continuous von Neumann algebras, *J. Funct. Anal.*, **50** (1983), 151–166.

[17] G. Robertson and T. Steger, Maximal subalgebras of the group factor of an $\tilde{A}_2$ group, *J. Operator Theory*, **36** (1996), 317–334.

[18] M. A. Ronan, *Lectures on Buildings*, Perspectives in Math. **7**, Academic Press, London 1989.

[19] J.-P. Serre, *Arbres, amalgames, SL_2*, 3^e ed., Astérisque **46**, Soc. Math. France, 1983.

[20] A. M. Sinclair and R. R. Smith, Strongly singular masas in type II_1 factors, *Geom. and Funct. Anal.*, **12** (2002), 199–216.

[21] C. Sutherland, Type analysis of the regular representation of a nonunimodular group, Pacific J. Math., **79** (1978), 225–250.