## MALNORMAL SUBGROUPS OF LATTICES AND THE PUKÁNSZKY INVARIANT IN GROUP FACTORS

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ABSTRACT. Let G be a connected semisimple real algebraic group. Assume that  $G(\mathbb{R})$  has no compact factors and let  $\Gamma$  be a torsionfree uniform lattice subgroup of  $G(\mathbb{R})$ . Then  $\Gamma$  contains a malnormal abelian subgroup A. This implies that the II<sub>1</sub> factor VN( $\Gamma$ ) contains a masa  $\mathfrak{A}$  with Pukánszky invariant { $\infty$ }.

### 1. INTRODUCTION

A subgroup  $\Gamma_0$  of a group  $\Gamma$  is malnormal if  $x\Gamma_0 x^{-1} \cap \Gamma_0 = \{1\}$  for all  $x \in \Gamma - \Gamma_0$ . An abelian malnormal subgroup is necessarily maximal abelian. The main result of this article is Theorem 1.1, which rests upon work of Prasad and Rapinchuk [PrR].

**Theorem 1.1.** Let G be a connected semisimple real algebraic group and let d be the  $\mathbb{R}$ -rank of G. Assume that  $G(\mathbb{R})$  has no compact factors and let  $\Gamma$  be a torsion-free uniform lattice subgroup of  $G(\mathbb{R})$ . Then  $\Gamma$ contains a malnormal abelian subgroup  $A \cong \mathbb{Z}^d$ .

Theorem 1.1 will be applied to the group factor  $VN(\Gamma)$ . Recall that if  $\Gamma$  is a group, then the von Neumann algebra  $VN(\Gamma)$  is the convolution algebra

$$VN(\Gamma) = \{ f \in \ell^2(\Gamma) : f \star \ell^2(\Gamma) \subseteq \ell^2(\Gamma) \}.$$

It is well known that if  $\Gamma$  is an infinite conjugacy class group then  $VN(\Gamma)$  is a factor of type II<sub>1</sub>. This is true if  $\Gamma$  is a lattice in a semisimple Lie group [GHJ, Lemma 3.3.1]. If  $\Gamma_0$  is a subgroup of  $\Gamma$ , then  $VN(\Gamma_0)$  embeds naturally as a subalgebra of  $VN(\Gamma)$  via  $f \mapsto \overline{f}$ , where

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

This article is concerned with examples where  $\Gamma_0 = A$  is an abelian subgroup of  $\Gamma$  and  $\mathfrak{A} = \operatorname{VN}(A)$  is a maximal abelian  $\star$ -subalgebra (masa) of  $\mathfrak{M} = \operatorname{VN}(\Gamma)$ . Recall that  $\mathfrak{A}$  is the von Neumann subalgebra of  $B(\ell^2(\Gamma))$  defined by the left convolution operators

$$\lambda(f):\phi\mapsto f\star\phi$$

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where  $f \in \ell^2(A)$  and  $f \star \ell^2(\Gamma) \subseteq \ell^2(\Gamma)$ . The algebra  $\mathfrak{A}$  also acts on  $\ell^2(\Gamma)$  by right convolution

$$\rho(f):\phi\mapsto\phi\star f.$$

Let  $\mathfrak{A}^{opp}$  be the von Neumann subalgebra of  $B(\ell^2(\Gamma))$  defined by this right action of  $\mathfrak{A}$ . Let  $\mathfrak{B}$  be the von Neumann subalgebra of  $B(\ell^2(\Gamma))$ generated by  $\mathfrak{A} \cup \mathfrak{A}^{opp}$  and let p denote the orthogonal projection of  $\ell^2(\Gamma)$  onto the closed subspace generated by  $\mathfrak{A}$ . Then p is in the centre of the commutant  $\mathfrak{B}'$ , and  $\mathfrak{B}'p$  is abelian. The von Neumann algebra  $\mathfrak{B}'(1-p)$  is of type I and may therefore be expressed as a direct sum  $\mathfrak{B}_{n_1} \oplus \mathfrak{B}_{n_2} \oplus \ldots$  of algebras  $\mathfrak{B}_{n_i}$  of type  $I_{n_i}$ , where  $1 \leq n_1 < n_2 < \cdots \leq \infty$ . The *Pukánszky invariant* [SS2, Chapter 7] is the set  $\{n_1, n_2, \ldots\}$ . It is an isomorphism invariant of the pair  $(\mathfrak{A}, \mathfrak{M})$ . It has been shown [NeS, Corollary 3.3] that each nonempty subset S of the natural numbers containing 1 can be realized as the Pukánszky invariant of some masa in the hyperfinite  $II_1$  factor R. This was extended [SS1, DSS] to subsets Scontaining  $\infty$  for R and for the free group factor. It was later extended [Whi] to arbitrary subsets  $S \subset \mathbb{N} \cup \{\infty\}$  for R (and for certain other McDuff factors).

It is known that every factor of type II<sub>1</sub> contains a singular masa [Po1]. S. Popa [Po2, Remark 3.4] showed that if the Pukánszky invariant of  $(\mathfrak{A}, \mathfrak{M})$  does not contain 1, then  $\mathfrak{A}$  is a singular masa in  $\mathfrak{M}$ . K. Dykema [Dyk] has shown (using Voiculescu's free entropy dimension) that the Pukánszky invariant of any masa in the free group factor must either contain  $\infty$  or be unbounded. This means that it is not possible for any singleton other than  $\{\infty\}$  to be a possible Pukánszky invariant occurring in every II<sub>1</sub> factor. Jolissaint [Jol] has shown that if  $F_0$  is the cyclic subgroup generated by the first generator of Thompson's group F then  $VN(F_0)$  has Pukánszky invariant  $\{\infty\}$ . A natural question arises.

• Does every II<sub>1</sub> factor  $\mathfrak{M}$  contain a masa  $\mathfrak{A}$  with Pukánszky invariant  $\{\infty\}$ ?

This article uses Theorem 1.1 to provide an affirmative answer for  $\mathfrak{M} = \operatorname{VN}(\Gamma)$ , where  $\Gamma$  is a torsion-free uniform lattice subgroup of a connected semisimple real algebraic group G without compact factors. If G has  $\mathbb{R}$ -rank  $\geq 2$ , then  $\operatorname{VN}(\Gamma)$  has Kazhdan's property T. This is the first result on possible values of the Pukánszky invariant in a II<sub>1</sub> factor with property T.

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#### 2. Malnormal Abelian subgroups of lattices

This section is devoted to the proof of Theorem 1.1. Let G be a connected semisimple real algebraic group and let d be the  $\mathbb{R}$ -rank of G. Assume that  $G(\mathbb{R})$  has no compact factors and let  $\Gamma$  be a torsionfree uniform lattice subgroup of  $G(\mathbb{R})$ .

Since  $\Gamma$  is finitely generated,  $\Gamma < G(\mathbb{K})$  for some finitely generated subfield  $\mathbb{K}$  of  $\mathbb{R}$ . By the Borel Density Theorem [Mar, Chapter II, Corollary 4.4],  $\Gamma$  is Zariski dense in  $G(\mathbb{R})$ . Therefore, according to Theorems 1 and 2 of [PrR], there exists a maximal abelian torus subgroup T of G with the following properties.

- (1) The  $\mathbb{R}$ -rank of T is d.
- (2)  $A = T(\mathbb{R}) \cap \Gamma$  is a uniform lattice in  $T(\mathbb{R})$ .
- (3) T has no proper algebraic subgroups defined over  $\mathbb{K}$ .

Moreover, T is the K-Zariski closure of a single R-regular element  $x_0$ in A [PrR]. In fact there are many such elements  $x_0$  [PrR, Remark 2]. We claim that A is a malnormal subgroup of  $\Gamma$ . To this end, fix an arbitrary element  $x \in \Gamma - A$ . We must show that  $xAx^{-1} \cap A = \{1\}$ .

Let  $T_s$  (respectively  $T_a$ ) be the maximal  $\mathbb{R}$ -split (respectively  $\mathbb{R}$ anisotropic) subtorus of T. Then  $T = T_s \cdot T_a$  (an almost direct product) [Bor, Proposition 8.15] and  $T_s$  is a maximal  $\mathbb{R}$ -split torus in G [PrR, Remark 1]. Thus  $T(\mathbb{R}) = S \cdot C$ , where  $S = T_s(\mathbb{R}) \cong \mathbb{R}^d$  and  $C = T_a(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^r$ , where r is the dimension of  $T_a$ . Since  $\Gamma$  is torsion free and discrete, A is a uniform lattice in S. In particular,  $A \cong \mathbb{Z}^d.$ 

Since T is the  $\mathbb{K}$ -Zariski closure of A, it follows that T is defined over K. Since  $x \in \Gamma \subseteq G(\mathbb{K})$ ,  $xTx^{-1}$  is also defined over K. According to condition (3), there are only two possibilities:

- $T \cap xTx^{-1} = \{1\};$   $T = xTx^{-1}.$

In the first case we also have  $T(\mathbb{K}) \cap xT(\mathbb{K})x^{-1} = \{1\}$ , and a fortiori  $A \cap xAx^{-1} = \{1\}$ , as required.

To show that the second case does not occur, assume that  $T = xTx^{-1}$ . This implies that  $T(\mathbb{R})$  is stable under conjugation by x. Also  $\Gamma$  is stable under conjugation by x. Therefore  $xAx^{-1} = A$ , since  $A = \Gamma \cap T(\mathbb{R})$ . There are two possibilities to consider for the action  $\alpha_x : a \mapsto xax^{-1}$ on A.

(a)  $\alpha_x$  fixes only the trivial element of A;

(b)  $\alpha_x$  fixes some nontrivial element of A.

Since conjugation by x stabilizes T, it also stabilizes  $T_s$  and  $T_a$  separately [Bor, Proposition 8.15(3)]. Thus  $xSx^{-1} = S$ . The symmetric space of  $G(\mathbb{R})$  is  $X = G(\mathbb{R})/K$ , where K is a maximal compact subgroup of  $G(\mathbb{R})$ . The group  $\Gamma$  acts freely on X, since it is torsion free. Since  $T_s$  is a maximal  $\mathbb{R}$ -split torus of G, there is a unique flat F in X

such that SF = F and S acts simply transitively on F [Mos, Lemma 5.1]. Now xF is another such flat, since

$$SxF = (xSx^{-1})xF = xSF = xF.$$

Hence xF = F.

The action of x on F is by some rigid motion and the action of A on F is by translations. No nontrivial element in  $\Gamma$  can act trivially on F, so we can calculate the conjugation by x of any element  $y \in A$  by considering the actions of these elements on F. The two cases above correspond to:

- (a) x acts on F by a rigid motion whose linear part has trivial 1-eigenspace;
- (b) x acts on F by a rigid motion whose linear part has nontrivial 1-eigenspace.

In case (a), x necessarily has a fixed point in F. Therefore x = 1, since  $\Gamma$  acts freely on X.

Consider case (b). The algebraic subgroup  $Z_G(x) \cap T$ , which consists of the elements commuting with x, is defined over  $\mathbb{K}$ . In case (b),  $Z_G(x) \cap T$  contains nontrivial elements of A. That is, it has nontrivial  $\mathbb{K}$ -points. Hence, it must be nontrivial (as an algebraic group). By condition (3) it must be all of T. Hence x commutes with every element of T. Therefore the algebraic closure of  $\{x\} \cup T$  over  $\mathbb{K}$  is commutative. However T is a maximal abelian subgroup over  $\mathbb{K}$ , and so  $x \in T(\mathbb{K})$ . Therefore  $x \in A$ , contrary to assumption. This completes the proof of Theorem 1.1.

# 3. The Pukánszky invariant

The following result was proved in [RoS, Proposition 3.6] and later extended in [SS1, Theorem 4.1].

**Proposition 3.1.** Suppose that A is an abelian subgroup of a countable group  $\Gamma$  such that  $\mathfrak{A} = \mathrm{VN}(A)$  is a masa of  $\mathrm{VN}(\Gamma)$ . If A is malnormal in  $\Gamma$  then the Pukánszky invariant of  $\mathfrak{A}$  contains precisely one element  $n = \#(A \setminus \Gamma/A - \{A\}).$ 

In view of Theorem 1.1, the next result is enough to provide examples of masas with Pukánszky invariant  $\{\infty\}$ .

**Theorem 3.2.** Let G be a connected semisimple real algebraic group. Assume that  $G(\mathbb{R})$  has no compact factors. Let  $\Gamma$  be a torsion free uniform lattice subgroup of  $G(\mathbb{R})$  and let  $A < \Gamma$  be an abelian subgroup. Then

$$#(A\backslash\Gamma/A) = \infty.$$

*Proof.* Suppose that  $\#(A \setminus \Gamma/A) < \infty$ . Then

$$\Gamma = \bigcup_{x \in F} AxA$$

where  $F \subset \Gamma$  is finite. Taking Zariski closures, it follows from the Borel Density Theorem that

(1) 
$$G(\mathbb{R}) = \overline{\Gamma} = \bigcup_{x \in F} \overline{AxA}.$$

For each  $x \in F$ , AxA is locally closed in the Zariski topology, since it is an orbit of  $\overline{A}$  acting on  $G(\mathbb{R})/\overline{A}$  [Zim, Corollary 3.1.5]. This means that  $\overline{A}x\overline{A} = U \cap E$  where U is Zariski-open and E is Zariski closed. Therefore

$$\overline{AxA} = \overline{U \cap E} \subseteq E \subseteq (U \cap E) \cup (G(\mathbb{R}) \setminus U) = (\overline{AxA}) \cup (G(\mathbb{R}) \setminus U).$$

Since  $U \subseteq G(\mathbb{R})$  is Zariski open (and  $G(\mathbb{R})$  is Zariski connected),  $G(\mathbb{R}) \setminus U$  has measure zero, relative to Haar measure  $\mu$  on  $G(\mathbb{R})$  [Mar, Chapter I, Proposition 2.5.3]. Therefore,

(2) 
$$\mu(\overline{AxA}) = \mu(\overline{Ax}\overline{A}).$$

Now we show that  $\mu(\bar{A}x\bar{A}) = 0$ . Each element of A is semisimple, since each element of  $\Gamma$  is [Mos, Section 11]. Therefore each element of the Zariski closure  $\bar{A}$  is also semisimple; in other words,  $\bar{A}$  is a torus subgroup. The dimension of  $\bar{A}$  as a Lie group is no larger than the absolute rank d' of G (the rank over  $\mathbb{C}$  of the Lie algebra of G). The number of positive roots of the compexified Lie algebra of G is at least d', and the total number of roots is at least 2d'. Thus the total dimension of the root spaces is at least 2d'. This means that d' is at most one third of the dimension of G. Thus the dimension of  $\bar{A} \times \bar{A}$ is at most two thirds the dimension of G. The map  $(a_1, a_2) \mapsto a_1 x a_2$ from  $\bar{A} \times \bar{A}$  to  $G(\mathbb{R})$  is  $C^{\infty}$  (in fact polynomial). Therefore, by the above dimension count, its image has measure zero. It follows from (2) that  $\mu(\bar{A}x\bar{A}) = 0$ . However, this contradicts (1), thereby proving the result.

An immediate consequence of Theorem 1.1 and Theorem 3.2 is

**Corollary 3.3.** Let G be a connected semisimple real algebraic group such that  $G(\mathbb{R})$  has no compact factors. Let  $\Gamma$  be a uniform lattice subgroup of  $G(\mathbb{R})$ . Then there exists an abelian subgroup  $A < \Gamma$  such that VN(A) is a masa of  $VN(\Gamma)$  with Pukánszky invariant  $\{\infty\}$ .

*Proof.* Let A be the malnormal abelian subgroup of  $\Gamma$  whose existence is assured by Theorem 1.1. By [SS2, p.123], VN(A) is a masa of VN( $\Gamma$ ). The result follows from Proposition 3.1.

**Remark 3.4.** A similar result was obtained by geometrical methods in [Rob, Theorem 4.6], if  $\Gamma$  is the fundamental group of a compact locally

symmetric space M of constant negative curvature and A is generated by the homotopy class of a simple closed geodesic in M.

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