Buildings

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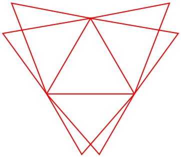
1. Examples.
2. What is a Building?
3. Groups acting on buildings
4. Interactions

Examples



Buildings are geometric objects, invented by J Tits (c. 1965).

He combined

- his own work on geometries associated to matrix groups ;
- previous algebraic work of F Bruhat and C Chevalley.

Reference: K Brown, *Buildings*, Springer 1996.

Buildings are simplicial complexes satisfying certain axioms.

Example 1

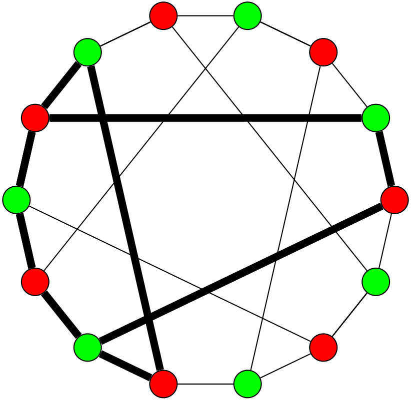
If V is a vector space of dimension $n+1$ over a field k, construct the flag complex Δ of proper linear subspaces of V:

1. Vertices of Δ are proper linear subspaces of V.
2. Two vertices V_1, V_2 of Δ are incident if $V_1 \subset V_2$.
3. Simplices are chains $V_1 \subset V_2 \subset \cdots \subset V_q$.

Note that

- Maximal simplices (called chambers) have n vertices.
- Therefore Δ has dimension $n-1$.

A projective plane



$k = \mathbb{Z}/2\mathbb{Z}$, $\dim V = 3$.

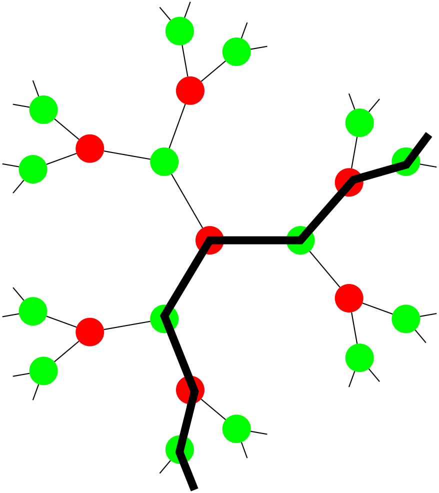
Vertices of Δ are lines and planes in V.

Δ contains 28 hexagonal apartments.

Any two chambers lie in a common apartment.

$\mathrm{GL}_3(k)$ is a group of symmetries of Δ of order 168.

Example 2



Δ is a tree and every vertex has degree ≥ 3.

Apartments are lines (infinite both ways).

There are uncountably many apartments.

Any two chambers lie in a common apartment.

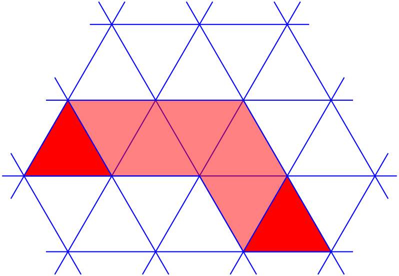
Is there a natural group of symmetries of Δ?

Chamber Complexes

Let Δ be a simplicial complex such that all maximal simplices (**chambers**) have the same dimension d.

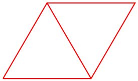
Chambers C, D are **adjacent** if $C \cap D$ has codimension 1.

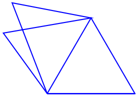
Δ is a **chamber complex** if any two chambers are connected by a sequence of adjacent chambers (a **gallery**).



Buildings

A chamber complex is thin [thick] if each simplex of codimension 1 is a face of exactly 2 [at least 3] chambers.

thin 

 thick

A **building** is a thick chamber complex Δ which is a union of thin chamber complexes, called apartments, satisfying:

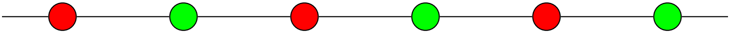
1. Apartments have the same dimension as Δ.
2. Any two simplices of Δ lie in a common apartment.
3. If apartments Σ, Σ' contain a common chamber, then there is an isomorphism $\Sigma \to \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.

Types of buildings

- All apartments are isomorphic.
- If apartments are spheres, Δ is called spherical.
- If apartments are euclidean spaces, Δ is called euclidean.

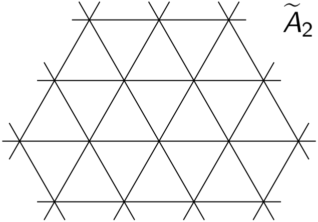
A projective plane is a 1-dimensional spherical building.

A 1-dimensional euclidean building is a tree and its apartments are lines (infinite both ways).

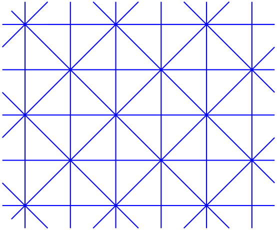


Focus now on euclidean buildings . . .

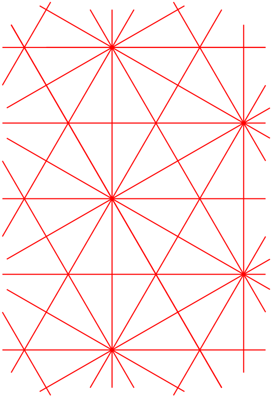
Apartments in euclidean buildings of dimension 2



$\widetilde{A}_2$



$\widetilde{B}_2$



$\widetilde{G}_2$

The p-adic numbers,

If p is prime, $\mathbb{Q}_p$ is the field of formal sums

$$x = a_j p^j + a_{j+1} p^{j+1} + \cdots ,$$

where each $a_i \in \{0, 1, \cdots , p-1\}$.

Define $$|x| = \begin{cases} p^{-j} & \text{if} \quad a_j \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Example

Let

$$y = 1 + 5 + 5^2 + \cdots \in \mathbb{Q}_5,$$

then

$$|y| = 1 \quad \text{and} \quad y = -\frac{1}{4} .$$

Reason: $y = 1 + 5y$

The ring of p-adic integers is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| \leq 1\}.$$

A p-adic integer has the form

$$x = a_0 + a_1 p + a_2 p^2 + \cdots .$$

Example

$$-\frac{1}{4} \in \mathbb{Z}_5 .$$

The tree

Observation

A ball of radius p^n is a disjoint union of p balls of radius p^{n-1}.

Example ($p = 3$)

$\mathbb{Z}_3 = \{a_0 + a_1 3 + a_2 3^2 + \cdots\}$ is a disjoint union of three balls:

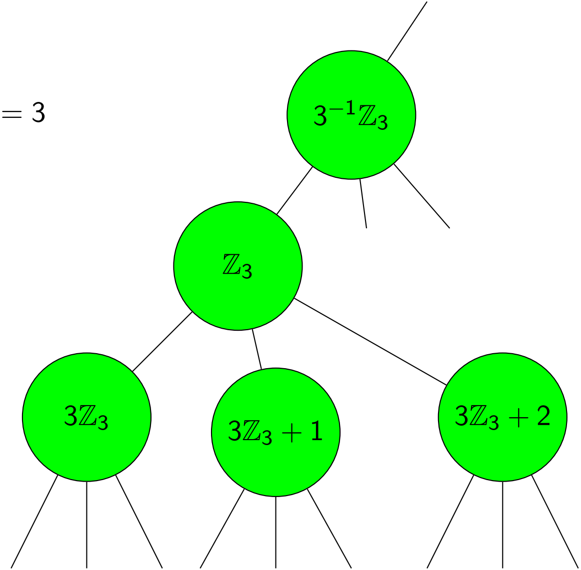
$$3\mathbb{Z}_3 + 0 = \{0 + a_1 3 + a_2 3^2 + \cdots\}$$
$$3\mathbb{Z}_3 + 1 = \{1 + a_1 3 + a_2 3^2 + \cdots\}$$
$$3\mathbb{Z}_3 + 2 = \{2 + a_1 3 + a_2 3^2 + \cdots\}.$$

Consequence

The set of balls in $\mathbb{Q}_p$ is the vertex set of a $(p+1)$-regular tree Δ.

Example

$p = 3$



The euclidean building of $SL_2(\mathbb{Q}_p)$

The group

$$\mathrm{SL}_2(\mathbb{Q}_p)$$

acts on its euclidean building Δ, which is a $(p+1)$-regular tree.

A different description of Δ:

A **vertex** is a maximal compact subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$.
e.g. $K = \mathrm{SL}_2(\mathbb{Z}_p)$.

An **edge** is (K, K') where $K \cap K'$ is a maximal proper subgroup of K and K'.

$\mathrm{SL}_2(\mathbb{Q}_p)$ acts on Δ via

$$K \mapsto g^{-1}Kg$$

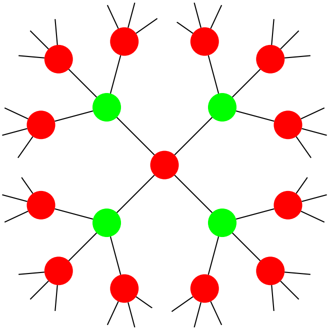
Example ($p = 3$)

In the building of $\mathrm{SL}_2(\mathbb{Q}_3)$, a vertex K has four neighbours

$$g^{-1}Kg$$

where

$$g = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$



The building of $SL_3(\mathbb{Q}_p)$

The group $\mathrm{SL}_3(\mathbb{Q}_p)$ acts on its euclidean building Δ.

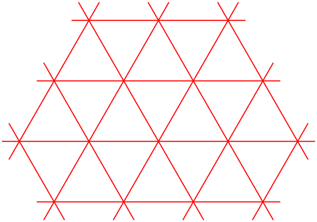
A **vertex** is a maximal compact subgroup K of $\mathrm{SL}_3(\mathbb{Q}_p)$.
e.g. $K = \mathrm{SL}_3(\mathbb{Z}_p)$.

An **edge** is (K, K') where $K \cap K'$ is a maximal proper subgroup of K and K'.

$\mathrm{SL}_3(\mathbb{Q}_p)$ acts on Δ via

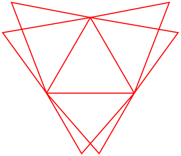
$$K \mapsto g^{-1}Kg$$

Δ has type $\widetilde{A}_2$: its apartments are $\widetilde{A}_2$ Coxeter complexes.

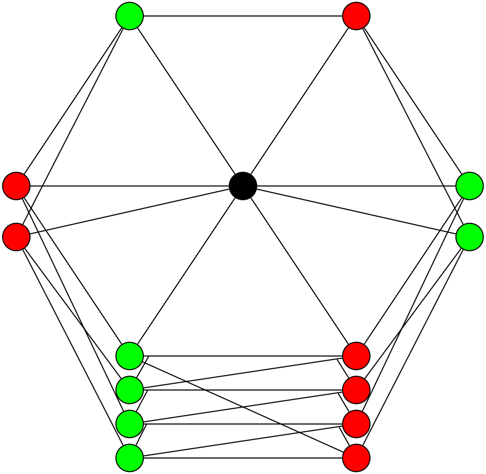


Each edge lies on $p+1$ triangles.

$p = 2$:



The neighbours of a vertex ($p = 2$)



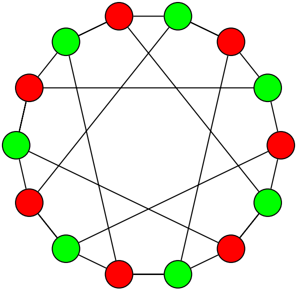
A ball of radius one.
The link of a vertex is a projective plane.

Local Structure and Global Structure

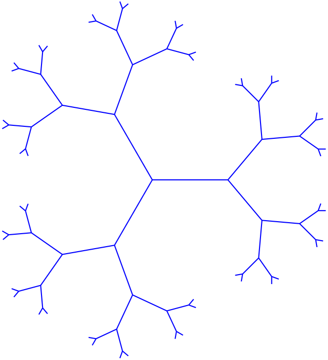
One can replace $\mathbb{Q}_p$ by an algebraic extension $\mathbb{K}$.

If $\mathbb{K}_1 \neq \mathbb{K}_2$, then the buildings of $\mathrm{SL}_3(\mathbb{K}_1)$ and $\mathrm{SL}_3(\mathbb{K}_2)$ are not isomorphic.

However, spheres of radius 1 are isomorphic :

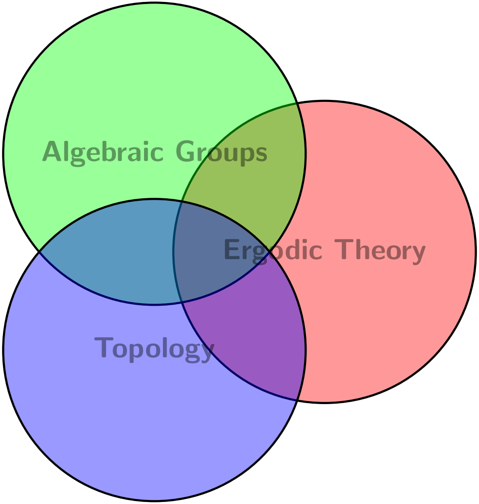


SL_2 is different: e.g. the buildings of $\mathrm{SL}_2(\mathbb{Q}_2)$, $\mathrm{SL}_2(\mathbb{Q}_2(\sqrt{-1}))$ are isomorphic.



SL_2 is atypical. Semisimple algebraic groups of higher relative rank behave like SL_3.

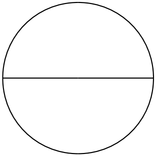
Interactions



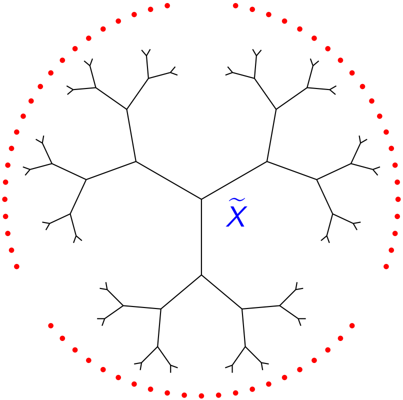
Example

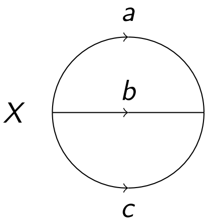
X : A finite connected graph.
$\widetilde{X}$: The universal covering space (a tree).
$\partial\widetilde{X}$: The boundary of $\tilde{X}$.

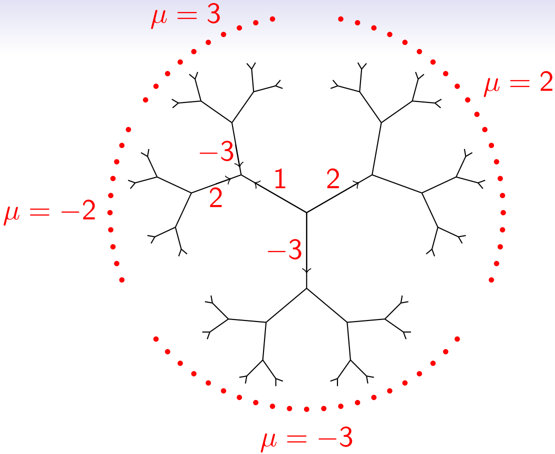
X



$\Gamma = \pi(X) < \mathrm{SL}_2(\mathbb{Q}_2)$.







Consider a cycle $\alpha \in H_1(X, \mathbb{Z})$.
Then α is a 1-chain and $\partial\alpha = 0$. e.g. $\alpha = a + 2b - 3c$

There is an associated harmonic cocycle on $\widetilde{X}$

and a corresponding Γ-invariant measure μ_α on $\partial\widetilde{X}$.

Definition

$\mathfrak{M}_\Gamma(\partial\Delta)$: the group of Γ-invariant $\mathbb{Z}$-valued measures on $\partial\Delta$.

Theorem

The map $\alpha \mapsto \mu_\alpha$ is an isomorphism

$$H_1(X, \mathbb{Z}) \cong \mathfrak{M}_\Gamma(\partial\Delta).$$

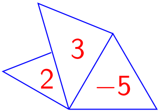
The proof uses noncommutative geometry.

Problem

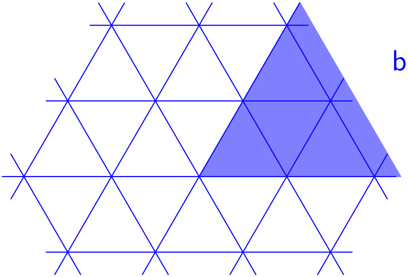
Extend this result to higher dimensional euclidean buildings.

$\widetilde{A}_2$ case

How to extend the notions of harmonic cocycle and boundary?



harmonic cocycle



boundary point