BOUNDARY C^*-ALGEBRAS FOR ACYLINDRICAL GROUPS

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Abstract. Let Δ be an infinite, locally finite tree with more than two ends. Let $\Gamma < \mathrm{Aut}(\Delta)$ be an acylindrical uniform lattice. Then the boundary algebra $\mathcal{A}_\Gamma = C(\partial\Delta) \rtimes \Gamma$ is a simple Cuntz-Krieger algebra whose K-theory is determined explicitly.

1. Introduction

Let Δ be an infinite, locally finite tree with more than two ends and with boundary $\partial\Delta$. Let k be a positive integer and let Γ be a group of automorphisms of Δ without inversion and with no proper invariant subtree. Say that Γ is k-acylindrical if the stabilizer of any path of length k in Δ is trivial [BP, Sel]. The group Γ is *acylindrical* if it is k-acylindrical for some integer $k \geq 1$. The main result of this article is

Theorem 1.1. *Let* $\Gamma < \mathrm{Aut}(\Delta)$ *be an acylindrical uniform lattice. Then the boundary algebra* $\mathcal{A}_\Gamma = C(\partial\Delta) \rtimes \Gamma$ *is a simple Cuntz-Krieger algebra.*

The action of Γ on $\partial\Delta$ is amenable, so the maximal crossed product $\mathcal{A}_\Gamma = C(\partial\Delta) \rtimes \Gamma$ coincides with the reduced crossed product and is nuclear by Proposition 4.8 and Théorème 4.5 in [AD]. The algebra $\mathcal{A}_\Gamma$ is described in Section 4 below. Cuntz-Krieger algebras were introduced in [CK] and are classified up to isomorphism by their K-theory [K]. A special case, where Γ is a free uniform tree lattice, was studied in [R2] by different methods. The K-groups of the boundary algebra $\mathcal{A}_\Gamma$ are isomorphic to the Bowen-Franks invariants of flow equivalence for a certain subshift of finite type associated with the geodesic flow [C3]. This subshift was studied in [BP, 6.3].

The K-groups of the algebra $\mathcal{A}_\Gamma$ may be computed explicitly. For example, if $\Gamma = \mathbb{Z}_{l+1} * \mathbb{Z}_{m+1}$ acts on its Bass-Serre tree, where $l, m \geq 1$, then $K_0(\mathcal{A}_\Gamma) = \mathbb{Z}_{lm-1}$ via an isomorphism sending the class of the identity idempotent to $l+m$. It follows that $\mathcal{A}_\Gamma \cong M_{l+m} \otimes \mathcal{O}_{lm}$, where $\mathcal{O}_n$ denotes the Cuntz algebra, which is generated by n isometries on a Hilbert space whose range projections sum to the identity operator [C1]. It follows that if $\Gamma' = \mathbb{Z}_{l'+1} * \mathbb{Z}_{m'+1}$ and $\mathcal{A}_\Gamma \cong \mathcal{A}_{\Gamma'}$, then $\{l, m\} = \{l', m'\}$.

Remark 1.2. The algebra $\mathcal{A}_\Gamma$ depends only on Γ. The group Γ and the tree Δ are quasi-isometric; more precisely, for any base vertex in Δ, the natural mapping from Γ onto the orbit of this vertex is a quasi-isometry from Γ to Δ. This is a special case of the "Fundamental Observation of Geometric Group Theory" [Ha, Theorem IV.23]. The mapping from Γ to Δ induces a Γ-equivariant homeomorphism of the boundaries of Γ and Δ.

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2. Background

The edges of the tree Δ are directed, and each geometric edge of Δ corresponds to two directed edges. Let Δ^0 denote the set of vertices and Δ^1 the set of directed edges of Δ. There is a distance function d defined on the geometric realization of Δ which assigns unit length to each edge. Choose an orientation on the set of edges which is invariant under Γ. This orientation consists of a partition of Δ^1 and a bijective involution

$$e \mapsto \overline{e} : \Delta^1 \to \Delta^1$$

which interchanges the two components of Δ^1. Each directed edge e has an initial vertex $o(e)$ and a terminal vertex $t(e)$ such that $o(\overline{e}) = t(e)$.

Let Γ be a group of automorphisms of Δ without inversion and with no proper invariant subtree. Say that Γ is k-acylindrical, where $k \geq 1$, if the stabilizer of any path of length k in Δ is trivial [BP]. (In [Sel] such a group Γ is said to be $(k-1)$-acylindrical.) To say that Γ is 1-acylindrical is the same as saying that Γ acts freely on Δ^1. For example, the action of a free product $\Gamma = \Gamma_1 * \Gamma_2$ on the associated Bass-Serre tree [Ser, I.4.1] is 1-acylindrical. If $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$ is a free product with amalgamation over Γ_0, then the action of Γ on the associated Bass-Serre tree is 2-acylindrical if Γ_0 is malnormal in each Γ_j, i.e. $g^{-1}\Gamma_0 g \cap \Gamma_0 = \{1\}$ for all $g \in \Gamma_j - \Gamma_0$, $j = 1, 2$. Every small splitting of a torsion free hyperbolic group gives rise to a 3-acylindrical action [Sel].

The boundary $\partial\Delta$ is the set of equivalence classes of infinite semi-geodesics in Δ, where two semi-geodesics are said to be equivalent if they agree except on finitely many edges. For the rest of this article we make the following assumptions.

Standing Hypotheses

- **(1):** Δ is an infinite locally finite tree with more than two boundary points.
- **(2):** $\Gamma < \mathrm{Aut}(\Delta)$ is a uniform tree lattice.
- **(3):** Γ acts without inversion and with no proper invariant subtree.
- **(4):** Γ is k-acylindrical, where $k \geq 1$.

Remark 2.1. The standing hypotheses imply that $\partial\Delta$ is uncountable.

Remark 2.2. The assumption that Γ has no proper invariant subtree is part of the definition of "acylindrical" in [Sel] (but not in [BP], which is why it is emphasised separately here). It implies that the action of Γ on $\partial\Delta$ is minimal. The assumption could have been omitted here if $\partial\Delta$ were replaced throughout by the limit set Λ_Γ.

Remark 2.3. The assumption that Γ is a *uniform* lattice is natural and necessary in the context of the theory of [RS]. It would be interesting to study $\mathcal{A}_\Gamma$ for non-uniform tree lattices. For example, [BP] provides an example of a 5-acylindrical action of the Nagao lattice $\mathrm{PGL}_2(\mathbb{F}_q[t])$.

3. Cuntz-Krieger algebras

It is convenient to use the approach to Cuntz-Krieger algebras developed in [RS]. Choose a nonzero matrix M with entries in $\{0, 1\}$. For $m \geq 0$, let W_m denote the set of all words of length $m+1$ based on the alphabet A and the transition matrix M. A word $w \in W_m$ is a formal product $w = a_0 a_1 \dots a_m$, where $a_j \in A$ and $M(a_{j+1}, a_j) = 1$, $0 \leq j \leq m-1$. Define $o(w) = a_0$ and $t(w) = a_m$.

Fix a nonempty finite or countable set D (whose elements are "decorations") and a map $\delta : D \to A$. Let $\overline{W}_m = \{(d, w) \in D \times W_m;\ o(w) = \delta(d)\}$, the set of "decorated words" of length $m+1$, and identify D with $\overline{W}_0$ via the map $d \mapsto (d, \delta(d))$. Let $W = \bigcup_m W_m$ and $\overline{W} = \bigcup_m \overline{W}_m$, the sets of all words and all decorated words respectively. Define $o : \overline{W}_m \to D$ and $t : \overline{W}_m \to A$ by $o(d,w) = d$ and $t(d,w) = t(w)$.

Let $u = a_0a_1 \dots a_m \in W_m$ and $v = b_0b_1 \dots b_n \in W_n$. If $t(u) = o(v)$, then there exists a unique product $uv \in W_{m+n}$ defined by

$$uv = a_0a_1 \dots a_m b_1 \dots b_n.$$

If $w = a_0a_1 \dots a_l \in W_l$ where $l \geq 0$ and if $p \neq 0$, we say that w is *p-periodic* if $a_{j+p} = a_j$ whenever both sides are defined. Assume that the nonzero $\{0,1\}$-matrix M has been chosen so that the following conditions from [RS] hold.

(H2): If $a, b \in A$ then there exists $w \in W$ such that $o(w) = a$ and $t(w) = b$.
(H3): For each nonzero integer p, there exists some $w \in W$ which is not p-periodic.

Definition 3.1. [RS] The C^*-algebra $\mathcal{A}_D = \mathcal{A}(A, D, M)$ is the universal C^*-algebra generated by a family of partial isometries $\{s_{u,v};\ u, v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfying the relations

$$s_{u,v}{}^* = s_{v,u} \tag{3.1a}$$

$$s_{u,v}s_{v,w} = s_{u,w} \tag{3.1b}$$

$$s_{u,v} = \sum_{\substack{w \in W_1, \\ o(w) = t(u) = t(v)}} s_{uw,vw} \tag{3.1c}$$

$$s_{u,u}s_{v,v} = 0, \text{ for } u, v \in \overline{W}_0, u \neq v. \tag{3.1d}$$

Remark 3.2. If D is finite (as is the case in this article), then $\mathcal{A}_D$ is isomorphic to a simple Cuntz-Krieger algebra with identity element $\mathbf{1} = \sum_{u \in \overline{W}_0} s_{u,u}$ [R1].

Remark 3.3. Two decorations $\delta_1 : D_1 \to A$ and $\delta_2 : D_2 \to A$ are said to be *equivalent* [RS, Section 5] if there is a bijection $\eta : D_1 \to D_2$ such that $\delta_1 = \delta_2\eta$. Equivalent decorations δ_1, δ_2 give rise to isomorphic algebras $\mathcal{A}_{D_1}, \mathcal{A}_{D_2}$.

Remark 3.4. Denote by $\mathcal{A}_A$ the Cuntz-Krieger algebra with decorating set A and with δ the identity map. The algebra $\mathcal{A}_A$ is isomorphic to the algebra $\mathcal{O}_{M^t}$ generated by a set of partial isometries $\{S_a; a \in A\}$ satisfying the relations $S_a^*S_a = \sum_b M(b,a)S_bS_b^*$ [RS, Remark 3.11]. If A contains n elements and $M(b,a) = 1$, for all $a, b \in A$, then $\mathcal{A}_A$ is the Cuntz algebra $\mathcal{O}_n$ generated by n isometries whose range projections sum to the identity operator [C1].

4. The algebra associated with an acylindrical group

The Standing Hypotheses (1)–(4) are now in force. A *geodesic* γ in Δ is a sequence $(s_j)_{j=-\infty}^{\infty}$ of vertices such that $d(s_i, s_j) = |i-j|$. A *directed segment* σ of length n is a sequence $(s_0, s_1, \dots, s_n)$ of vertices such that $d(s_i, s_j) = |i-j|$. Denote such a directed segment by $[s_0, s_n]$ and let $\mathfrak{S}_n$ be the set of directed segments of length n in Δ. Since the group Γ is k-acylindrical, Γ acts freely on the set $\mathfrak{S}_k$.

The alphabet A is defined to be $\Gamma\backslash\mathfrak{S}_{k+1}$, the set of Γ-orbits of directed segments of length $k+1$ in Δ. Since Γ is a uniform lattice, A is finite.

Define a matrix M with entries in $\{0,1\}$ as follows. If $a,b \in A$, we say that $M(b,a) = 1$ if and only if $a = \Gamma\sigma$ and $b = \Gamma\tau$, where $\sigma = (s_0, s_1, \dots, s_{k+1})$, $\tau = (t_0, t_1, \dots, t_{k+1})$ are directed segments such that $s_{j+1} = t_j$, $0 \le j \le k$. The definition is illustrated in Figure 1.

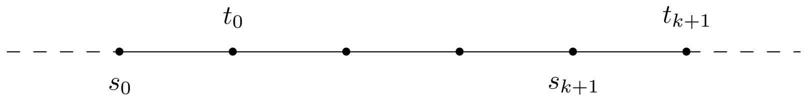


FIGURE 1. The condition $M(b,a) = 1$.

As in Section 3, W_m denotes the set of all words of length $m+1$ based on the alphabet A and transition matrix M. Let $\mathfrak{W}_m = \Gamma\backslash\mathfrak{S}_{m+k+1}$ and let $\mathfrak{W} = \bigcup_m \mathfrak{W}_m$. There is a map

$$\alpha : \mathfrak{W}_m \to W_m$$

defined by

$$\alpha(\Gamma(s_0, s_1, \dots, s_{m+k+1})) = (\Gamma[s_0, s_{k+1}])(\Gamma[s_1, s_{k+2}])\dots(\Gamma[s_m, s_{m+k+1}]).$$

Lemma 4.1. *The map α is a bijection from $\mathfrak{W}_m$ onto W_m.*

Proof. Suppose that $\alpha(\Gamma\sigma) = \alpha(\Gamma\tau)$, where $\sigma = (s_0, s_1, \dots, s_{m+k+1})$ and $\tau = (t_0, t_1, \dots, t_{m+k+1})$. Then $\Gamma[s_j, s_{j+k+1}] = \Gamma[t_j, t_{j+k+1}]$, $0 \le j \le m$. For each j there exists $g_j \in \Gamma$ such that $g_j[s_j, s_{j+k+1}] = [t_j, t_{j+k+1}]$, and g_j is uniquely determined, since Γ acts freely on the set of segments of length k. Now if $0 \le j \le m-1$, then $[s_j, s_{j+k+1}] \cap [s_{j+1}, s_{j+k+2}] = [s_{j+1}, s_{j+k+1}]$ is a segment of length k. Also $g_j[s_{j+1}, s_{j+k+1}] = [t_{j+1}, t_{j+k+1}] = g_{j+1}[s_{j+1}, s_{j+k+1}]$. Therefore $g_j = g_{j+1}$. It follows that $g_0\sigma = \tau$ and so $\Gamma\sigma = \Gamma\tau$. This proves injectivity.

To prove surjectivity, suppose that $w = a_0a_1\dots a_m \in W_m$. Then by definition $a_j = \Gamma\sigma_j$ for $0 \le j \le m$, where $\sigma_j \cap \sigma_{j+1}$ is a segment of length k, for $1 \le j \le m-1$. It follows that there is a directed segment $\sigma = (s_0, s_1, \dots, s_{m+k+1}) \in \mathfrak{S}_{m+k+1}$ such that $\sigma_j = [s_j, s_{j+k+1}], 0 \le j \le m$. Thus $\alpha(\Gamma\sigma) = w$. $\square$

Now fix a vertex $P \in \Delta^0$. Let $\overline{\mathfrak{W}}_m$ denote the set of directed segments of length $m+k+1$ which begin at P and let $\overline{\mathfrak{W}} = \bigcup_{m\ge 0} \overline{\mathfrak{W}}_m$. The decorating set is $D = \overline{\mathfrak{W}}_0$, and the decorating map $\delta : D \to A$ is defined by $\delta(d) = \Gamma d$. Define $\overline{\alpha} : \overline{\mathfrak{W}} \to \overline{W}$ by

$$\overline{\alpha}(\sigma) = (o(\sigma), \alpha(\Gamma\sigma)),$$

where $o(\sigma) = (s_0, s_1, \dots, s_{k+1})$ is the initial segment of length $k+1$ of σ. Also define $t(\sigma)$ to be the final segment of length $k+1$ of σ.

Lemma 4.2. *The map $\overline{\alpha}$ is a bijection from $\overline{\mathfrak{W}}_m$ onto $\overline{W}_m$, for each $m \ge 0$.*

Proof. If $\overline{\alpha}(\sigma_1) = \overline{\alpha}(\sigma_2)$, then $o(\sigma_1) = o(\sigma_2)$; moreover $\Gamma\sigma_1 = \Gamma\sigma_2$ by Lemma 4.1. Since Γ acts freely on $\mathfrak{S}_k$, it follows that $\sigma_1 = \sigma_2$. Therefore $\overline{\alpha}$ is injective.

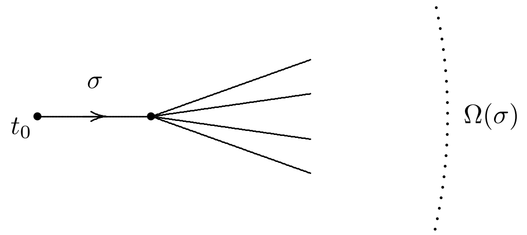
To see that $\overline{\alpha}$ is surjective, let $\overline{w} = (d, w) \in \overline{W}_m$, where $w \in W_m$ and $d \in D$. By Lemma 4.1, there exists $\sigma \in \mathfrak{S}_{m+k+1}$ such that $\alpha(\Gamma\sigma) = w$. Now

$$\Gamma d = \delta(d) = o(w) = o(\alpha(\Gamma\sigma)) = \Gamma o(\sigma).$$

Replacing σ by $g\sigma$ for suitable $g \in \Gamma$ ensures that $o(\sigma) = d$. Then $\sigma \in \overline{\mathfrak{W}}_m$ and $\overline{\alpha}(\sigma) = \overline{w}$. $\square$

Recall that each $\omega \in \partial\Delta$ is represented by a unique semi-geodesic $[s_0, \omega)$ with initial vertex $s_0 \in \Delta^0$. If σ is a directed segment with initial vertex t_0, let

$$\Omega(\sigma) = \{\omega \in \partial\Delta : [t_0, \omega) \text{ contains } \sigma\}.$$



The boundary $\partial\Delta$ has a natural compact totally disconnected topology generated by sets of the form $\Omega(\sigma)$ where $\sigma \in \overline{\mathfrak{W}}$ [Ser, I.2.2]. The group Γ acts on $\partial\Delta$, and one can form the crossed product C^*-algebra $C(\partial\Delta) \rtimes \Gamma$. This is the universal C^*-algebra generated by the commutative C^*-algebra $C(\partial\Delta)$ and the image of a unitary representation π of Γ, satisfying the covariance relation

$$f(g^{-1}\omega) = \pi(g) \cdot f \cdot \pi(g)^{-1}(\omega) \tag{4.1}$$

for $f \in C(\partial\Delta)$, $g \in \Gamma$ and $\omega \in \partial\Delta$. It is convenient to denote $\pi(g)$ simply by g. Equation (4.1) implies that for each clopen set $E \subset \partial\Delta$,

$$\chi_{gE} = g \cdot \chi_E \cdot g^{-1}. \tag{4.2}$$

The indicator function χ_E is continuous and is regarded as an element of the crossed product algebra via the embedding $C(\partial\Delta) \subset C(\partial\Delta) \rtimes \Gamma$. The following is a more precise version of Theorem 1.1.

Theorem 4.3. *Let $\mathcal{A}_\Gamma = C(\partial\Delta) \rtimes \Gamma$. Then $\mathcal{A}_\Gamma$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{A}_D$ associated with the alphabet A, the decorating set D and the transition matrix M.*

Proof. The isomorphism $\phi : \mathcal{A}_D \to C(\partial\Delta) \rtimes \Gamma$ is defined as follows. Let $\overline{w}_j = \overline{\alpha}(\sigma_j) \in \overline{W}$, $j = 1, 2$, with $t(\overline{w}_1) = t(\overline{w}_2)$. By the definition of $\overline{\alpha}$, there is an element $g \in \Gamma$ such that $gt(\sigma_1) = t(\sigma_2)$. Recall that $t(\sigma_1)$ and $t(\sigma_2)$ are segments of length $k + 1$. Therefore g is unique, since Γ acts freely on $\mathfrak{S}_k$. Define the homomorphism ϕ by

$$\phi(s_{\overline{w}_2, \overline{w}_1}) = g\chi_{\Omega(\sigma_1)} = \chi_{\Omega(\sigma_2)}g.$$

This equation defines a $*$-homomorphism of $\mathcal{A}_\Gamma$ because the operators of the form $\phi(s_{\overline{w}_2, \overline{w}_1})$ are easily seen to satisfy the relations (3.1). Since the algebra $\mathcal{A}_D$ is simple [RS, Theorem 5.9], ϕ is injective.

Now $\chi_{\Omega(\sigma)} = \phi(s_{\overline{w}, \overline{w}})$, where $\sigma \in \overline{\mathfrak{W}}$ and $\overline{w} = \overline{\alpha}(\sigma)$. Since the sets $\Omega(\sigma)$, $\sigma \in \overline{\mathfrak{W}}$, form a basis for the topology of Ω, the linear span of $\{\chi_{\Omega(\sigma)}; \sigma \in \overline{\mathfrak{W}}\}$ is dense in $C(\partial\Delta)$. It follows that the range of ϕ contains $C(\partial\Delta)$. To show that ϕ is surjective, it therefore suffices to show that the range of ϕ contains Γ.

Let $g \in \Gamma$ and choose an integer $m \geq d(P, g^{-1}P)$. Let $\sigma \in \overline{\mathfrak{W}}_m$. Then σ is a directed segment of length $m + k + 1$ with initial vertex P and final vertex Q, say.

Let σ'' be the directed segment with initial vertex $g^{-1}P$ and final vertex Q. Since $m \geq d(P, g^{-1}P)$, it follows that $t(\sigma'') = t(\sigma)$.

Let $\sigma' = g\sigma''$. Then σ' is a path beginning at P and $t(\sigma') = gt(\sigma)$. Let $\overline{w}_1 = \overline{\alpha}(\sigma)$ and $\overline{w}_2 = \overline{\alpha}(\sigma')$. Then $t(\overline{w}_1) = t(\overline{w}_2)$ and $g\chi_{\Omega(\sigma)} = \phi(s_{\overline{w}_2,\overline{w}_1}) \in \phi(\mathcal{A}_D)$. This holds for each $\sigma \in \overline{\mathfrak{W}}_m$. Therefore

$$g = \sum_{\sigma \in \overline{\mathfrak{W}}_m} g\chi_{\Omega(\sigma)} \in \phi(\mathcal{A}_D).$$

This shows that the range of ϕ contains Γ, as required. $\square$

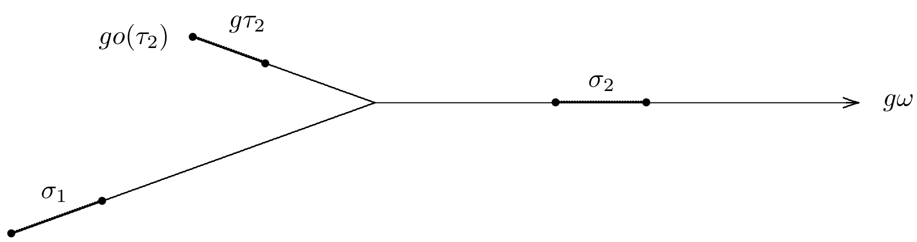
We now prove that conditions (H2) and (H3) are satisfied. To verify condition (H2), it is enough to show that if $a, b \in A$, then there is a directed segment σ such that

$$a = \Gamma o(\sigma), \quad b = \Gamma t(\sigma). \tag{4.3}$$

Let $a = \Gamma\sigma_1$, $b = \Gamma\tau_2$, where $\sigma_1, \tau_2 \in \mathfrak{S}_{k+1}$. By [Ch, Proposition 1 (iii)], there is a Γ-periodic geodesic γ containing τ_2. By definition, this means that there is a subgroup of Γ which leaves the geodesic γ invariant and acts upon it by translation. Choose $\omega \in \partial\Delta$ to be the boundary point of γ with

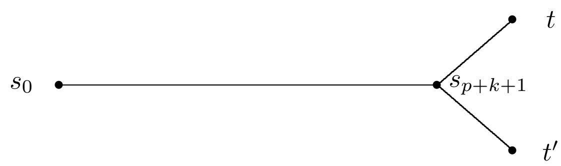
$$\tau_2 \subset [o(\tau_2), \omega).$$

Since the action of Γ on $\partial\Delta$ is minimal, there exists $g \in \Gamma$ such that $g\omega \in \Omega(\sigma_1)$. The geodesic $g\gamma$ is Γ-periodic. Therefore the semi-geodesic $[go(\tau_2), g\omega)$ contains infinitely many directed segments σ_2 which are Γ-translates of τ_2. Choose such a segment σ_2 far enough away from $g\tau_2$ so that $\sigma_2 \in \Omega(\sigma_1)$. Let σ be the directed segment with $o(\sigma) = \sigma_1$ and $t(\sigma) = \sigma_2$. Then (4.3) is satisfied.



To prove that condition (H3) holds, let $p > 0$. Since Δ has more than two ends, there exist vertices of Δ which have degree greater than 2. Let $\sigma = (s_0, s_1, \dots, s_{p+k}) \in \mathfrak{S}_{p+k}$ be a directed segment whose final vertex s_{p+k} has degree greater than 2. Extend σ to two different segments:

$$(s_0, s_1, \dots, s_{p+k}, t),\ (s_0, s_1, \dots, s_{p+k}, t') \in \mathfrak{S}_{p+k+1}.$$



Let $w = \alpha(\Gamma(s_0, \dots, s_{p+k}, t))$ and $w' = \alpha(\Gamma(s_0, \dots, s_{p+k}, t'))$. Then $w, w' \in W_p$, $o(w) = o(w')$ and $t(w) \neq t(w')$ since Γ acts freely on $\mathfrak{S}_k$ and α is injective. Therefore at least one of the words w, w' is not p-periodic.

5. K-theory and examples

The Standing Hypotheses (1)–(4) remain in force. Thus the algebra $\mathcal{A}_\Gamma$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{A}_D$ associated with the alphabet A, the set of decorations D and transition matrix M.

The simple Cuntz-Krieger algebras $\mathcal{A}_D$ are purely infinite, nuclear and satisfy the Universal Coefficient Theorem [RS, Remark 6.5]. They are therefore classified by their K-theory [K]. It is convenient to consider the related algebra $\mathcal{A}_A$ which is stably isomorphic to $\mathcal{A}_D$. Recall from Remark 3.4 that $\mathcal{A}_A$ is isomorphic to the algebra $\mathcal{O}_{M^t}$. The groups $K_0(\mathcal{A}_A)$, $K_1(\mathcal{A}_A)$ are the Bowen-Franks invariants of flow equivalence for a certain subshift associated with (Γ, Δ). More precisely, according to [C3, Proposition 3.1] the group $K_0(\mathcal{A}_A)$ is isomorphic to the abelian group

$$\mathcal{G}_\Gamma = \left\langle A \;\middle|\; a = \sum_{b \in A} M(a,b)b,\ a \in A \right\rangle . \tag{5.1}$$

Note that, as the notation suggests, $\mathcal{G}_\Gamma$ depends only on Γ, by Remark 1.2. Also $K_1(\mathcal{A}_A)$ is the torsion free part of $\mathcal{G}_\Gamma$. Therefore $\mathcal{A}_A$ is classified up to stable isomorphism by the group $\mathcal{G}_\Gamma$. Since the algebra $\mathcal{A}_D$ is stably isomorphic to $\mathcal{A}_A$ [RS, Corollary 5.15], we obtain the following result.

Theorem 5.1. *Under the Standing Hypotheses (1)–(4),* $K_0(\mathcal{A}_\Gamma) \cong \mathcal{G}_\Gamma$.

To completely classify $\mathcal{A}_A$ up to isomorphism, we need to identify the class $[\mathbf{1}]$ of the identity idempotent in $K_0(\mathcal{A}_A)$ [K]. By Remark 3.2, this class corresponds to the element

$$\varepsilon = \sum_{a \in A} a \in \mathcal{G}_\Gamma. \tag{5.2}$$

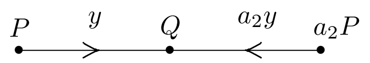
Here are explicit calculations in the case where Γ is a free product of finite cyclic groups.

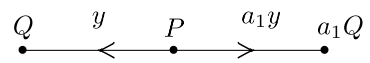
5.1. **Example:** The group $\Gamma = \mathbb{Z}_{l+1} * \mathbb{Z}_{m+1}$ acts on its Bass-Serre tree [Ser, I.4] with an edge $y = [P, Q]$ as its fundamental domain. The stabilizer Γ_P of P is isomorphic to $\mathbb{Z}_{l+1}$ and the stabilizer Γ_Q of Q is isomorphic to $\mathbb{Z}_{m+1}$.

$$P \xrightarrow{\;y\;} Q$$

By construction, Γ acts freely and transitively on the geometric edges of Δ. In other words, Γ is 1-acylindrical. The theory applies, with $k = 1$, and the alphabet A is the set of Γ-orbits of directed segments of length 2 in Δ.

Let $A_1 = \Gamma_P - \{1\}$ and $A_2 = \Gamma_Q - \{1\}$, so that $|A_1| = l$, $|A_2| = m$. Each directed segment of length 2 in Δ lies in the Γ-orbit of one of the directed segments $[P, a_2P]$, $[Q, a_1Q]$, for some $a_1 \in A_1$, $a_2 \in A_2$. Let $\hat{a}_2 = \Gamma[P, a_2P]$, $\hat{a}_1 = \Gamma[Q, a_1Q]$ be the corresponding elements of A_2, A_1.





The map $a \mapsto \hat{a}$ is a bijection from $A_1 \cup A_2$ onto A. The $\{0, 1\}$-matrix M is defined by $M(\hat{a}, \hat{b}) = 1 \iff$ either $a \in A_1, b \in A_2$ or $b \in A_1, a \in A_2$.

The condition $M(\hat{a}_1, \hat{a}_2) = 1$.

By Theorem 5.1,

$$K_0(\mathcal{A}_\Gamma) = \left\langle \hat{A}_1 \cup \hat{A}_2 \;\middle|\; \hat{a} = \sum_{b \in A_{3-j}} \hat{b}, \quad a \in A_j, \quad j = 1, 2 \right\rangle . \tag{5.3}$$

The relations on the right side of (5.3) show that all the generators of $\hat{A}_1$ are equal and all the generators of $\hat{A}_2$ are equal. Therefore

$$K_0(\mathcal{A}_\Gamma) = \langle \hat{a}_1, \hat{a}_2 \mid \hat{a}_2 = l\hat{a}_1, \hat{a}_1 = m\hat{a}_2 \rangle = \langle \hat{a}_1 \mid \hat{a}_1 = lm\hat{a}_1 \rangle = \mathbb{Z}_{lm-1}.$$

Recall that the classical Cuntz algebra $\mathcal{O}_n$ is generated by n isometries whose range projections sum to the identity operator [C1]. Now $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ [C2]. It follows from the classification theorem [K] that $\mathcal{A}_\Gamma$ is stably isomorphic to $\mathcal{O}_{lm}$.

In order to classify $\mathcal{A}_D$ up to isomorphism, the class of the identity idempotent in $K_0(\mathcal{A}_\Gamma)$ must be identified. This is done by showing that $\mathcal{A}_\Gamma \cong \mathcal{A}_A$. Choose the decorating set D to be the set of directed segments of length 2 which begin at P. The map $\delta : d \mapsto \Gamma d$ has domain D and range $\hat{A}_2$. Each of the $l + 1$ edges with origin P is the initial edge of precisely one directed segment of length 2 from each Γ-orbit in $\hat{A}_2$, and so $\delta^{-1}(\hat{a}_2)$ contains $l + 1$ elements, for each $\hat{a}_2 \in \hat{A}_2$.

Define a new decorating set

$$D' = \{(\hat{a}, w) \in (\hat{A}_2 \times W_0) \cup (\hat{A}_1 \times W_1) : o(w) = \hat{a}\}$$

and a decorating map $\delta' : D' \to A$ by $\delta'(\hat{a}, w) = t(w)$. The range of δ' is $\hat{A}_2$ and for each $\hat{a}_2 \in \hat{A}_2$, $\delta'^{-1}(\hat{a}_2)$ contains $l + 1$ elements: one element from $\hat{A}_2 \times W_0$, and l elements from $\hat{A}_1 \times W_1$. It follows that δ' is equivalent to δ, in the sense of Remark 3.3, and so the algebras $\mathcal{A}_D, \mathcal{A}_{D'}$ are isomorphic.

On the other hand, there is an isomorphism $\phi : \mathcal{A}_{D'} \to \mathcal{A}_A$ defined by

$$\phi(s_{(\overline{w}_1, u_1),(\overline{w}_2, u_2)}) = s_{\overline{w}_1 u_1, \overline{w}_2 u_2}$$

for $\overline{w}_1, \overline{w}_2 \in D'$, $u_1, u_2 \in W$, $o(u_i) = \delta'(\overline{w}_i) = t(\overline{w}_i)$, and $t(u_1) = t(u_2)$. The proof of this fact is contained in [RS, Lemma 5.15]. Therefore

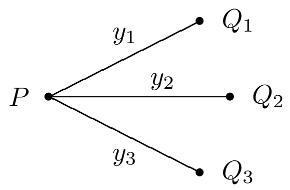
$$\mathcal{A}_\Gamma \cong \mathcal{A}_D \cong \mathcal{A}_{D'} \cong \mathcal{A}_A.$$

Since A contains $l+m$ elements, it follows from (5.2) that the class $[\mathbf{1}]$ in $K_0(\mathcal{A}_\Gamma)$ corresponds to the element $l+m \in \mathbb{Z}_{lm-1}$. Now it is known [C2] that

$$(K_0(M_k \otimes \mathcal{O}_n), [\mathbf{1}]) \cong (\mathbb{Z}_{n-1}, k).$$

This proves that $\mathcal{A}_\Gamma \cong M_{l+m} \otimes \mathcal{O}_{lm}$. Note further that $\mathcal{A}_\Gamma \cong \mathcal{O}_{lm}$ if and only if $l+m$ is a generator of $\mathbb{Z}_{lm-1}$; that is, $(l+m, lm-1) = 1$.

5.2. **Example:** More generally, a free product of finite groups $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$ acts on its Bass-Serre tree. The fundamental domain is a tree consisting of n edges y_i with terminal vertex Q_i emanating from a common vertex P. The stabilizer of P and of each of the edges y_j is trivial, and the stabilizer of Q_j is isomorphic to Γ_j.



The group Γ is 1-acylindrical and acts freely (but not transitively, in contrast to Example 5.1) on the set of edges of Δ, with finitely many orbits.

Let $A_i = \Gamma_i - \{1\}$ and $\gamma_i = |A_i|$, $1 \le i \le n$. Each directed segment of length 2 in Δ lies in the Γ-orbit of a directed segment of the form (P, Q_i, aP), $a \in A_i$, or (Q_j, P, Q_k), $j \neq k$, as illustrated below.

Let $\hat{A}_i = \{\hat{a} = \Gamma(P, Q_i, aP) : a \in A_i\}$, and let $\hat{B} = \{\hat{b}_{jk} = \Gamma(Q_j, P, Q_k) : j \neq k\}$. Then by Theorem 5.1

$$K_0(\mathcal{A}_\Gamma) = \left\langle \bigcup_i \hat{A}_i \cup \hat{B} \;\middle|\; \hat{a} = \sum_{j \neq i} \hat{b}_{ij} \quad (\hat{a} \in \hat{A}_i), \quad \hat{b}_{jk} = \sum_{\hat{a} \in \hat{A}_k} \hat{a} \right\rangle. \tag{5.4}$$

The relations on the right side of (5.4) show that $\hat{b}_{jk}$ depends only on k. Therefore, for each i, all the generators in $\hat{A}_i$ are equal. It follows that

$$K_0(\mathcal{A}_\Gamma) = \langle \hat{a}_i \mid \hat{a}_i = \sum_{j \neq i} \gamma_j \hat{a}_j \rangle. \tag{5.5}$$

It is easy to see from (5.5) that $K_0(\mathcal{A}_\Gamma)$ is a torsion group and therefore that $K_1(\mathcal{A}_\Gamma) = 0$. In other words, the unitary group of $\mathcal{A}_\Gamma$ is connected [C2]. If all the groups Γ_i have the same order, say $\gamma+1$, and $\delta = \gamma(n-1)-1$, then (5.5) simplifies to

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}_{(\gamma+1)\delta} \oplus (\mathbb{Z}_{\gamma+1})^{n-2}$$

with canonical generators $\hat{a}_1, \hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_{n-1} - \hat{a}_1$.

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