TORSION IN BOUNDARY COINVARIANTS AND K-THEORY FOR AFFINE BUILDINGS

GUYAN ROBERTSON

Abstract. Let $(G, \mathfrak{I}, N, S)$ be an affine topological Tits system, and let Γ be a torsion free cocompact lattice in G. This article studies the coinvariants $H_0(\Gamma; C(\Omega, \mathbb{Z}))$, where Ω is the Furstenberg boundary of G. It is shown that the class $[\mathbf{1}]$ of the identity function in $H_0(\Gamma; C(\Omega, \mathbb{Z}))$ has finite order, with explicit bounds for the order.

A similar statement applies to the K_0 group of the boundary crossed product C^*-algebra $C(\Omega) \rtimes \Gamma$. If the Tits system has type $\widetilde{A}_2$, exact computations are given, both for the crossed product algebra and for the reduced group C^*-algebra.

1. Introduction

This article is concerned with coinvariants for group actions on the boundary of an affine building. The results are most easily stated for subgroups of linear algebraic groups. Let k be a non-archimedean local field with finite residue field $\overline{k}$ of order q. Let G be the group of k-rational points of an absolutely almost simple, simply connected linear algebraic k-group. Then G acts on its Bruhat-Tits building Δ, and on its Furstenberg boundary Ω.

Let Γ be a torsion free lattice in G. The abelian group $C(\Omega, \mathbb{Z})$ of continuous integer-valued functions on Ω has the structure of a Γ-module. The module of Γ-coinvariants $\Omega_\Gamma = H_0(\Gamma; C(\Omega, \mathbb{Z}))$ is a finitely generated group. We prove that the class $[\mathbf{1}]$ in Ω_Γ of the constant function $\mathbf{1} \in C(\Omega, \mathbb{Z})$ has finite order. If G is not one of the exceptional types $\widetilde{E}_8, \widetilde{F}_4$ or $\widetilde{G}_2$, then the order of $[\mathbf{1}]$ is less than $\mathrm{covol}(\Gamma)$, where the Haar measure μ on G is normalized so that an Iwahori subgroup of G has measure 1. There is a weaker estimate for groups of exceptional type. If G has rank 2 then the estimates are significantly improved.

Date: January 15, 2005.

2000 *Mathematics Subject Classification.* 51E24, 46L80.

Key words and phrases. affine building, Furstenberg boundary, K-theory, C^*-algebra.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

The topological action of Γ on the Furstenberg boundary is encoded in the crossed product C^*-algebra $\mathcal{A}_\Gamma = C(\Omega) \rtimes \Gamma$. Embedded in $\mathcal{A}_\Gamma$ is the reduced group C^*-algebra $C^*_r(\Gamma)$, which is the completion of the complex group algebra of Γ in the regular representation as operators on $\ell^2(\Gamma)$. The action of Γ on Ω is amenable, so the K-theory of $\mathcal{A}_\Gamma$ is computable by known results, in contrast to that of $C^*_r(\Gamma)$, which rests on the validity of the Baum-Connes conjecture. The natural embedding $C(\Omega) \to \mathcal{A}_\Gamma$ induces a homomorphism

$$\varphi : \Omega_\Gamma \to K_0(\mathcal{A}_\Gamma)$$

and $\varphi([\mathbf{1}]) = [\mathbf{1}]_{K_0}$, the class of $\mathbf{1}$ in the K_0-group of $\mathcal{A}_\Gamma$. Therefore $[\mathbf{1}]_{K_0}$ has finite order in $K_0(\mathcal{A}_\Gamma)$.

If Γ is a torsion free lattice in $G = \mathrm{SL}_3(k)$ then exact computations can be performed. The Baum-Connes Theorem of V. Lafforgue [La] is used to compute $K_*(C^*_r(\Gamma))$ and the results of [RS] are used to compute $K_*(\mathcal{A}_\Gamma)$. In particular $K_0(C^*_r(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)}$, a free abelian group, one of whose generators is the class $[\mathbf{1}]$. The embedding of $C^*_r(\Gamma)$ into $\mathcal{A}_\Gamma$ induces a homomorphism $\psi : K_*(C^*_r(\Gamma)) \to K_*(\mathcal{A}_\Gamma)$. This homomorphism is not injective, since $[\mathbf{1}]$ has finite order in $K_0(\mathcal{A}_\Gamma)$. The computations at the end of the article suggest that the *only* reason for failure of injectivity of the homomorphism ψ is the fact that $[\mathbf{1}]$ has finite order in $K_0(\mathcal{A}_\Gamma)$.

Much of this article considers the more general case where Γ is a subgroup of a topological group G with a BN-pair, and Γ acts on the boundary Ω of the affine building of G.

The results are organized as follows. Sections 2 and 3 state and prove the main result concerning the class $[\mathbf{1}]$ in Ω_Γ. Section 4 gives improved estimates in the rank 2 case. Section 5 studies the connection with the K-Theory of the boundary algebra $\mathcal{A}_\Gamma$. Comparison with K-theory of the reduced C^*-algebra $C^*_r(\Gamma)$ is made in Section 6, which contains some exact computational results for buildings of type $\widetilde{A}_2$.

2. Torsion in Boundary Coinvariants

Let $(G, \mathfrak{I}, N, S)$ be an affine topological Tits system [Gd, Definition 2.3]. Then G is a group with a BN-pair in the usual algebraic sense [Ti1, Section 2] and the Weyl group $W = N/(\mathfrak{I} \cap N)$ is an infinite Coxeter group with generating set S. The subgroup $\mathfrak{I}$ of G is called an *Iwahori* subgroup. A subgroup of G is *parahoric* if it contains a conjugate of $\mathfrak{I}$. The topological requirements are that G is a second countable locally compact group and that all proper parahoric subgroups of G are open and compact [Gd, Definition 2.3].

Let $n+1 = |S|$ be the *rank* of the Tits system. The group G acts on the Tits complex Δ, which is an affine building of dimension n. It will be assumed throughout that Δ is irreducible; in other words, the Coxeter group W is not a direct product of nontrivial Coxeter groups. Denote by Δ^i the set of i-simplices of Δ, $(0 \leq i \leq n)$. The vertices of Δ are the maximal proper parahoric subgroups of G, and a finite set of such subgroups spans a simplex in Δ if and only if its intersection is parahoric. The action of G on Δ is by conjugation of subgroups. The building Δ is a union of n-dimensional subcomplexes, called *apartments*. Each apartment is a Coxeter complex, with Coxeter group W.

Associated with the Coxeter system (W, S) there is a Coxeter diagram of type $\widetilde{X}_n$ $(X = A, B, \dots, G)$, whose vertex set I is a set of $n+1$ *types*, which are in natural bijective correspondence with the elements of S. Each vertex $v \in \Delta^0$ has a *type* $\tau(v) \in I$. The type of a simplex in Δ is the set of types of its vertices. By construction, the action of G on Δ preserves types. A type $\mathfrak{t} \in I$ is *special* if deleting $\mathfrak{t}$ and all the edges containing $\mathfrak{t}$ from the diagram of type $\widetilde{X}_n$ results in the diagram of type X_n (the diagram of the corresponding finite Coxeter group). A vertex $v \in \Delta$ is said to be *special* if its type $\tau(v)$ is special [BT, 1.3.7].

A simplex of maximal dimension n in Δ is called a *chamber*. Every chamber has exactly one vertex of each type. If σ is any chamber containing the vertex v then the codimension-1 face of σ which does not contain v has type $I - \{\tau(v)\}$.

The action of G on Δ is *strongly transitive*, in the sense that G acts transitively on the set of pairs (σ, A) where σ is a chamber contained in an apartment A of Δ. The building Δ is *locally finite*, in the sense that the number of chambers containing any simplex is finite, and it is *thick*, in the sense that each simplex of dimension $n-1$ is contained in at least three chambers. If τ is a simplex in Δ of dimension $n-1$ and type $I - \{\mathfrak{t}\}$, then the number of chambers of Δ which contain τ is $q_{\mathfrak{t}} + 1$ where $q_{\mathfrak{t}} \geq 2$. The integer $q_{\mathfrak{t}}$ depends only on $\mathfrak{t}$; not on τ.

Associated with the group G there is also a spherical building, the building at infinity Δ_∞. The *boundary* Ω of Δ is the set of chambers of Δ_∞, endowed with a natural compact totally disconnected topology, which we shall describe later on. Since G acts transitively on the chambers of Δ_∞, Ω may be identified with the topological homogeneous space G/B, where the *Borel subgroup* B is the stabilizer of a chamber of Δ_∞.

Example 2.1. A standard example is $G = \mathrm{SL}_{n+1}(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers. In this case B is the subgroup of upper triangular matrices in G, and Ω is the Furstenberg boundary of G.

If Γ is a subgroup of G, then Γ acts on Ω, and the abelian group $C(\Omega, \mathbb{Z})$ of continuous integer-valued functions on Ω has the structure of a Γ-module. The module of Γ-coinvariants, $C(\Omega, \mathbb{Z})_\Gamma$, is the quotient of $C(\Omega, \mathbb{Z})$ by the submodule generated by $\{g\cdot f - f : g \in \Gamma, f \in C(\Omega, \mathbb{Z})\}$. Recall that $C(\Omega, \mathbb{Z})_\Gamma$ is the homology group $H_0(\Gamma; C(\Omega, \mathbb{Z}))$. For the rest of this article, $C(\Omega, \mathbb{Z})_\Gamma$ will be denoted simply by Ω_Γ. Define $c(\Gamma) \in \mathbb{Z}_+ \cup \{\infty\}$ to be the number of Γ-orbits of chambers in Δ.

If Γ is a torsion free cocompact lattice in G, then $c(\Gamma)$ is the number of n-cells of the finite cell complex $\Delta\backslash\Gamma$. Suppose that the Haar measure μ on G has the Tits normalization $\mu(\mathfrak{I}) = 1$ [Ti2, §3.7]. Then $c(\Gamma) = \mathrm{covol}(\Gamma)$.

We shall see below that if Γ is a torsion free cocompact lattice in G then Ω_Γ is a finitely generated abelian group. Note that such a torsion free lattice Γ acts freely and properly on Δ [Gd, Lemma 2.6, Lemma 3.3]. If $f \in C(\Omega, \mathbb{Z})$ then $[f]$ will denote its class in Ω_Γ. Also, $\mathbf{1}$ will denote the constant function defined by $\mathbf{1}(\omega) = \omega$ for $\omega \in \Omega$.

Theorem 2.2. *Let $(G, \mathfrak{I}, N, S)$ be an affine topological Tits system and let Γ be a torsion free lattice in G. Then Ω_Γ is a finitely generated abelian group and the following statements hold.*

1. *The element $[\mathbf{1}]$ has finite order in Ω_Γ.*
2. *If $\mathfrak{s} \in I$ is a special type, then the order of $[\mathbf{1}]$ in Ω_Γ satisfies*
$$\mathrm{ord}([\mathbf{1}]) < q_{\mathfrak{s}} \cdot \mathrm{covol}(\Gamma)\,.$$
3. *If, in addition, G is not one of the exceptional types $\widetilde{G}_2, \widetilde{F}_4, \widetilde{E}_8$, then*
$$\mathrm{ord}([\mathbf{1}]) < \mathrm{covol}(\Gamma)\,.$$

Remark 2.3. A torsion free lattice in G is automatically cocompact [Se2, II.1.5].

Remark 2.4. Suppose that Γ is isomorphic to a subgroup of a group Γ' and that the action of Γ on Ω extends to an action of Γ' on Ω. Then there is a natural surjection $\Omega_\Gamma \to \Omega_{\Gamma'}$. It follows that Theorem 2.2 remains true if Γ is replaced by any such group Γ'.

Remark 2.5. The group Ω_Γ depends only on Γ and not on the ambient group G. This follows from the rigidity results of [KL], if $n \geq 2$, and from [Gr] if $n = 1$.

We now describe briefly how Theorem 2.2 applies to algebraic groups. Let k be a non-archimedean local field and let G be the group of k-rational points of an absolutely almost simple, simply connected linear algebraic k-group : e.g. $k = \mathbb{Q}_p$, $G = \mathrm{SL}_{n+1}(\mathbb{Q}_p)$. Associated with G there is a topological Tits system of rank $n+1$, where G has k-rank n [IM]. Now G acts properly on the corresponding Bruhat-Tits building Δ [Ti2, §2.1], and on the boundary $\Omega = G/B$, where B is a Borel subgroup [BM, Section 5].

Let q be the order of the residue field $\overline{k}$. For each type $\mathfrak{t} \in I$ there is an integer $d(\mathfrak{t})$ such that $q_\mathfrak{t} = q^{d(\mathfrak{t})}$. That is, any simplex τ of codimension one and type $I - \{\mathfrak{t}\}$ is contained in $q^{d(\mathfrak{t})} + 1$ chambers [Ti2, §2.4]. If G is k-split (i.e. there is a maximal torus $T \subset G$ which is k-split) then $d(\mathfrak{t}) = 1$ for all $\mathfrak{t} \in I$ [Ti2, §3.5.4].

If k has characteristic zero, then the condition that Γ is torsion free can be omitted from Theorem 2.2. Recall that a non-archimedean local field of characteristic zero is a finite extension of $\mathbb{Q}_p$, for some prime p.

Corollary 2.6. *Let k be a non-archimedean local field of characteristic zero. Let G be the group of k-rational points of an absolutely almost simple, simply connected linear algebraic k-group. If Γ is a lattice in G, then the class $[\mathbf{1}]$ has torsion in Ω_Γ.*

Proof. A lattice Γ in G is automatically cocompact [M, Proposition IX, 3.7]. By Selberg's Lemma [Gd, Theorem 2.7], Γ has a torsion free subgroup Γ_0 of finite index. Now Theorem 2.2 implies that $[\mathbf{1}]$ has finite order in Ω_{Γ_0}. The result follows from the observation that there is a natural surjection $\Omega_{\Gamma_0} \to \Omega_\Gamma$. □

3. Proof of Theorem 2.2

Throughout this section, the assumptions of Theorem 2.2 are in force. Before proving Theorem 2.2, we require some preliminaries. Recall that a gallery of type $i = (i_1, \dots, i_k)$ is a sequence of chambers $(\sigma_0, \sigma_1, \dots \sigma_k)$ such that each pair of successive chambers σ_{j-1}, σ_j meet in a common face of type $I - \{i_j\}$. Choose a special type $\mathfrak{s} \in I$, which will remain fixed throughout this section. Fix once and for all the following data.

- (**A1**) An apartment A in Δ.
- (**A2**) A sector S in A with base vertex v of type $\mathfrak{s}$ and base chamber C.
- (**A3**) The unique vertex $v' \in S$ of type $\mathfrak{s}$, obtained by reflecting v in a codimension-1 face of C.
- (**A4**) The unique chamber C' containing v' which is the base chamber of a subsector of S.

- (**A5**) A minimal gallery of type $i = (i_1, \dots, i_k)$ from C to C', where $i_1 = \mathfrak{s}$. This minimal gallery necessarily lies inside S.

These data are illustrated by Figure 1, which shows part of an apartment in a building of type $\widetilde{G}_2$ and a minimal gallery from C to C'. Special vertices are indicated by large points.

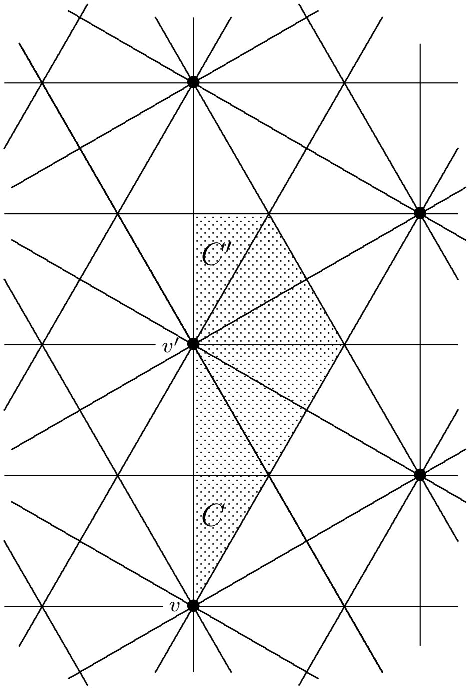


FIGURE 1. Part of an apartment A in a building of type $\widetilde{G}_2$.

Now let $\mathfrak{D} = \Delta^n/\Gamma$, the set of Γ-orbits of chambers of Δ. Since Γ acts freely and cocompactly on Δ, $\mathfrak{D}$ is finite and elements of $\mathfrak{D}$ are in $1-1$ correspondence with the set of n-cells of the finite cell complex Δ/Γ.

If x, $y \in \mathfrak{D}$, let $M_i(x,y)$ denote the number of Γ-orbits of galleries of type i which have initial chamber in x and final chamber in y. If $\sigma_0 \in x$ is fixed then $M_i(x,y)$ is equal to the number of galleries $(\sigma_0, \sigma_1, \dots \sigma_k)$ of type i with final chamber $\sigma_k \in y$. To see this, note that any gallery of type i with initial chamber in x and final chamber in y lies in the Γ-orbit of such a gallery $(\sigma_0, \sigma_1, \dots \sigma_k)$. Moreover, two distinct galleries of this form lie in different Γ-orbits. For suppose that $(\sigma_0, \sigma_1, \dots, \sigma_j, \tau_{j+1}, \dots \tau_k)$ is another such gallery, with $\tau_{j+1} \neq \sigma_{j+1}$, the first chamber at which they differ. Then τ_{j+1} and σ_{j+1} have a common face of codimension one, and so lie in different Γ-orbits, since the action of Γ is free. (If $g\tau_{j+1} = \sigma_{j+1}$, then g must fix every point in the common codimension one face and so $g = 1$.) A similar argument shows that if $\sigma_k \in y$ is fixed then then $M_i(x, y)$ is equal to the number of galleries $(\sigma_0, \sigma_1, \dots \sigma_k)$ of type i with initial chamber $\sigma_0 \in x$. Cocompactness of the Γ-action implies that $M_i(x, y)$ is finite.

If σ is a chamber in Δ, then the number N_i of galleries $(\sigma_0, \sigma_1, \dots, \sigma_k)$ of type i, with final chamber $\sigma_k = \sigma$, is independent of σ. This follows, since G acts transitively on the set Δ^n of chambers of Δ. Note that $N_i > 1$, by thickness of the building Δ.

Two different galleries of type i which have final chamber σ are necessarily in different Γ-orbits, by freeness of the action of Γ. It follows that if $y \in \mathfrak{D}$, then the number of Γ-orbits of galleries $(\sigma_1, \dots, \sigma_k)$ of type i, with $\sigma_k \in y$, is equal to N_i. In other words, for each $y \in \mathfrak{D}$,

$$\sum_{x \in \mathfrak{D}} M_i(x, y) = N_i\,. \tag{1}$$

Recall that if τ is a simplex of Δ of codimension one and type $I - \{\mathfrak{t}\}$, then the number of chambers of Δ which contain τ is $q_{\mathfrak{t}} + 1$ where $q_{\mathfrak{t}} \geq 2$. Thus the number of galleries $(\sigma_0, \sigma_1, \dots, \sigma_k)$ of type i, with final chamber $\sigma_k = \sigma$ (fixed, but arbitrary), is equal to $q_{i_k} q_{i_{k-1}} \dots q_{i_1}$, where $i_1 = \mathfrak{s}$. On the other hand, this number is also equal to the number $q_{i_1} q_{i_2} \dots q_{i_k}$ of galleries $(\sigma_0, \sigma_1, \dots, \sigma_k)$ of type i, with initial chamber $\sigma_0 = \sigma$ (fixed, but arbitrary). It follows that, for each $x \in \mathfrak{D}$,

$$\sum_{y \in \mathfrak{D}} M_i(x, y) = N_i\,. \tag{2}$$

Definition 3.1. Fix a type $\mathfrak{s} \in I$. Let $\alpha_{\mathfrak{s}}$ denote the number of chambers of Δ which contain a fixed vertex u of type $\mathfrak{s}$. Since G acts transitively on the set of vertices of type $\mathfrak{s}$, $\alpha_{\mathfrak{s}}$ does not depend on the choice of the vertex u.

Remark 3.2. The Iwahori subgroup $\mathfrak{I}$ is a chamber of Δ. Let the parahoric subgroup $P_{\mathfrak{s}} < G$ be the vertex of the type $\mathfrak{s}$ of $\mathfrak{I}$. Then $P_{\mathfrak{s}}$ is a maximal compact subgroup of G containing $\mathfrak{I}$ and $\alpha_{\mathfrak{s}} = |P_{\mathfrak{s}} : \mathfrak{I}|$. (In [Gd, Section 3], $\alpha_{\mathfrak{s}}$ is denoted $\tau_{\{\mathfrak{s}\}}$.)

Lemma 3.3. *Let $\mathfrak{s} \in I$ be a special type. Then*

$$N_i < q_{\mathfrak{s}} \cdot \alpha_{\mathfrak{s}}. \tag{3}$$

Proof. Fix a chamber σ_0. We must estimate the number of galleries $(\sigma_0, \sigma_1, \dots, \sigma_k)$ of type i (with initial chamber σ_0).

There are $q_{\mathfrak{s}}$ possible choices for σ_1. Suppose that σ_1 has been chosen and let u be the vertex of σ_1 not belonging to σ_0. By construction, σ_k

also contains u (Figure 2) and so there are less than $\alpha_{\mathfrak{s}}$ possible choices for σ_k. (Note the $\sigma_k \neq \sigma_1$.) Once σ_k has been chosen, there is a unique (minimal) gallery of type $(i_2, \dots, i_k)$ with initial chamber σ_1 and final chamber σ_k. In other words, the gallery $(\sigma_0, \sigma_1, \dots, \sigma_k)$ is uniquely determined, once σ_1 and σ_k are chosen. There are therefore at most $q_{\mathfrak{s}}(\alpha_{\mathfrak{s}} - 1)$ choices for this gallery. □

Remark 3.4. An easy calculation in $\widetilde{A}_2$ buildings shows that the estimate (3) cannot be improved to $N_i \leq \alpha_{\mathfrak{s}}$.

Definition 3.5. Let Γ be a torsion free cocompact lattice in G. If $\mathfrak{s} \in I$, let $n_{\mathfrak{s}}(\Gamma)$ (or simply $n_{\mathfrak{s}}$, if Γ is understood) denote the number of Γ-orbits of vertices of type $\mathfrak{s}$ in Δ.

Recall that $\operatorname{covol}(\Gamma)$ is equal to the number of Γ-orbits of chambers in Δ.

Lemma 3.6. *Fix a type* $\mathfrak{s} \in I$. *Then* $\operatorname{covol}(\Gamma) = n_{\mathfrak{s}}(\Gamma) \cdot \alpha_{\mathfrak{s}}$.

Proof. Choose a set $\mathcal{S}$ of representative vertices from the Γ-orbits of vertices of type $\mathfrak{s}$ in Δ. Thus $|\mathcal{S}| = n_{\mathfrak{s}}(\Gamma)$. For $v \in \mathcal{S}$, let R_v denote the set of chambers containing v. Each R_v contains $\alpha_{\mathfrak{s}}$ chambers. We claim that the number of chambers in $R = \bigcup_{v \in \mathcal{S}} R_v$ equals $\operatorname{covol}(\Gamma)$.

Each chamber in Δ is clearly in the Γ-orbit of some chamber in R. Moreover, any two distinct chambers in R lie in different Γ-orbits. For suppose that $\sigma_v \in R_v$, $\sigma_w \in R_w$ and $g\sigma_v = \sigma_w$, where $g \in \Gamma$. Then $gv = w$, since the action of Γ is type preserving and every chamber contains exactly one vertex of type $\mathfrak{s}$. Therefore $v = w$, since distinct vertices in $\mathcal{S}$ lie in different Γ-orbits. Moreover $g = 1$, since the action of Γ is free. Thus $\sigma_v = \sigma_w$. This shows that there are $\operatorname{covol}(\Gamma)$ chambers in R. □

Before proving Theorem 2.2, we provide more details of the structure of the boundary Ω. Let σ be a chamber in Δ^n and let s be a special vertex of σ. The codimension one faces of σ having s as a vertex determine roots containing σ, and the intersection of these roots is a sector in Δ with base vertex s and base chamber σ. Two sectors are *parallel* if the Hausdorff distance between them is finite. This happens if and only if they contain a common subsector. The boundary Ω of Δ is the set of parallel equivalence classes of sectors in Δ [Ron, Chap. 9.3]. If $\omega \in \Omega$ and if s is a special vertex of Δ then there exists a unique sector $[s, \omega)$ in ω with base vertex s, [Ron, Lemma 9.7].

If $\sigma \in \Delta^n$, let $o(\sigma)$ denote the vertex of σ of type $\mathfrak{s}$. Recall that vertices of type $\mathfrak{s}$ are special. Let $\Omega(\sigma)$ denote the set of boundary

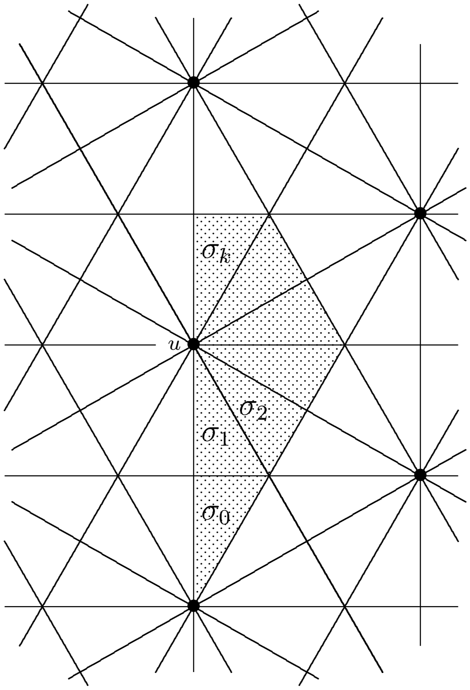


FIGURE 2. A minimal gallery $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ in a $\widetilde{G}_2$ building.

points ω whose representative sectors have base vertex $o(\sigma)$ and base chamber σ. That is,

$$\Omega(\sigma) = \{\omega \in \Omega : \sigma \subset [o(\sigma), \omega)\} .$$

The sets $\Omega(\sigma)$, $\sigma \in \Delta^n$, form a basis for the topology of Ω. Moreover, each $\Omega(\sigma)$ is a clopen subset of Ω. Let γ_i denote the set of ordered pairs $(\sigma, \sigma') \in \Delta^n \times \Delta^n$ such that there exists a gallery of type i from σ to σ'. Then for each $\sigma \in \Delta^n$, $\Omega(\sigma)$ can be expressed as a disjoint union

$$\Omega(\sigma) = \bigsqcup_{(\sigma,\sigma') \in \gamma_i} \Omega(\sigma') . \tag{4}$$

For if $\omega \in \Omega(\sigma)$, then the sector $[o(\sigma), \omega)$ is strongly isometric, in the sense of [Gt, 15.5] to the sector S in the apartment A, as described at the beginning of this section. Let σ' be the image under this strong isometry of the chamber C' in A. Then $(\sigma, \sigma') \in \gamma_i$ and $\omega \in \Omega(\sigma')$. Thus $\Omega(\sigma)$ is indeed a subset of the right hand side of (4). Conversely, each set $\Omega(\sigma')$ on the right hand side of (4) is contained in $\Omega(\sigma)$. For if $(\sigma, \sigma') \in \gamma_i$ and $\omega \in \Omega(\sigma')$ then the strong isometry from $[o(\sigma'), \omega)$

onto S' extends to a strong isometry from $[o(\sigma),\omega)$ onto S [Gt, §15.5 Lemma]. Thus $\omega \in \Omega(\sigma)$.

To check that the union on the right of (4) is disjoint, suppose that $\omega \in \Omega(\sigma_1') \cap \Omega(\sigma_2')$, where $(\sigma,\sigma_1'),(\sigma,\sigma_2') \in \gamma_i$. Then the strong isometry from $[o(\sigma_1'),\omega)$ onto $[o(\sigma_2'),\omega)$ extends to a strong isometry from $[o(\sigma),\omega)$ onto itself, which is necessarily the identity map. In particular, $\sigma_1' = \sigma_2'$.

If $\sigma \in \Delta^n$, let $\chi_\sigma \in C(\Omega,\mathbb{Z})$ denote the characteristic function of $\Omega(\sigma)$. That is

$$\chi_\sigma(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi_\sigma - \chi_{g\sigma} = \chi_\sigma - g.\chi_\sigma$ for each $g \in \Gamma$, the class $[\chi_\sigma]$ of χ_σ in Ω_Γ depends only on the Γ-orbit of σ in Δ^n. If $x = \Gamma\sigma \in \mathfrak{D}$, it therefore makes sense to define

$$[x] = [\chi_\sigma] \in \Omega_\Gamma\,. \tag{5}$$

Now it follows from (4) that, for each $\sigma \in \Delta^n$,

$$\chi_\sigma = \sum_{(\sigma,\sigma')\in\gamma_i} \chi_{\sigma'} = \sum_{y\in\mathfrak{D}} \sum_{\substack{\sigma'\in y \\ (\sigma,\sigma')\in\gamma_i}} \chi_{\sigma'}\,. \tag{6}$$

Passing to equivalence classes in Ω_Γ gives, for each $x \in \mathfrak{D}$,

$$[x] = \sum_{y\in\mathfrak{D}} M_i(x,y)[y]\,. \tag{7}$$

We can now proceed with the proof of the Theorem 2.2. If s is a vertex of type $\mathfrak{s}$ of Δ, then each element $\omega \in \Omega$ lies in $\Omega(\sigma)$ where σ is the base chamber of the sector $[s,\omega)$. Moreover ω lies in precisely one such set $\Omega(\sigma)$, with $\sigma \in \Delta^n$, $o(\sigma) = s$. Therefore

$$\mathbf{1} = \sum_{\substack{\sigma\in\Delta^n \\ o(\sigma)=s}} \chi_\sigma\,. \tag{8}$$

Since the action of Γ on Δ is free and type preserving, no two chambers $\sigma \in \Delta^n$ with $o(\sigma) = s$ lie in the same Γ-orbit. To simplify notation, let $n_\mathfrak{s} = n_\mathfrak{s}(\Gamma)$, the number of Γ-orbits of vertices of type $\mathfrak{s}$ in Δ. If we choose a representative set $\mathcal{S}$ of vertices of type $\mathfrak{s}$ in Δ then the chambers containing these vertices form a representative set of chambers, by the proof of Lemma 3.6. It follows that in Ω_Γ,

$$\begin{aligned} n_{\mathfrak{s}} \cdot [\mathbf{1}] &= \sum_{s\in\mathcal{S}} \sum_{\substack{\sigma\in\Delta^n \\ o(\sigma)=s}} [\chi_\sigma] \qquad \text{(by (8))} \\ &= \sum_{x\in\mathfrak{D}} [x]\,. \end{aligned}$$

Therefore

$$\begin{aligned} n_{\mathfrak{s}} \cdot [\mathbf{1}] &= \sum_{x\in\mathfrak{D}} \sum_{y\in\mathfrak{D}} M_i(x,y)[y] \qquad \text{(by (7))} \\ &= \sum_{y\in\mathfrak{D}} \left(\sum_{x\in\mathfrak{D}} M_i(x,y) \right) [y] \\ &= \sum_{y\in\mathfrak{D}} N_i \cdot [y] \qquad \text{(by (1))} \\ &= N_i n_{\mathfrak{s}} \cdot [\mathbf{1}]\,. \end{aligned}$$

It follows that

$$n_{\mathfrak{s}}(N_i - 1) \cdot [\mathbf{1}] = 0, \tag{9}$$

which proves the first assertion of Theorem 2.2.

Using Lemmas 3.3, 3.6, we can estimate the order of the element $[\mathbf{1}]$.

$$n_{\mathfrak{s}}(N_i - 1) < n_{\mathfrak{s}} \cdot (q_{\mathfrak{s}}\alpha_{\mathfrak{s}} - 1) = q_{\mathfrak{s}} \cdot \mathrm{covol}(\Gamma) - n_{\mathfrak{s}}. \tag{10}$$

This proves the second assertion of Theorem 2.2. The next Lemma proves the final assertion of Theorem 2.2 by showing that the estimate of the order of $[\mathbf{1}]$ can be improved if certain exceptional cases are excluded.

Lemma 3.7. *Suppose that the Weyl group is not one of the exceptional types $\widetilde{E}_8, \widetilde{F}_4, \widetilde{G}_2$. Then*

$$\mathrm{ord}([\mathbf{1}]) < \mathrm{covol}(\Gamma)\,.$$

Proof. An examination of the possible Coxeter diagrams [Bou, Chap VI, No 4.4, Théorème 4] shows that if the diagram is not one of the types $\widetilde{E}_8, \widetilde{F}_4, \widetilde{G}_2$, then it contains at least two special types. Therefore every chamber of Δ contains at least two special vertices. Choose two such vertices and suppose that they have types $\mathfrak{s}$ and $\mathfrak{t}$, say. In that case the condition **(A3)** on the apartment A in Δ can be changed to read:

- (**A3**$'$) The unique special vertex $v' \in S$ of type $\mathfrak{t}$ which lies in C.

Assume that the remaining conditions **(A1)**, **(A2)**, **(A4)**, **(A5)** are unchanged. Figure 3 illustrates the setup in the $\widetilde{B}_2$ case.

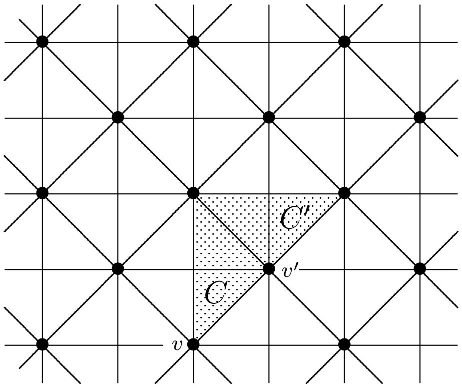


FIGURE 3. Part of an apartment A in a building of type $\widetilde{B}_2$, and a minimal gallery from C to C'.

The proof proceeds exactly as before, except that all the chambers in a gallery $(\sigma_0, \sigma_1, \dots, \sigma_k)$ of type i now contain a common vertex u of type $\mathfrak{t}$. Therefore equation(3) becomes

$$N_i < \alpha_{\mathfrak{t}}\,.$$

Observe that one must be careful with the notation. For example in equation (6), the function χ_σ on the left is now defined in terms of sectors based at the vertex of type $\mathfrak{s}$ of σ, whereas the functions $\chi_{\sigma'}$ on the right will now be defined in terms of sectors based at the vertex of type $\mathfrak{t}$ of σ'. Equation (9) becomes

$$(n_{\mathfrak{t}} N_i - n_{\mathfrak{s}}) \cdot [\mathbf{1}] = 0\,. \tag{11}$$

The order of the element $[\mathbf{1}]$ is bounded by

$$n_{\mathfrak{t}} \cdot \alpha_{\mathfrak{t}} - n_{\mathfrak{s}} = \operatorname{covol}(\Gamma) - n_{\mathfrak{s}}. \tag{12}$$

□

Finally, we verify that Ω_Γ is a finitely generated group. Sets of the form $\Omega(\sigma)$, $\sigma \in \Delta^n$, form a basis of clopen sets for the topology of Ω. It follows that the abelian group $C(\Omega, \mathbb{Z})$ is generated by the set of characteristic functions $\{\chi_\sigma : \sigma \in \Delta^n\}$. We show that Ω_Γ is generated by $\{[x] : x \in \mathfrak{D}\}$.

Lemma 3.8. *Every clopen set V in Ω may be expressed as a finite disjoint union of sets of the form $\Omega(\sigma)$, $\sigma \in \Delta^n$.*

Proof. Fix a special vertex s of type $\mathfrak{s}$ in Δ. For each $\omega \in \Omega$, sets of the form $\Omega(\sigma)$ with $\sigma \in \Delta^n$ and $\sigma \subset [s, \omega)$ form a basic family of open neighbourhoods of ω. Therefore, for each $\omega \in V$, there exists a chamber $\sigma_\omega \in \Delta^n$ with $\sigma_\omega \subset [s, \omega)$ and $\omega \in \Omega(\sigma_\omega) \subseteq V$. The clopen set V, being compact, is a finite union of such sets:

$$V = \Omega(\sigma_{\omega_1}) \cup \cdots \cup \Omega(\sigma_{\omega_k}) .$$

Fix a sector Q in Δ, with base vertex s. For each j, $1 \leq j \leq k$, let C_j be the chamber in Q which is the image of σ_{ω_j}, under the unique strong isometry from $[s, \omega_j)$ onto Q. Let Q_j be the subsector of Q with base chamber C_j $(1 \leq j \leq k)$, and choose a chamber C in $\bigcap_{j=1}^k Q_j$. Informally, C is chosen to be sufficiently far away from the base vertex s.

For $1 \leq j \leq k$, let τ_j be the chamber in $[s, \omega_j)$ which is the image of C under the strong isometry from Q onto $[s, \omega_j)$. For each $\omega \in \Omega(\sigma_{\omega_j})$ there is a retraction from $[s, \omega)$ onto $[s, \omega_j)$ [Gt, 4.2]. Let $\tau_j(\omega)$ be the inverse image of the chamber τ_j under this retraction. By local finiteness of Δ, there are only finitely many such chambers $\tau_j(\omega)$, $\omega \in \Omega(\sigma_{\omega_j})$. Call them $\tau_{j,l}$, $1 \leq l \leq n_j$. Thus $\Omega(\sigma_{\omega_j})$ may be expressed as a finite disjoint union:

$$\Omega(\sigma_j) = \bigsqcup_l \Omega(\tau_{j,l}) .$$

Moreover, if $\omega \in \Omega(\tau_{j,l})$ then the strong isometry from $[s, \omega)$ onto Q maps $\tau_{j,l}$ to the chamber C. Finally, V may be expressed as a disjoint union:

$$V = \bigsqcup_{j,l} \Omega(\tau_{j,l}) .$$

To check that this union is indeed disjoint, suppose that $\omega \in \Omega(\tau_{j,l}) \cap \Omega(\tau_{r,s})$. Then, under the strong isometry from Q onto $[s, \omega)$, the image of the chamber C is equal to both $\tau_{j,l}$ and $\tau_{r,s}$. In particular, $\tau_{j,l} = \tau_{r,s}$. ☐

Proposition 3.9. *Let $(G, \mathfrak{I}, N, S)$ be an affine topological Tits system, and let Γ be a subgroup of G. Then*

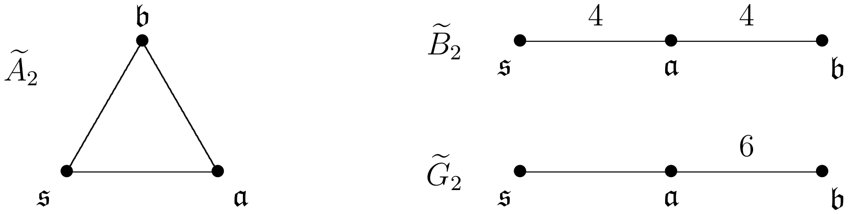
- *(a) The abelian group $C(\Omega, \mathbb{Z})$ is generated by the set of characteristic functions $\{\chi_\sigma : \sigma \in \Delta^n\}$.*
- *(b) Ω_Γ is generated by $\{[x] : x \in \mathfrak{D}\}$.*

Proof. (a) Any function $f \in C(\Omega, \mathbb{Z})$ is bounded, by compactness of Ω, and so takes finitely many values $n_i \in \mathbb{Z}$. Now $V_i = \{\omega \in \Omega : f(\omega) = n_i\}$ is a clopen set in Ω. It follows from the preceding Lemma that f may be expressed as a finite sum $f = \sum_j m_j \chi_{\sigma_j}$, with $\sigma_j \in \Delta^n$.

(b) This is an immediate consequence of (a). ☐

4. Further calculations in the rank 2 case

This section is devoted to showing that the estimate for the order of $[\mathbf{1}]$ given by Theorem 2.2 can be improved if the building Δ is 2-dimensional. The group G has type $\widetilde{A}_2$, $\widetilde{B}_2$ or $\widetilde{G}_2$. Denote the type set by $I = \{\mathfrak{s}, \mathfrak{a}, \mathfrak{b}\}$, where $\mathfrak{s}$ is a special type of the corresponding Coxeter diagram, as indicated below. Note that in the $\widetilde{B}_2$ case, the vertex $\mathfrak{b}$ is also special. In the $\widetilde{A}_2$ case, all vertices are special and $q_\mathfrak{t} = q$ for all $\mathfrak{t} \in I$.



Proposition 4.1. *Under the preceding assumptions, let* Γ *be a torsion free lattice in* G. *Then*

$$(q_\mathfrak{a}^2 - 1)n_\mathfrak{s} \cdot [\mathbf{1}] = 0 \tag{13}$$

in Ω_Γ.

Proof. We prove the $\widetilde{G}_2$ case. For the minimal gallery of type i between C and C' described in Figure 1, we obtain $N_i = q_\mathfrak{s} q_\mathfrak{a}^3 q_\mathfrak{b}^2$, so that

$$q_\mathfrak{s} q_\mathfrak{a}^3 q_\mathfrak{b}^2 n_\mathfrak{s} \cdot [\mathbf{1}] = n_\mathfrak{s} \cdot [\mathbf{1}]. \tag{14}$$

On the other hand, for a minimal gallery of type j between C and C'' described in Figure 4 below, we obtain $N_j = q_\mathfrak{s}^2 q_\mathfrak{a}^4 q_\mathfrak{b}^4$, so that

$$q_\mathfrak{s}^2 q_\mathfrak{a}^4 q_\mathfrak{b}^4 \; n_\mathfrak{s} \cdot [\mathbf{1}] = n_\mathfrak{s} \cdot [\mathbf{1}]. \tag{15}$$

Equations (14), (15) imply that

$$q_\mathfrak{a}^2 n_\mathfrak{s} \cdot [\mathbf{1}] = n_\mathfrak{s} \cdot [\mathbf{1}],$$

thereby proving (13).

The $\widetilde{B}_2$ and $\widetilde{A}_2$ cases follow by similar calculations, using the configurations in Figure 5 below. ☐

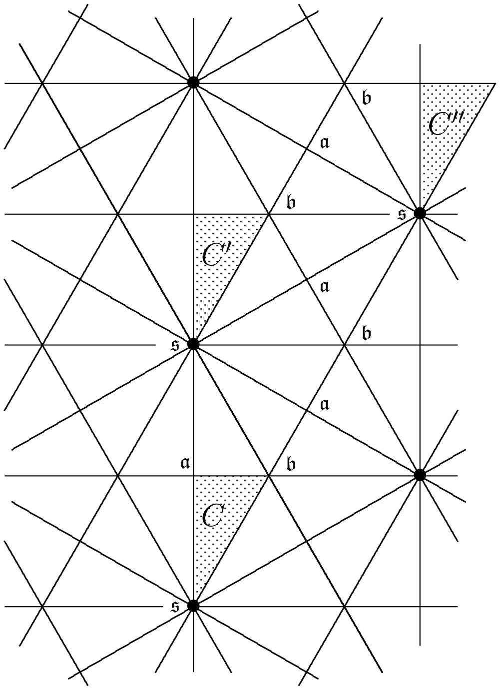
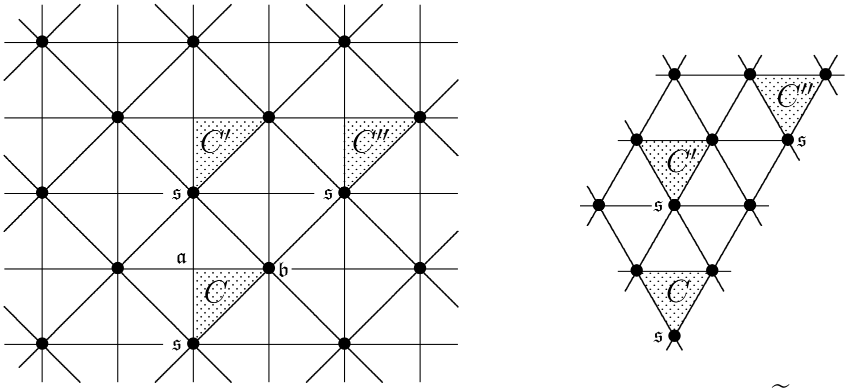


FIGURE 4. The $\widetilde{G}_2$ case.



The $\widetilde{B}_2$ case.

The $\widetilde{A}_2$ case.

FIGURE 5

Let k be a non-archimedean local field with residue field $\overline{k}$ of order q. Let L be a simple, simply connected linear algebraic k-group and assume that L is k-split and has k-rank 2. Let G be the group of k-rational points of L and let Γ be a torsion free lattice in G. Then $q_{\mathfrak{t}} = q$ for all $\mathfrak{t} \in I$ [Ti2, §3.5.4], and equation (13) becomes

$$(q^2 - 1)n_{\mathfrak{s}} \cdot [\mathbf{1}] = 0. \tag{16}$$

A parahoric subgroup $P_{\mathfrak{s}}$ corresponding to a *hyperspecial* vertex of type $\mathfrak{s}$ has maximal volume among compact subgroups of G [Ti2, 3.8.2]. This volume is $[P_{\mathfrak{s}} : \mathfrak{I}]$, by Remark 3.2. In particular, all such subgroups have the same volume. It follows that $n_{\mathfrak{s}} = \mathrm{covol}(\Gamma)/[P_{\mathfrak{s}} : \mathfrak{I}]$ has the same value for all hyperspecial types $\mathfrak{s}$.

Suppose, for example, that G is the symplectic group $Sp_2(k)$, which has type $\widetilde{B}_2$ (or, equivalently, $\widetilde{C}_2$). Examination of the tables at the end of [Ti2] shows that the diagram of G has two hyperspecial types $\mathfrak{s}$, $\mathfrak{t}$. Thus $n_{\mathfrak{s}} = n_{\mathfrak{t}}$, and it follows from (11) and Figure 3, that

$$(q^3 - 1)n_{\mathfrak{s}} \cdot [\mathbf{1}] = 0.$$

Combining this with (16) gives the following improvement to (16), for the case $G = Sp_2(k)$:

$$(q - 1)n_{\mathfrak{s}} \cdot [\mathbf{1}] = 0. \tag{17}$$

Remark 4.2. An interesting problem is to find the exact value of the order of $[\mathbf{1}]$. This is known in the case where the group G has k-rank 1, and Δ is a tree. In that case a torsion free lattice Γ in G is a free group of finite rank r, and it follows from [R1, R2] that $[\mathbf{1}]$ has order $r - 1 = -\chi(\Gamma)$, where $\chi(\Gamma)$ denotes the Euler-Poincaré characteristic of Γ. If $G = \mathrm{SL}_2(k)$, then $-\chi(\Gamma) = (q - 1)n_{\mathfrak{s}}(\Gamma)$.

In the rank 2 case, the order of $[\mathbf{1}]$ is in general smaller than $\chi(\Gamma)$. For by [Se1, p. 150, Théorème 7], $\chi(\Gamma) = (q - 1)(q^m - 1)n_{\mathfrak{s}}(\Gamma)$, where $m = 2, 3, 5$ according as G has type $\widetilde{A}_2$, $\widetilde{B}_2$, $\widetilde{G}_2$. Note however that by (16), (17), we do have $\chi(\Gamma) \cdot [\mathbf{1}] = 0$ if $G = \mathrm{SL}_3(k)$ or $G = Sp_2(k)$.

5. K-THEORY OF THE BOUNDARY ALGEBRA $\mathcal{A}_\Gamma$

We retain the general assumptions of Theorem 2.2. Thus G is a locally compact group acting strongly transitively by type preserving automorphisms on the affine building Δ, and Γ is a torsion free discrete subgroup of G.

As in [RS], [R1], the group Γ acts on the commutative C^*-algebra $C(\Omega)$, and one can form the full crossed product C^*-algebra $\mathcal{A}_\Gamma =$

$C(\Omega) \rtimes \Gamma$, [R1, Section 1]. The inclusion map $C(\Omega) \to \mathcal{A}_\Gamma$ induces a natural homomorphism from $C(\Omega, \mathbb{Z}) = K_0(C(\Omega))$ to $K_0(\mathcal{A}_\Gamma)$, which maps χ_σ to the class of the corresponding idempotent in $\mathcal{A}_\Gamma$. The covariance relations in $\mathcal{A}_\Gamma$ imply that for each $g \in \Gamma$ and $\sigma \in \Delta^n$, the functions χ_σ and $g \cdot \chi_\sigma = \chi_{g\sigma}$ map to the same element of $K_0(\mathcal{A}_\Gamma)$. Thus there is an induced homomorphism $\varphi : \Omega_\Gamma \to K_0(\mathcal{A}_\Gamma)$. Moreover $\varphi([\mathbf{1}]) = [\mathbf{1}]_{K_0}$, the class of $\mathbf{1}$ in the K_0-group of $\mathcal{A}_\Gamma$. We have the following immediate consequence of Theorem 2.2.

Corollary 5.1. *If Γ is a torsion free lattice in G then $[\mathbf{1}]_{K_0}$ has finite order in $K_0(\mathcal{A}_\Gamma)$.*

Remark 5.2. Clearly the bounds for the order of $[\mathbf{1}]$ obtained in the preceding sections apply also to $[\mathbf{1}]_{K_0}$. If G has type $\widetilde{A}_n$, Corollary 5.1 was proved in [RS], [R1]. In that case $q_{\mathfrak{t}} = q$ for all $\mathfrak{t} \in I$. For $n = 1$, it follows from [R1, R2] that the order of $[\mathbf{1}]_{K_0}$ is actually

$$\operatorname{ord}([\mathbf{1}]_{K_0}) = (q-1) \cdot n_{\mathfrak{s}}. \tag{18}$$

The computational evidence at the end of Section 6 below indicates that (18) also holds for $n = 2$.

Return now to the general assumptions of Theorem 2.2. It is important that Γ is amenable at infinity in the sense of [AR, Section 5.2]. Since the action of G on Δ is strongly transitive, its action on the boundary Ω is transitive. Therefore Ω may be identified, as a topological Γ-space, with G/B, where the Borel subgroup B is the stabilizer of some point $\omega \in \Omega$. The next result shows that the group B is amenable and so the action of Γ on Ω is amenable [AR, Section 2.2]. Moreover the crossed product algebra $\mathcal{A}_\Gamma$ is unique : the full and reduced crossed products coincide.

Proposition 5.3. *Let $\omega \in \Omega$ and let $B = \{g \in G : g\omega = \omega\}$. Then B is amenable and so (Γ, Ω) is amenable as a topological Γ-space, if Γ is a closed subgroup of G.*

Proof. Let $s \in \Delta^0$ be a special vertex and let A be an apartment in Δ containing the sector $[s, \omega)$. Let N_{trans} denote the subgroup of G consisting of elements which stabilize A and act by translation on A.

If $g \in B$, then the sectors $[gs, \omega)$ and $g[s, \omega)$ both have base vertex gs and both represent the same boundary point ω. Therefore $g[s, \omega) = [gs, \omega)$. Now the sectors $[gs, \omega)$, $[s, \omega)$ and $[g^{-1}s, \omega)$ are all equivalent, and so contain a common subsector S. The sectors S and gS, being subsectors of $[s, \omega)$, are parallel sectors in the apartment A. Let σ be the base chamber of S. Since G acts strongly transitively on Δ, there exists an element $g' \in G$ such that $g'A = A$ and $g'\sigma = g\sigma$. In particular $g'\omega = \omega$.

Since the action of G is type preserving, it follows from [Gt, Theorem 17.3] that $g' \in N_{\text{trans}}$. Moreover $gv = g'v$, for all $v \in S$. Let $\lambda_\omega(g) = g'|_A$, the restriction of g' to A. Then $\lambda_\omega(g)$ is the unique translation of A such that $gv = \lambda_\omega(g)v$, for all $v \in S$. As the notation suggests, $\lambda_\omega(g)$ depends on g and ω, but not on S.

It is easy to check that the mapping $\lambda_\omega : g \mapsto \lambda_\omega(g)$ is a homomorphism from B onto the group T_0 of type preserving translations on A. Since $T_0 \cong \mathbb{Z}^n$ is an amenable group, it will follow that B is amenable if $\ker \lambda_\omega$ can be shown to be amenable.

For each vertex v of $[s,\omega)$, let $B_v = \{g \in B : gv = v\}$. Then

$$\ker \lambda_\omega = \bigcup_{v \in [s,\omega)} B_v \,.$$

Each of the groups B_v is compact, being a closed subgroup of a parahoric subgroup. The group $\ker \lambda_\omega$ may thus be expressed as the inductive limit of the family of compact groups $\{B_v : v \in [s,\omega)\}$, directed by inclusion. Therefore $\ker \lambda_\omega$ is amenable. ☐

Remark 5.4. If G is the group of k-rational points of an absolutely almost simple, simply connected linear algebraic k-group, this result is well known. For then the Borel subgroup B is solvable, hence amenable.

The amenability of the Γ-space Ω has the consequence that the Baum-Connes conjecture, with coefficients in $C(\Omega)$ has been verified [Tu, Théorème 0.1]. Consequently $K_*(\mathcal{A}_\Gamma)$ can be calculated by means of the Kasparov-Skandalis spectral sequence [KaS, 5.6, 5.7]. This has initial terms

$$\begin{aligned} E^2_{p,q} &= H_p(\Gamma, K_q(C(\Omega))) \\ &= \begin{cases} H_p(\Gamma, C(\Omega,\mathbb{Z}))\,, & \text{if } 0 \le p \le n \text{ and } q \text{ is even,} \\ 0\,, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $H_p = 0$ for $p > n$, since Γ has homological dimension $\le n$. Moreover $K_1(C(\Omega)) = 0$, since Ω is totally disconnected.

Suppose now that Δ has dimension $n = 2$. Some of the nonzero terms in the first quadrant are shown in (19).

$$\begin{array}{llllll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
E^2_{04} & E^2_{14} & E^2_{24} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
E^2_{02} & E^2_{12} & E^2_{22} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
E^2_{00} & E^2_{10} & E^2_{20} & 0 & 0 & \cdots
\end{array} \tag{19}$$

Recall that for $r \geq 2$ there are differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$, and $E^{r+1}_{p,q}$ is the homology of E^r_* at the position of $E^r_{p,q}$. Since the differentials d^2 go up one row, it is clear that $d^2 = 0$ and $E^3_{p,q} = E^2_{p,q}$. Since the differentials d^3 go three units to the left, $d^3 = 0$ and $E^4_{p,q} = E^3_{p,q}$. Continuing in this way we see that $E^\infty_{p,q} = E^2_{p,q}$. Therefore the spectral sequence degenerates with $E^\infty_{p,q} = E^2_{p,q}$. Convergence of the spectral sequence to $K_*(\mathcal{A}_\Gamma)$ means that

$$K_1(\mathcal{A}_\Gamma) = H_1(\Gamma, C(\Omega, \mathbb{Z}))$$

and that there is a short exact sequence

$$0 \longrightarrow H_0(\Gamma, C(\Omega, \mathbb{Z})) \longrightarrow K_0(\mathcal{A}_\Gamma) \longrightarrow H_2(\Gamma, C(\Omega, \mathbb{Z})) \longrightarrow 0\,.$$

In particular, $\Omega_\Gamma = H_0(\Gamma, C(\Omega, \mathbb{Z}))$ is isomorphic to a subgroup of $K_0(\mathcal{A}_\Gamma)$.

6. $\widetilde{A}_2$ BUILDINGS AND REDUCED GROUP C^*-ALGEBRAS

The reduced group C^*-algebra of a group Γ is the completion $C^*_r(\Gamma)$ of the complex group algebra of Γ in the regular representation as operators on $\ell^2(\Gamma)$. Let Γ be a discrete torsion free group acting properly on the affine building Δ, satisfying the hypotheses of Theorem 2.2. By Proposition 5.3, Γ acts amenably on the compact space Ω. It follows that the Baum-Connes assembly map is injective [Hig] and so the Novikov conjecture is true. (This also follows from [KaS].) Therefore the class $[\mathbf{1}]$ in $K_0(C^*_r(\Gamma))$ does not have finite order.

Since $C^*_r(\Gamma)$ embeds in $\mathcal{A}_\Gamma$, there is a natural homomorphism

$$K_*(C^*_r(\Gamma)) \to K_*(\mathcal{A}_\Gamma).$$

This homomorphism is not injective, by Theorem 2.2, since $[\mathbf{1}]$ does have finite order in $K_0(\mathcal{A}_\Gamma)$. It is therefore worth comparing the K-theories of these two algebras. If the building is type $\widetilde{A}_2$, everything can be calculated explicitly.

The computation required is a corollary of [La], which states that the Baum-Connes conjecture holds for any discrete group Γ satisfying the following properties.

(1) Γ acts continuously, isometrically and properly with compact quotient on a uniformly locally finite affine building or on a complete riemannian manifold of nonpositive curvature;
(2) Γ has property (RD) of Jolissaint.

For a group Γ satisfying these conditions, $K_*(C_r^*(\Gamma))$ is isomorphic to the geometric group $K_*^\Gamma(\Delta) = KK_*^\Gamma(C_0(\Delta), \mathbb{C})$. (The notation is consistent with [BCH], because Δ is Γ-compact.) This provides a way of calculating the groups $K_*(C_r^*(\Gamma))$.

Assume therefore that all the conditions of Theorem 2.2 hold, together with the condition that Δ has type $\widetilde{A}_2$. This is the case, for example, if Γ is a torsion free lattice in $G = \mathrm{SL}_3(k)$.

Condition (1) is clearly satisfied and condition (2) is also satisfied by the main result of [RRS]. The finite cell complex $B\Gamma = \Delta/\Gamma$ is a $K(\Gamma, 1)$ space [Br, I.4], so the group homology $H_*(\Gamma, \mathbb{Z})$ is isomorphic to the usual simplicial homology $H_*(B\Gamma)$ [Br, Proposition II.4.1]. Thus $H_0(\Gamma, \mathbb{Z}) = \mathbb{Z}$ and $H_1(\Gamma, \mathbb{Z}) = \Gamma_{ab}$, the abelianization of Γ. Moreover, since $B\Gamma$ is 2-dimensional, the group Γ has homological dimension at most 2 [Br, VIII.2 Proposition (2.2) and VIII.6 Exercise 6]. It follows that $H_2(\Gamma, \mathbb{Z})$ is free abelian and $H_p(\Gamma, \mathbb{Z}) = 0$ for $p > 2$. Since Γ satisfies the Baum-Connes conjecture, $K_*(C_r^*(\Gamma))$ coincides with its "γ-part" [Ka, Definition-corollary 3.12]. Therefore $K_*(C_r^*(\Gamma))$ may be computed as the limit of a spectral sequence $E_{p,q}^r$ [KaS, Theorem 5.6 and Remark 5.7(a)]. Since Γ is torsion free, Γ acts freely on Δ. According to [KaS, Remarks 5.7(b)] the initial terms of the spectral sequence are

(20)
$$E_{p,q}^2 = H_p(\Gamma, K_q(\mathbb{C})) = \begin{cases} H_p(\Gamma, \mathbb{Z}) & \text{if } p \in \{0,1,2\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The nonzero terms in the first quadrant are shown in (19). Exactly as for (19), the spectral sequence degenerates with $E_{p,q}^\infty = E_{p,q}^2$. Convergence of the spectral sequence to $K_*(C_r^*(\Gamma))$ means that

$$K_1(C_r^*(\Gamma)) = H_1(\Gamma, \mathbb{Z}) \tag{21}$$

and that there is a short exact sequence

$$0 \longrightarrow H_0(\Gamma, \mathbb{Z}) \longrightarrow K_0(C_r^*(\Gamma)) \longrightarrow H_2(\Gamma, \mathbb{Z}) \longrightarrow 0. \tag{22}$$

Since $H_2(\Gamma, \mathbb{Z})$ is free abelian, (22) splits and we have

$$\begin{aligned} K_0(C_r^*(\Gamma)) &= H_0(\Gamma, \mathbb{Z}) \oplus H_2(\Gamma, \mathbb{Z}), \\ K_1(C_r^*(\Gamma)) &= H_1(\Gamma, \mathbb{Z}). \end{aligned} \tag{23}$$

Now $H_1(\Gamma, \mathbb{Z})$ is a finite group, because Γ has Kazhdan's property (T) [BS, Corollary 1]. It follows that $K_0(C_r^*(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)}$, where $\chi(\Gamma)$ is the Euler-Poincaré characteristic of Γ. This proves

Theorem 6.1. *Let Γ be a torsion free cocompact lattice in G, where $(G, \mathfrak{I}, N, S)$ is an affine topological Tits system of type $\widetilde{A}_2$. Then*

$$K_0(C_r^*(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)} \quad \textit{and} \quad K_1(C_r^*(\Gamma)) = \Gamma_{ab}. \tag{24}$$

The value of $\chi(\Gamma)$ is easily calculated [Se1, p. 150, Théorème 7],[R1, Section 4]. It is

$$\chi(\Gamma) = (q-1)(q^2-1) \cdot n_{\mathfrak{s}}(\Gamma), \tag{25}$$

where q is the order of the building Δ and $n_{\mathfrak{s}}(\Gamma)$ is the number of Γ-orbits of vertices of type $\mathfrak{s}$, where $\mathfrak{s} \in I$ is fixed.

In [CMSZ] a detailed study was undertaken of groups of type rotating automorphisms of $\widetilde{A}_2$ buildings, subject to the condition that the group action is free and transitive on the vertex set of the building. For $\widetilde{A}_2$ buildings of orders $q = 2, 3$, the authors of that article give a complete enumeration of the possible groups with this property. These groups are called $\widetilde{A}_2$ groups. Some, but not all, of the $\widetilde{A}_2$ groups are cocompact lattices in $\mathrm{PGL}_3(k)$ for some local field k with residue field of order q. It is an empirical fact that either $k = \mathbb{Q}_p$ or $k = \mathbb{F}_q((X))$ in all the examples constructed so far.

For each $\widetilde{A}_2$ group $\widetilde{\Gamma} < \mathrm{PGL}_3(k)$, consider the unique type preserving subgroup $\Gamma < \widetilde{\Gamma}$ of index 3. Each such Γ is torsion free and acts freely and transitively on the set of vertices of a fixed type $\mathfrak{s}$. That is $n_{\mathfrak{s}} = 1$. Therefore

$$\chi(\Gamma) = (q-1)(q^2-1) = 1 + \operatorname{rank} H_2(\Gamma, \mathbb{Z})\,.$$

Remark 6.2. There are eight such groups Γ if $q = 2$, and twenty-four if $q = 3$. Using the results of [RS] and the MAGMA computer algebra package, one can compute $K_0(\mathcal{A}_\Gamma)$. One checks that in all these examples,

$$\operatorname{rank} K_0(\mathcal{A}_\Gamma) = 2 \cdot \operatorname{rank} H_2(\Gamma, \mathbb{Z}) = \begin{cases} 4 & \text{if } q = 2, \\ 30 & \text{if } q = 3. \end{cases}$$

Furthermore, the class of $[\mathbf{1}]$ in the $K_0(\mathcal{A}_\Gamma)$ has order $q-1$. Note that for $q=2$ this means that $[\mathbf{1}]=0$.

These values also appear to be true for higher values of q. In particular, they have been verified for a number of groups with $q=4,5,7$. Here is an example with $q=4$.

Example 6.3. Consider the Regular $\widetilde{A}_2$ group Γ_r, with $q=4$. This is a torsion free cocompact subgroup of $\mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is the Laurent series field $F_4((X))$ with coefficients in the field F_4 with four elements. It is described in [CMSZ, Part I, Section 4], and its embedding in $\mathrm{PGL}_3(F_4((X)))$ is essentially unique, by the Strong Rigidity Theorem of Margulis. The group Γ_r is torsion free and has 21 generators $x_i, 0 \le i \le 20$, and relations (written modulo 21):

$$\begin{cases} x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & 0 \le j \le 6, \\ x_j x_{j+3} x_{j-6} = 1 & 0 \le j \le 20. \end{cases}$$

Let $\Gamma < \mathrm{PSL}_3(\mathbb{K})$ be the type preserving index three subgroup of Γ_r. The group Γ has generators $x_j x_0^{-1}$, $1 \le j \le 20$. Using the results of [RS] one obtains

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{88} \oplus (\mathbb{Z}/2\mathbb{Z})^{12} \oplus (\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/7\mathbb{Z})^4 \oplus (\mathbb{Z}/9\mathbb{Z}),$$

and the class of $[\mathbf{1}]$ in $K_0(\mathcal{A}_\Gamma)$ is $3+\mathbb{Z}/9\mathbb{Z}$, which has order $q-1=3$. It also follows from [RS, Theorem 2.1] that $K_0(\mathcal{A}_\Gamma) = K_1(\mathcal{A}_\Gamma)$.

According to Theorem 6.1,

$$K_0(C_r^*(\Gamma)) = \mathbb{Z}^{45} = \mathbb{Z}^{44} \oplus \langle[\mathbf{1}]\rangle \quad \text{and} \quad K_1(C_r^*(\Gamma)) = (\mathbb{Z}/2\mathbb{Z})^6 \oplus (\mathbb{Z}/3\mathbb{Z}).$$

The second equality was obtained using the MAGMA computer algebra package. This, and similar, examples suggest that the *only* reason for failure of injectivity of the natural homomorphism

$$K_0(C_r^*(\Gamma)) \to K_0(\mathcal{A}_\Gamma)$$

is the fact that $[\mathbf{1}]$ has finite order in $K_0(\mathcal{A}_\Gamma)$.

Example 6.4. For completeness, here are the results of the computations for one of the groups with $q=3$. The Regular group 1.1 of [CMSZ], with $q=3$, has 13 generators x_i, $0 \le i \le 12$, and relations (written modulo 13):

$$\begin{cases} x_j^3 = 1 & 0 \le j \le 13, \\ x_j x_{j+8} x_{j+6} = 1 & 0 \le j \le 13. \end{cases}$$

Let Γ be the type preserving index three subgroup. The group Γ has generators $x_j x_0^{-1}$, $1 \le j \le 12$. Note that the group 1.1 has torsion, but its type preserving subgroup Γ is torsion free. One obtains

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{30} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^6 \oplus (\mathbb{Z}/13\mathbb{Z})^4,$$

and the class of $[\mathbf{1}]$ in $K_0(\mathcal{A}_\Gamma)$ is $1 + \mathbb{Z}/2\mathbb{Z}$, which has order $q - 1 = 2$. It also follows from Theorem 6.1 that

$$K_0(C_r^*(\Gamma)) = \mathbb{Z}^{16} \quad \text{and} \quad K_1(C_r^*(\Gamma)) = (\mathbb{Z}/3\mathbb{Z})^3 \oplus (\mathbb{Z}/13\mathbb{Z}).$$

References

[AR] C. Anantharaman-Delaroche and J. Renault, *Amenable Groupoids*, Monographs of L'Enseignement Mathématique, Geneva, 2000.

[BCH] P. Baum, A. Connes and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras, in *C^*-algebras: 1943-1993 A Fifty Year Celebration*, 241–291, Contemp. Math. **167**, Amer. Math. Soc., 1994.

[BM] M. Burger and S. Mozes, CAT(-1)–spaces, divergence groups and their commensurators, *J. Amer. Math. Soc.* **9** (1996), 57-93.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie. Chap IV–VI.* Éléments de mathématique, Fasc. XXXIV. Actualités Scientifiques et Industrielles, No 1337, Hermann, Paris 1968.

[Br] K. Brown, *Cohomology of Groups*, Springer-Verlag, New York, 1982.

[BS] W. Ballmann and J. Swiatkowski, On L^2-cohomology and Property (T) for automorphism groups of polyhedral cell complexes, *Geometric and Funct. Anal.* **7** (1997), 615-645.

[BT] F. Bruhat and J. Tits, Groupes réductifs sur un corps local : I. Données radicielles valuées, *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5–251.

[CMSZ] D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I,II, *Geom. Ded.* **47** (1993), 143–166 and 167–223.

[Gd] H. Garland, p-adic curvature and the cohomology of discrete subgroups of p-adic groups, *Ann. of Math.* **97** (1973), 375–423.

[Gt] P. Garrett, *Buildings and Classical Groups*, Chapman and Hall, London, 1997.

[Gr] M. Gromov, Hyperbolic groups. *Essays in group theory*, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer, New York, 1987.

[Hig] N. Higson, Bivariant K-theory and the Novikov conjecture, *Geometric and Funct. Anal.* **10** (2000), 563–581.

[IM] N. Iwahori and H. Matsumoto, On some Bruhat decompostion and the structure of the Hecke ring of p-adic Chevalley groups, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 5–48.

[KaS] G. G. Kasparov and G. Skandalis , Groups acting on buildings, operator K-theory, and Novikov's conjecture, *K-theory* **4** (1991), 303–337.

[Ka] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.* **91** (1988), 147–201.

[KL] B. Kleiner and B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 115–197.

[La] V. Lafforgue, Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété (T), *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), 439–444.

[M] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, Berlin, 1991.

[R1] G. Robertson, Torsion in K-theory for boundary actions on affine buildings of type $\widetilde{A}_n$, *K-theory* **22** (2001), 251–269.

[R2] G. Robertson, Boundary operator algebras for free uniform tree lattices, *Houston J. Math.*, to appear.

[RRS] J. Ramagge, G. Robertson and T. Steger, A Haagerup inequality for $\widetilde{A}_1 \times \widetilde{A}_1$ and $\widetilde{A}_2$ buildings, *Geometric and Funct. Anal.* **8** (1998), 702–731.

[RS] G. Robertson and T. Steger, Asymptotic K-theory for groups acting on $\widetilde{A}_2$ buildings, *Canad. J. Math.* **53** (2001), 809–833.

[Ron] M. A. Ronan, *Lectures on Buildings*, Perspectives in Math. 7, Academic Press, London 1989.

[Se1] J.-P. Serre, Cohomologie des groupes discrets, *Ann. of Math. Studies* **70** (1971), 77–169.

[Se2] J.-P. Serre, *Arbres, amalgames, SL_2*, Astérisque **46**, Soc. Math. France, 1977.

[Ti1] J. Tits, Algebraic and abstract simple groups, *Ann. of Math.* **80** (1964), 313–329.

[Ti2] J. Tits, *Reductive groups over local fields*, Proceedings of Symposia in Pure Mathematics **33** (1979), 29–69.

[Tu] J. L. Tu, La conjecture de Baum-Connes pour les feuilletages moyennables, *K-theory* **17** (1999), 215–264.

School of Mathematics and Statistics, University of Newcastle, NE1 7RU, U.K.

E-mail address: `a.g.robertson@newcastle.ac.uk`