INVARIANT BOUNDARY DISTRIBUTIONS FOR FINITE GRAPHS

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Abstract. Let Γ be the fundamental group of a finite connected graph $\mathcal{G}$. Let $\mathfrak{M}$ be an abelian group. A *distribution* on the boundary $\partial\Delta$ of the universal covering tree Δ is an $\mathfrak{M}$-valued measure defined on clopen sets. If $\mathfrak{M}$ has no $\chi(\mathcal{G})$-torsion then the group of Γ-invariant distributions on $\partial\Delta$ is isomorphic to $H_1(\mathcal{G}, \mathfrak{M})$.

1. Introduction

Let $\mathcal{G}$ be a finite connected graph, and let Δ be its universal covering tree. The edges of Δ are directed and each geometric edge corresponds to two directed edges δ and $\overline{\delta}$. Let Δ^0 be the set of vertices and Δ^1 the set of directed edges of Δ. The boundary $\partial\Delta$ is the set of equivalence classes of infinite semi-geodesics in Δ, where two semi-geodesics are equivalent if they agree except on finitely many edges. The boundary has a natural compact totally disconnected topology. The fundamental group Γ of $\mathcal{G}$ is a free group which acts on Δ and on $\partial\Delta$. If $\mathfrak{M}$ is an abelian group then an $\mathfrak{M}$-valued *distribution* on $\partial\Delta$ is a finitely additive $\mathfrak{M}$-valued measure μ defined on the clopen subsets of $\partial\Delta$. By integration, a distribution may be regarded as an $\mathfrak{M}$-linear function on the group $C^\infty(\partial\Delta, \mathfrak{M})$ of locally constant $\mathfrak{M}$-valued functions on $\partial\Delta$. Let $\mathfrak{D}^\Gamma(\partial\Delta, \mathfrak{M})$ be the additive group of all Γ-invariant $\mathfrak{M}$-valued distributions on $\partial\Delta$ and let

$$\mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M}) = \{\mu \in \mathfrak{D}^\Gamma(\partial\Delta, \mathfrak{M}) : \mu(\partial\Delta) = 0\}.$$

Theorem 1.1. *There is an isomorphism of abelian groups*

$$H_1(\mathcal{G}, \mathfrak{M}) \cong \mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M}).$$

Let $\chi(\mathcal{G})$ denote the Euler characteristic of $\mathcal{G}$. If $\mathfrak{M}$ has no $\chi(\mathcal{G})$-torsion then each element of $\mathfrak{D}^\Gamma(\partial\Delta, \mathfrak{M})$ has total mass zero [Proposition 2.6]. Setting $\mathfrak{M} = \mathbb{Z}$ gives:

Corollary 1.2. *There is an isomorphism of abelian groups*

$$H_1(\mathcal{G}, \mathbb{Z}) \cong \mathfrak{D}^\Gamma(\partial\Delta, \mathbb{Z}).$$

A. Haefliger and L. Banghe have proved a continuous analogue of Theorem 1.1: if Γ is the fundamental group of a compact surface of genus g then the space of Γ-invariant (classical) distributions on S^1 which vanish on constant functions has dimension $2g$ [3, Theorem 5.A.2].

The motivation for this article came from C^*-algebraic K-theory, as explained in Section 3 below.

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2. Construction of distributions

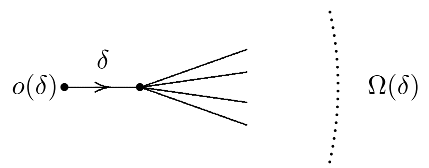
Choose an orientation on Δ^1 which is invariant under Γ. This orientation consists of a partition of Δ^1 and a bijective involution

$$\delta \mapsto \overline{\delta} : \Delta^1 \to \Delta^1$$

which interchanges the two components of Δ^1. Each directed edge δ has an initial vertex $o(\delta)$ and a terminal vertex $t(\delta)$, such that $o(\overline{\delta}) = t(\delta)$. The maps $\delta \mapsto \overline{\delta}$, $\delta \mapsto o(\delta)$ and $\delta \mapsto t(\delta)$ are Γ-equivariant. The quotient graph $\mathcal{G} = \Gamma\backslash\Delta$ has vertex set $V = \Gamma\backslash\Delta^0$ and directed edge set $E = \Gamma\backslash\Delta^1$. There are induced maps $x \mapsto \overline{x}$, $x \mapsto o(x)$ and $x \mapsto t(x)$ on the quotient and the partition of Δ^1 passes to a partition

$$E = E_+ \sqcup \overline{E_+}.$$

If $\delta \in \Delta^1$, let Ω_δ be the clopen subset of $\partial\Delta$ corresponding to the set of all semi-geodesics with initial edge δ and initial vertex $o(\delta)$.



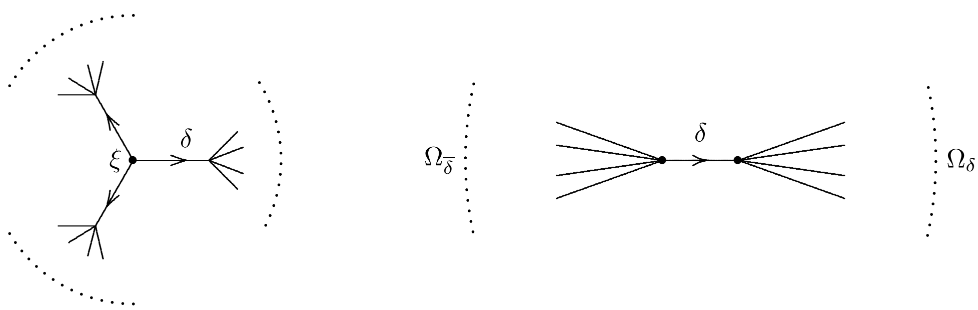
The sets Ω_δ, $\delta \in \Delta^1$ form a basis for a totally disconnected compact topology on $\partial\Delta$ which is described as an inverse limit in [6, I.2.2]. Any clopen set V in $\partial\Delta$ is a finite disjoint union of sets of the form Ω_δ. Indeed, choose a base vertex ξ. Then, for all sufficiently large n, V is a disjoint union of sets of the form Ω_δ, where δ is directed away from o and $d(o(\delta), \xi) = n$. The following relations are satisfied :

$$\Omega_\delta = \bigsqcup_{\substack{o(\delta')=t(\delta)\\ \delta'\neq\overline{\delta}}} \Omega_{\delta'}. \tag{1}$$

Let $\mathfrak{M}$ be an abelian group and let μ be an $\mathfrak{M}$-valued distribution on $\partial\Delta$. Then

$$\sum_{\substack{\delta\in\Delta^1\\ o(\delta)=\xi}} \mu(\Omega_\delta) = \mu(\partial\Delta), \qquad \text{for } \xi \in \Delta^0; \tag{2a}$$

$$\mu(\Omega_\delta) + \mu(\Omega_{\overline{\delta}}) = \mu(\partial\Delta), \qquad \text{for } \delta \in \Delta^1. \tag{2b}$$



For each $\alpha = \sum_{x \in E_+} n_x x \in H_1(\mathcal{G}, \mathfrak{M})$, define a Γ-invariant distribution μ_α by

$$\mu_\alpha(\Omega_\delta) = \langle \alpha - \overline{\alpha}, \Gamma\delta \rangle, \tag{3}$$

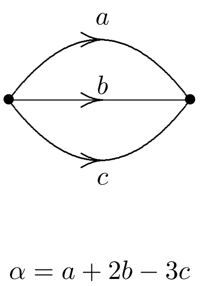
where $\langle \cdot , \cdot \rangle$ is the standard inner product $\mathfrak{M}E \times \mathfrak{M}E \to \mathfrak{M}$ and $\overline{\alpha} = \sum_{x \in E_+} n_x \overline{x}$. Thus, if $x \in E_+$,

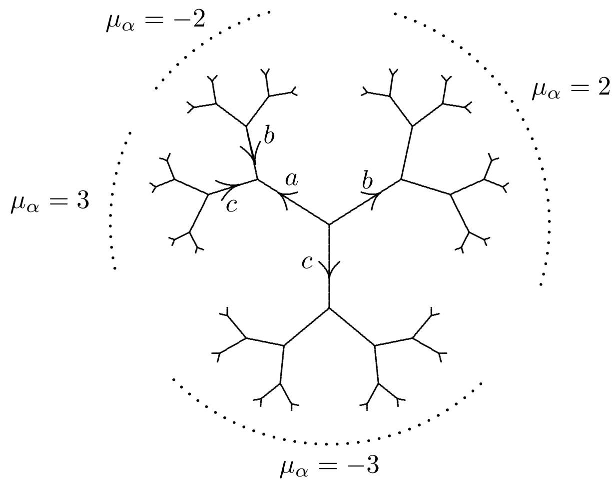
$$\mu_\alpha(\Omega_\delta) = \begin{cases} n_x & \text{if} \quad \Gamma\delta = x, \\ -n_x & \text{if} \quad \Gamma\delta = \overline{x}. \end{cases}$$

The verification that μ_α is well defined is given below and it is clear from (3) that μ_α is Γ-invariant.

Example 2.1. For the directed graph below, with two vertices and three edges, the boundary distribution corresponding to the 1-cycle $\alpha = a + 2b - 3c$ satisfies

$$\mu_\alpha(\Omega_\delta) = \begin{cases} 1 & \text{if} \quad \Gamma\delta = a, \\ 2 & \text{if} \quad \Gamma\delta = b \\ -3 & \text{if} \quad \Gamma\delta = c. \end{cases}$$





Recall that $H_1(\mathcal{G}, \mathfrak{M}) = \ker \partial$, where the boundary map $\partial : \mathfrak{M}E_+ \to \mathfrak{M}V$ is defined by $\partial x = t(x) - o(x)$ [6, Section II.2.8]. In order to check that μ_α is well defined, if $\alpha \in H_1(\mathcal{G}, \mathfrak{M})$, it is enough to show that equation (3) respects the relation (1). We must therefore show that, for all $x \in E$,

$$\langle \alpha - \overline{\alpha}, Tx \rangle = \langle \alpha - \overline{\alpha}, x \rangle, \tag{4}$$

where $T : \mathfrak{M}E \to \mathfrak{M}E$ is defined by

$$Tx = \sum_{\substack{o(y)=t(x) \\ y \neq \overline{x}}} y = \left(\sum_{o(y)=t(x)} y \right) - \overline{x}. \tag{5}$$

Equivalently, it is necessary that $(I - T^*)(\alpha - \overline{\alpha}) = 0$, where T^* is the adjoint of T. Define homomorphisms $\varphi_0 : \mathfrak{M}V \to \mathfrak{M}E$ and $\varphi_1 : \mathfrak{M}E_+ \to \mathfrak{M}E$ by

$$\varphi_0(v) = \sum_{o(y)=v} y \qquad \text{and} \qquad \varphi_1(x) = x - \overline{x}.$$

An easy calculation, using the identity

$$(I - T^*)x = x + \overline{x} - \left(\sum_{t(y)=o(x)} y \right),$$

shows that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{M}V & \xleftarrow{\partial} & \mathfrak{M}E_+ \\ {\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ \mathfrak{M}E & \xleftarrow[I-T^*]{} & \mathfrak{M}E \end{array} \tag{6}$$

If $\alpha \in H_1(\mathcal{G}, \mathfrak{M})$, it follows that $(I - T^*)(\alpha - \overline{\alpha}) = \varphi_0 \circ \partial(\alpha) = 0$, as required.

Lemma 2.2. *Let $\mathfrak{M}$ be an abelian group. The map $\alpha \mapsto \mu_\alpha$ is an injection from $H_1(\mathcal{G}, \mathfrak{M})$ into $\mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M})$.*

Proof. Fix $\alpha = \sum_{x \in E_+} n_x x \in H_1(\mathcal{G}, \mathfrak{M})$. Choose $\delta \in \Delta_+^1$ and let $y = \Gamma\delta \in E_+$. It follows from (2b) that

$$\begin{aligned} \mu_\alpha(\partial\Delta) &= \langle \alpha - \overline{\alpha}, y + \overline{y} \rangle \\ &= \langle \alpha - \overline{\alpha}, y \rangle + \langle \alpha - \overline{\alpha}, \overline{y} \rangle \\ &= n_y - n_y \\ &= 0. \end{aligned}$$

Thus $\mu_\alpha \in \mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M})$. The proof that the map is injective is straightforward. □

Remark 2.3. The map $\alpha \mapsto \mu_\alpha$ clearly depends on the choice of orientation on the tree Δ, although the group $H_1(\mathcal{G}, \mathfrak{M})$ does not [6, Section II.2.8].

The next result completes the proof of the Theorem 1.1.

Lemma 2.4. *Let $\mathfrak{M}$ be an abelian group. The map $\alpha \mapsto \mu_\alpha$ is a surjection from $H_1(\mathcal{G}, \mathfrak{M})$ onto $\mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M})$.*

Proof. Let $\mu \in \mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M})$. Since μ is Γ-invariant, we may define a function $\lambda : E \to \mathfrak{M}$ by $\lambda(\Gamma\delta) = \mu(\Omega_\delta)$. Let $\alpha = \sum_{x \in E_+} \lambda(x)x$. We show that $\alpha \in H_1(\mathcal{G}, \mathfrak{M})$ and that $\mu = \mu_\alpha$.

Since $\mu(\partial\Delta) = 0$, the relations (2) project to the following flow relations on $\mathcal{G}$.

$$\sum_{\substack{x \in E \\ o(x)=v}} \lambda(x) = 0, \qquad \text{for } v \in V; \tag{7a}$$

$$\lambda(x) + \lambda(\overline{x}) = 0, \qquad \text{for } x \in E. \tag{7b}$$

It follows from equations (7) that

$$\begin{aligned}
\partial\alpha &= \sum_{x\in E_+} \lambda(x)(t(x)-o(x)) \\
&= \sum_{x\in \overline{E}_+} \lambda(\overline{x})t(\overline{x}) - \sum_{x\in E_+} \lambda(x)o(x) \\
&= -\sum_{x\in \overline{E}_+} \lambda(x)o(x) - \sum_{x\in E_+} \lambda(x)o(x) \\
&= -\sum_{x\in E} \lambda(x)o(x) \\
&= -\sum_{v\in V} \left(\sum_{\substack{x\in E\\ o(x)=v}} \lambda(x) \right) v = 0.
\end{aligned}$$

Therefore $\alpha \in H_1(\mathcal{G}, \mathfrak{M})$. Finally, it follows from (7b) that

$$\alpha - \overline{\alpha} = \sum_{x\in E} \lambda(x)x$$

and, for each $\delta \in \Delta^1$,

$$\mu_\alpha(\Omega_\delta) = \langle \alpha - \overline{\alpha}, \Gamma\delta\rangle = \lambda(\Gamma\delta) = \mu(\Omega_\delta).$$

Therefore $\mu = \mu_\alpha$, as required. □

Remark 2.5. A special case occurs when $\mathbb{K}$ is a non-archimedean local field and Γ is a torsion free cocompact lattice in $\mathrm{SL}(2,\mathbb{K})$. The Bruhat-Tits building associated with $\mathrm{SL}(2,\mathbb{K})$ is a regular tree Δ whose boundary $\partial\Delta$ may be identified with the projective line $\mathbb{P}_1(\mathbb{K})$ [6, II.1.1]. In this context, the relations (7) assert that if $\mu \in \mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M})$ then the function $\delta \mapsto \mu(\Omega_\delta)$ is a Γ-invariant harmonic cocycle on Δ^1, in the sense of [2, 3.15]. The relationship between harmonic cocycles and boundary distributions has been studied in the p-adic case in [7].

In many cases, every Γ-invariant distribution on $\partial\Delta$ has total mass zero.

Proposition 2.6. *Let $\mathfrak{M}$ be an abelian group. If $\mathfrak{M}$ does not have $\chi(\mathcal{G})$-torsion then*

$$\mathfrak{D}^\Gamma(\partial\Delta, \mathfrak{M}) = \mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M}).$$

Proof. Let $\mu \in \mathfrak{D}^\Gamma(\partial\Delta, \mathfrak{M})$. For $x \in E$, define $\lambda(x) = \mu(\Omega_\delta)$ if $x = \Gamma\delta$. This is well defined, since μ is Γ-invariant. Let $\sigma = \mu(\partial\Delta)$. The relations (2) project to the quotient graph $\mathcal{G}$ as follows.

$$\sum_{\substack{x\in E\\ o(x)=v}} \lambda(x) = \sigma, \qquad \text{for } v \in V; \tag{8a}$$

$$\lambda(x) + \lambda(\overline{x}) = \sigma, \qquad \text{for } x \in E. \tag{8b}$$

Let n_0 $[n_1]$ be the number of vertices [edges] of $\mathcal{G}$, so that $\chi(\mathcal{G}) = n_0 - n_1$. Since the map $x \mapsto o(x) : E \to V$ is surjective, the relations (8) imply that

$$\begin{aligned} n_0\sigma &= \sum_{v\in V} \sum_{\substack{x\in E\\ o(x)=v}} \lambda(x) = \sum_{x\in E} \lambda(x) \\ &= \sum_{x\in E_+} (\lambda(x) + \lambda(\overline{x})) = \sum_{x\in E_+} \sigma \\ &= n_1\sigma. \end{aligned}$$

Therefore $\chi(\mathcal{G}).\sigma = 0$. The hypothesis on $\mathfrak{M}$ implies that $\sigma = 0$; in other words, $\mu \in \mathfrak{D}_0^\Gamma(\partial\Delta, \mathfrak{M})$. □

The following example of a nonzero Γ-invariant boundary distribution shows that the assumption that $\mathfrak{M}$ does not have $\chi(\mathcal{G})$-torsion cannot be removed.

Example 2.7. Let $\mathcal{G}$ be a $(q+1)$-regular graph, where $q > 3$, so that $\chi(\mathcal{G}) = \frac{n_0}{2}(1-q)$, where n_0 is the number of vertices of $\mathcal{G}$. Let $\mathfrak{M} = \mathbb{Z}_{q-1}$ and define $\mu \in \mathfrak{D}^\Gamma(\partial\Delta, \mathbb{Z}_{q-1})$ by $\mu(\Omega_\delta) = 1$, for all $\delta \in \Delta^1$. Then $\mu(\partial\Delta) = 2 \neq 0$.

3. The relation to K-theory

The motivation for this article came from the study of the K-theory of the crossed product C^*-algebra $C(\partial\Delta) \rtimes \Gamma$. Suppose that each vertex of $\mathcal{G}$ has at least three neighbours. Then the compact space $\partial\Delta$ is perfect (hence uncountable) and $\mathcal{A} = C(\partial\Delta) \rtimes \Gamma$ is a Cuntz-Krieger algebra [4]. The group $K_1(\mathcal{A})$ is isomorphic to $U(\mathcal{A})/U_0(\mathcal{A})$, the quotient of the unitary group of $\mathcal{A}$ by the connected component of the identity, and it follows from [1] that $K_1(\mathcal{A}) \cong \ker(T-1)$, where T is the map defined by equation (5), with $\mathfrak{M} = \mathbb{Z}$. We have the following result.

Proposition 3.1. *Suppose that each vertex of $\mathcal{G}$ has at least three neighbours. Then the map $\alpha \mapsto \alpha - \overline{\alpha}$ is an isomorphism from $H_1(\mathcal{G}, \mathbb{Z})$ onto* $\ker(T-1)$.

Proof. Let $\alpha = \sum_{x\in E} \lambda(x)x \in \ker(T-I)$. If $y \in E$, then the coefficient of y in the sum representing $(T-I)\alpha$ is

$$\left(\sum_{\substack{x\in E, x\neq\overline{y}\\ t(x)=o(y)}} \lambda(x) \right) - \lambda(y) = \left(\sum_{\substack{x\in E\\ t(x)=o(y)}} \lambda(x) \right) - \lambda(y) - \lambda(\overline{y}).$$

This coefficient is zero, since $\alpha \in \ker(T-I)$. Therefore

$$\lambda(y) + \lambda(\overline{y}) = \sum_{\substack{x\in E\\ t(x)=o(y)}} \lambda(x). \tag{9}$$

For any $y \in E$, define $\sigma(y) = \lambda(y) + \lambda(\overline{y})$. The right hand side of equation (9) depends only on $o(y)$, therefore $\sigma(y)$ depends only on $o(y)$. On the other hand, $\sigma(y) = \sigma(\overline{y})$, so that $\sigma(y)$ depends only on $t(y)$. Since the graph $\mathcal{G}$ is connected, it follows that $\sigma(y) = \sigma$, a constant, for all $y \in E$. The result follows easily, using the arguments of Proposition 2.6 and Lemma 2.4, together with the fact that the Euler characteristic of $\mathcal{G}$ is non-zero. □

The natural map from Γ into $U(\mathcal{A})$ induces a homomorphism from Γ into $K_1(\mathcal{A})$. The isomorphism $K_1(\mathcal{A}) \cong \ker(T-1)$ is described explicitly in [5, Section 2]. Combining this with Proposition 3.1 and [4, Section 1], it is easy to see that the homomorphism $\Gamma \to K_1(\mathcal{A})$ is surjective.

References

[1] J. Cuntz, A class of C^*-algebras and topological Markov chains: Reducible chains and the Ext-functor for C^*-algebras, *Invent. Math.* **63** (1981), 23–50.

[2] H. Garland, p-adic curvature and the cohomology of discrete subgroups of p-adic groups, *Ann. of Math.* **97** (1973), 375–423.

[3] J. Lott, Invariant currents on limit sets, *Comment. Math. Helv.* **75** (2000), 319–350.

[4] G. Robertson, Boundary operator algebras for free uniform tree lattices, *Houston J. Math.*, **31** (2005), 913–935.

[5] M. Rørdam, Classification of Cuntz-Krieger algebras, *K-Theory* **9** (1995), 31–58.

[6] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980.

[7] J. Teitelbaum, Values of p-adic L-functions and a p-adic Poisson kernel, *Invent. Math.* **101** (1990), 395–410.

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