NEGATIVE DEFINITE KERNELS AND A DYNAMICAL CHARACTERIZATION OF PROPERTY (T) FOR COUNTABLE GROUPS

GUYAN ROBERTSON AND TIM STEGER

Abstract. A class of negative definite kernels is defined in terms of measure spaces. Using this concept, property (T) for a countable group Γ is characterized in terms of measure preserving actions of Γ, as follows. If a set S is translated a finite amount by any fixed element of Γ, then there is a uniform bound on how far S is translated.

Introduction

A group G has Kazhdan's property (T) if the trivial representation is isolated in the unitary dual of G [K]. We refer to [HV] for an exposition of many of the remarkable properties of such groups.

Groups with property (T) are characterized by the absence of unbounded negative definite functions [HV, Chapter 5, Theorem 20]. This characterization was obtained in [D], [G] and, independently, in [AW]. We use this to prove the following dynamical characterization of property (T), which is the main result of this paper. All measure spaces are positive but not necessarily finite.

A countable group Γ has property (T) if and only if it satisfies the following condition: for every measure-preserving action of Γ on a measure space $(\Omega, \mathcal{B}, \mu)$ and every set $S \in \mathcal{B}$ such that $\mu(S \triangle gS) < \infty$ for all $g \in \Gamma$, we have

$$\sup_{g \in \Gamma} \mu(S \triangle gS) < \infty.$$

Remark 0.1. To see how the condition above can fail to hold, suppose that there exists a homomorphism from Γ onto $\mathbb{Z}$. (There is no such homomorphism, if Γ has property (T).) Take $\Omega = \mathbb{Z}$, $\mu =$ counting measure, $S = \{1, 2, 3, \dots\}$ and let Γ act on Ω by translation. Then $\mu(S \triangle gS) < \infty$ for all $g \in \Gamma$ but $\sup_{g \in \Gamma} \mu(S \triangle gS) = \infty$.

Definition 0.1.1. Let Γ be a countable or finite set. Say that a function $f : \Gamma \times \Gamma \longrightarrow [0, \infty)$ is a *measure definite kernel* if there is a

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measure space $(\Omega, \mathcal{B}, \mu)$ which contains sets $S_x \in \mathcal{B}$ for $x \in \Gamma$ such that $f(x,y) = \mu(S_x \triangle S_y)$.

Remark 0.2. If f is measure definite we can find a positive measure space (Ω, μ) which contains sets S_x for $x \in \Gamma$ such that $f(x,y) = \mu(S_x \backslash S_y) = \mu(S_y \backslash S_x)$. For this, simply replace Ω by $\Omega' = \Omega \times \{0,1\}$ with the product measure and replace each set S_x by $S'_x = (S_x \times \{0\}) \sqcup (S_x^c \times \{1\})$.

1. MEASURE DEFINITE KERNELS

We refer to [HV, Chapter 5] for the definition and properties of negative definite kernels. A negative definite kernel f is assumed to be *normalized*, i.e. $f(x,x) = 0$. A universal construction of a negative definite kernel on a set Γ is to take a mapping $x \mapsto v_x$ of Γ into a real Hilbert space $\mathcal{H}$ and to let the kernel $f(x,y) = \|v_x - v_y\|^2$. Moreover, for a given f, the embedding $x \mapsto v_x$ of Γ in $\mathcal{H}$ is unique up to rigid motions of the closed convex hull of $\{v_x : x \in \Gamma\}$ [HV, Chapter 5, Proposition 14].

Proposition 1.1. *Every measure definite kernel is negative definite.*

PROOF: Let f be a measure definite kernel. Using the notation of Definition 0.1.1, fix $x_0 \in \Gamma$ and define a function $\eta : \Gamma \to L^2(\Omega, \mu)$ by $\eta(x) = \chi_x - \chi_{x_0}$, where χ_x denotes the characteristic function of S_x. Then

$$\|\eta(x) - \eta(y)\|^2 = \mu(S_x \triangle S_y) = f(x,y)$$

and so f is negative definite [HV, Chapter 5.13, Exemples]. $\square$

Every measure definite kernel is a pseudometric. The negative definite kernel defined on $\mathbb{R}$ by $f(x,y) = (x-y)^2$ is clearly not a pseudometric. The converse of Proposition 1.1 is therefore false. However we shall show below that the square root of any negative definite kernel is measure definite.

For the rest of this section let Γ be a fixed countable set. Let $\Omega_{00} = \{0,1\}^\Gamma$ with the product topology. Then Ω_{00} is either a Cantor set or finite. Let $\Omega_0 = \Omega_{00} \backslash \{(0,0,0,\dots),(1,1,1,\dots)\}$ and let $\mathcal{B}_0$ be the σ-algebra of Borel sets in Ω_0. Given $x \in \Gamma$, let $S_x^0 = \{c \in \Omega_0 : c(x) = 1\}$. Our next result shows that $\mathcal{B}_0$ is a universal σ-algebra for measure definite kernels on Γ.

Proposition 1.2. *If $f : \Gamma \times \Gamma \longrightarrow [0,\infty)$ is a measure definite kernel then there exists a regular Borel measure μ_0 on $(\Omega_0, \mathcal{B}_0)$ such that $f(x,y) = \mu_0(S_x^0 \triangle S_y^0)$ for all $x, y \in \Gamma$.*

PROOF: Suppose that a measure space $(\Omega, \mathcal{B}, \mu)$ is given, which contains sets S_x, such that $\mu(S_x \triangle S_y)$ is finite for $x, y \in \Gamma$. We can and will delete from Ω all points belonging to all (respectively none) of the sets S_x without changing $\mu(S_x \triangle S_y)$. Define a map $\phi : \Omega \to \Omega_0$ by

$$(\phi(p))(x) = \begin{cases} 0 & \text{if} \quad p \notin S_x \\ 1 & \text{if} \quad p \in S_x. \end{cases}$$

Since ϕ is measurable, we may define a corresponding measure $\mu_0 = \phi_*(\mu)$ on $(\Omega_0, \mathcal{B}_0)$.

Then

$$\mu_0(S_x^0 \triangle S_y^0) = \mu(\phi^{-1}(S_x^0 \triangle S_y^0)) = \mu(S_x \triangle S_y).$$

In this way any measure definite kernel may be expressed as $\mu_0(S_x^0 \triangle S_y^0)$ for some measure μ_0 on $(\Omega_0, \mathcal{B}_0)$. We claim (a) every open set in Ω_0 is σ-compact ; (b) $\mu_0(K) < \infty$ for every compact set K in Ω_0. By [R, Theorem 2.18], these two assertions imply that the measure μ_0 is regular.

Let $\mathcal{C}$ denote the family of all cylinder sets of the form

$$\begin{aligned} A &= (S_{x_1}^0 \cap \cdots \cap S_{x_m}^0) \backslash (S_{y_1}^0 \cup \cdots \cup S_{y_n}^0) \\ &= \{c \in \Omega_0 : c(x_i) = 1, c(y_j) = 0, 1 \le i \le m, 1 \le j \le n\}. \end{aligned}$$

where $\{x_1, \ldots, x_m, y_1, \ldots, y_n\} \subset \Gamma$ and $m, n \ge 1$. Each set in $\mathcal{C}$ has finite measure since $\mu_0(S_x^0 \triangle S_y^0) < \infty$ for all $x, y \in \Gamma$. Assertion (a) follows from the fact that the family $\mathcal{C}$ is a countable base for the topology of Ω_0 consisting of compact open sets. Assertion (b) follows because every compact subset of Ω_0 is contained in a finite union of sets in $\mathcal{C}$, and hence is contained in a finite union of sets of finite measure. □

Proposition 1.3. *The class of measure definite kernels on Γ is closed under pointwise convergence.*

Proof: By Proposition 1.2 we may consider only measure definite kernels of the form $\mu(S_x^0 \triangle S_y^0)$ for regular Borel measures μ on Ω_0. Suppose that measures $(\mu_j)_{j=1}^\infty$ on Ω_0 give rise to measure definite kernels f_j which converge pointwise to f. We must show that f is measure definite. For fixed $x, y \in \Gamma$, $f_j(x, y) \to f(x, y)$, so the restricted measures

$$\mu_j|_{S_x^0 \triangle S_y^0}$$

are positive and of uniformly bounded finite mass. Since the set of positive Borel measures of total mass bounded by a given fixed constant form a compact metric space, we can, by passing to a subsequence, assume that $\mu_j|_{S_x^0 \triangle S_y^0}$ approaches some positive, finite mass Borel measure on $S_x^0 \triangle S_y^0$. Since Γ is countable, we may pass to a subsequence where this happens for all $x, y \in \Gamma$.

Write $\Omega_0 = \bigsqcup_{k=1}^\infty T_k$, where $\mu_j|_{T_k}$ has a limit for each T_k and where each $S_x^0 \triangle S_y^0$ is contained in a finite union of T_k's. To see that this is possible, suppose that the elements of Γ are labelled, x_1, x_2, etc. Let all sets of the two forms

$$\{c \in \Omega_0 : (c(x_1), \ldots, c(x_{n-1}), c(x_n)) = (1, \ldots, 1, 0)\}$$

and

$$\{c \in \Omega_0 : (c(x_1), \dots, c(x_{n-1}), c(x_n)) = (0, \dots, 0, 1)\}$$

make up the collection $(T_k)_k$. This collection will be countable or finite as Γ is countable or finite. The disjoint union of the T_k gives all of Ω_0.

Put the limit measures on the T_k together to form a measure μ on Ω_0. For any $x, y \in \Gamma$, we have

$$\begin{aligned}\mu(S_x^0 \triangle S_y^0) &= \sum_{k=1}^{\infty} \lim_j \mu_j(T_k \cap (S_x^0 \triangle S_y^0)) = \lim_j \sum_{k=1}^{\infty} \mu_j(T_k \cap (S_x^0 \triangle S_y^0)) \\ &= \lim_j \mu_j(S_x^0 \triangle S_y^0) = \lim_j f_j(x, y) = f(x, y).\end{aligned}$$

This shows that f is measure definite and so proves the result. □

Let $(u_j)_{j=1}^N \subset \mathcal{H}$ and $(u'_j)_{j=1}^N \subset \mathcal{H}'$ be two tuples of vectors in two real Hilbert spaces. We say that the two tuples are in the same *configuration* if $\|u_j - u_k\| = \|u'_j - u'_k\|$ for $1 \le j, k \le N$. If $\mathcal{H}$,$\mathcal{H}'$ have the same dimension, then the two tuples are in the same configuration if and only if there is an affine isometry from $\mathcal{H}$ to $\mathcal{H}'$ sending u_j to u'_j, $1 \le j \le N$.

Proposition 1.4. *Let f be a negative definite kernel on a set Γ and choose a mapping $x \mapsto v_x$ of Γ into a real Hilbert space $\mathcal{H}$ such that $f(x, y) = \|v_x - v_y\|^2$. Then*
(i) $\sqrt{f}$ is measure definite;
(ii) there exists a regular Borel measure μ_0 on Ω_0 such that $f(x, y) = \mu_0(S_x^0 \triangle S_y^0)$ for all $x, y \in \Gamma$ and having following property:

if $\{x_1, \dots, x_m, y_1, \dots, y_n\} \subset \Gamma$, where $m, n \ge 1$, then $\mu_0((S_{x_1}^0 \cap \dots \cap S_{x_m}^0) \backslash (S_{y_1}^0 \cup \dots \cup S_{y_n}^0))$ is determined canonically by the configuration of $(v_{x_1}, \dots, v_{x_m}, v_{y_1}, \dots, v_{y_n})$.

Proof: Suppose the result has been proved for finite Γ. In the infinite case, think of Γ as the increasing union of a sequence of finite sets, and for each of these finite sets construct a measure μ_0 satisfying the conditions of the statement insofar as they apply to elements of that set. Take limits as in the proof of Proposition 1.3 to obtain a measure μ_0 satisfying the conditions for all of Γ.

Suppose, therefore, that Γ is finite. Replace $\mathcal{H}$ by the linear subspace which the the v_x generate. This guarantees that $\mathcal{H}$ is finite dimensional. Let Ω be the homogeneous space of all half-spaces of $\mathcal{H}$. Then Ω may be realized as E_n/E_{n-1}, where E_n is the (unimodular) group of rigid motions of the n-dimensional real Hilbert space $\mathcal{H}$ and the stabilizer of a given half space is isomorphic to E_{n-1}. Endow Ω with its usual measure μ, which is invariant under rigid motions of $\mathcal{H}$. We can normalize μ so that

$$\mu\{M \in \Omega : v \in M, w \notin M\} = \|v - w\|.$$

(This follows, for example, from the explicit expression for the Haar measure on the affine group of $\mathcal{H}$ given in [B, Chap.VII §3 No.3 Ex.4].

It may also be seen geometrically.) Let $S_x = \{M \in \Omega : v_x \in M\}$. Then $\sqrt{f}(x,y) = \|v_x - v_y\| = \mu(S_x \backslash S_y) = \mu(S_y \backslash S_x)$ is measure definite. This proves (i).

To prove (ii) it suffices to show that the measure

$$\mu\{M \in \Omega : v_{x_j} \in M, v_{y_k} \notin M, 1 \le j \le m, 1 \le k \le n\}$$

depends only on the configuration of $(v_{x_1}, \dots, v_{x_m}, v_{y_1}, \dots, v_{y_n})$. This is clear except for a possible dependence on the dimension of $\mathcal{H}$. Suppose then that

$$\{v_{x_1}, \dots, v_{x_m}, v_{y_1}, \dots, v_{y_n}\} \subset \mathcal{H} \subset \mathcal{H}'$$

Let Ω' be the space of all half-spaces of $\mathcal{H}'$ and let μ' be the natural measure on Ω'. We must prove that

$$\begin{aligned}&\mu\{M \in \Omega : v_{x_j} \in M, v_{y_k} \notin M, 1 \le j \le m, 1 \le k \le n\}\\ &\qquad = \mu'\{M' \in \Omega' : v_{x_j} \in M', v_{y_k} \notin M', 1 \le j \le m, 1 \le k \le n\}.\end{aligned}$$

Define a measure ν on Ω by $\nu(E) = \mu'\{M' \in \Omega' : M' \cap \mathcal{H} \in E\}$. Since ν is invariant under rigid motions of $\mathcal{H}$ and satisfies

$$\nu\{M \in \Omega : v \in M, w \notin M\} = \mu'\{M' \in \Omega' : v \in M', w \notin M'\} = \|v - w\|$$

it follows that $\nu = \mu$. □

Remarks. (i) Let Γ be the set of vertices of a tree. The usual integer-valued distance function $d(x,y)$ between elements of Γ is a measure definite kernel. To see this, fix a vertex $o \in \Gamma$ and for each $x \in \Gamma$ let $S_x = [o,x]$, the geodesic path between o and x. Then $d(x,y) = \mu(S_x \triangle S_y)$, where μ is Lebesgue measure on the geometric realization of the tree. (However it is easy to see that not all measure definite kernels arise from the distance function on a tree.) A. Valette [V, Section 3] has given an appropriate construction of Lebesgue measure for real trees from which it follows similarly that the distance function in a real tree is measure definite.

(ii) It is not true that f^2 is negative definite whenever f is measure definite. Let $\Gamma = \{1,2,3,4\}$ be the set of vertices of the tree in Figure 1.

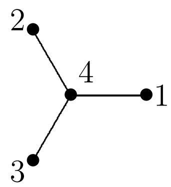

FIGURE 1. A tree with four vertices

Thus $d(1,4) = d(2,4) = d(3,4) = 1$ and $d(1,2) = d(2,3) = d(3,1) = 2$ By the previous remark, $d(x,y)$ is measure definite. There is clearly no set $\{v_1, v_2, v_3, v_4\}$ of vectors in a Hilbert space such that $d(x,y) = \|v_x - v_y\|$. Therefore d^2 is not negative definite.

2. PROPERTY (T)

We give an application of the preceding ideas to characterize countable groups with property (T) in terms of measure preserving actions on infinite measure spaces. Another characterization has been given by K. Schmidt [S, Proposition 2.10], A. Connes and B. Weiss [CW] in terms of measure preserving actions on finite measure spaces, using the notion of strong ergodicity.

We recall the Delorme–Guichardet characterization of groups with property (T) [HV, Chapter 5, Theorem 20], which may be expressed as follows: a group has property (T) if and only if every left Γ-invariant negative definite kernel on Γ is bounded.

Theorem 2.1. *A countable group Γ has property (T) if and only if it satisfies the following condition: for every measure-preserving action of Γ on a measure space $(\Omega, \mathcal{B}, \mu)$ and every set $S \in \mathcal{B}$ such that $\mu(S \triangle gS) < \infty$ for all $g \in \Gamma$, we have*

$$\sup_{g\in\Gamma} \mu(S \triangle gS) < \infty.$$

PROOF: Suppose that Γ has property (T), and that we are given a measure preserving action as described. Define a left Γ-invariant measure definite kernel on Γ by $f(x,y) = \mu(xS \triangle yS)$. By Proposition 1.1, f is negative definite and so uniformly bounded, by the Delorme–Guichardet result.

Conversely, suppose that the condition in the theorem holds, and let f be a left Γ-invariant negative definite kernel on $\Gamma \times \Gamma$. By Proposition 1.4 $\sqrt{f}$ is measure definite with corresponding regular Borel measure μ_0 defined on the space $\Omega_0 = \{0,1\}^\Gamma \backslash \{(0,0,0,\dots),(1,1,1,\dots)\}$. We must show that μ_0 is invariant under the natural action of Γ on Ω_0 defined by $(gc)(x_0) = c(g^{-1}x_0)$. Recall that $\mathcal{C}$ is the family of all cylinder sets of the form

$$\begin{aligned} A &= (S^0_{x_1} \cap \dots \cap S^0_{x_m}) \backslash (S^0_{y_1} \cup \dots \cup S^0_{y_n}) \\ &= \{c \in \Omega_0 : c(x_i) = 1, c(y_j) = 0, 1 \le i \le m, 1 \le j \le n\} \end{aligned}$$

where $\{x_1, \dots, x_m, y_1, \dots, y_n\} \subset \Gamma$ and $m, n \ge 1$. Observe that A is compact and open in Ω_0.

To pass from A to gA, one must apply g to each of the x_i's and y_j's. Since $\|v_{gx} - v_{gy}\|^2 = f(gx, gy) = f(x,y) = \|v_x - v_y\|^2$, it follows from Proposition 1.4(ii) that the measures $\mu_0(A)$ and $\mu_0(g(A))$ coincide. We have therefore shown that μ_0 and μ_0^g coincide on all sets in $\mathcal{C}$.

Suppose that the elements of G are labelled, x_1, x_2, etc. Let $\mathcal{C}_1$ be the subset of $\mathcal{C}$ containing those sets of the form

$$\{c \in \Omega_0 : (c(x_1), c(x_2), \dots, c(x_n)) = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)\}$$

where n is a positive integer, where each ϵ_j is either 0 or 1, and where each of the values 0 and 1 occurs at least once.

The sets in $\mathcal{C}_1$ satisfy the following properties: (1) $\mathcal{C}_1$ is a neigborhood base for Ω; (2) if two sets in $\mathcal{C}_1$ are not disjoint, then one contains the other; (3) $\mathcal{C}_1$ contains no infinite ascending chain.

Let U be an open subset of Ω. It follows from the three properties of $\mathcal{C}_1$ that U is the disjoint union of those sets in $\mathcal{C}_1$ maximal with respect to being contained in U. Thus $\mu_0(U) = \mu_0(g(U))$. Since the measures μ_0 and μ_0^g are regular and they coincide on open sets, they must agree on all Borel sets. Therefore the action of Γ on Ω_0 is measure-preserving.

Now write $S = S_e^0$. Since $\sqrt{f}$ is measure definite, $\mu_0(S \triangle gS) < \infty$ for all $g \in \Gamma$. The condition in the theorem implies that $\sup_{g\in\Gamma} \mu_0(S\triangle gS) < \infty$. The negative definite kernel f is therefore bounded and so Γ has property (T). $\square$

Questions. Several questions remain.

(i) Can measure definite kernels be characterized internally, without reference to measure spaces?

(ii) Is there a result analogous to Theorem 2.1 for locally compact groups?

(iii) Can the action of Γ in Theorem 2.1 be assumed to be ergodic?

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Department of Mathematics, University of Newcastle, NSW 2308, AUSTRALIA

E-mail address: `guyan@@maths.newcastle.edu.au`

Istituto di Matematica e Fisica, Università di Sassari, via Vienna 2, 07100 Sassari, ITALIA

E-mail address: steger@@ssmain.uniss.it