Free Groups
Products of Trees
Arithmetic of Quaternions

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Motivation

A Closed Surface $M$ has Universal Cover

$$\mathcal{H} = \{z \in \mathbb{C} : \Im z > 0\}$$

The fundamental group $\Gamma$ of $M$ acts on $\mathcal{H}$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

$\Gamma$ can be realized as a subgroup of

$$\operatorname{PSL}_2(\mathbb{R}) = \operatorname{SL}_2(\mathbb{R})/\{I, -I\}$$

and

$$\Gamma \backslash \mathcal{H} = M.$$
Discrete Analogue

$X$: A finite connected graph.

$T$: The universal covering space (a tree).

The fundamental group $\Gamma$ of $X$ is a **free group** and

$$\Gamma \backslash T = X.$$
Replace $\mathbb{R}$ by a **local field** ...

For $p \geq 2$ prime, $\mathbb{Q}_p$ is the field of formal sums:

$$x = a_j p^j + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots,$$

where each $a_i \in \{0, 1, \ldots, p-1\}$ and $a_j \neq 0$.

$$|x| = p^{-j} \quad \text{discrete valuation}$$

**The $p$-adic integers**

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| \leq 1\}$$

$$= \text{set of sums with } j \geq 0$$

$$= \overline{\mathbb{Z}} \quad \text{a compact open subring of } \mathbb{Q}_p$$

**Balls centre 0 :**

$$\cdots \supset p^{-1} \mathbb{Z}_p \supset \mathbb{Z}_p \supset p \mathbb{Z}_p \supset p^2 \mathbb{Z}_p \supset \cdots$$

The group

$$\text{PGL}_2(\mathbb{Q}_p) = \text{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times$$

acts on a $(p+1)$-regular tree $T_{p+1}$.  

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The tree $T_{p+1}$ of $\text{PGL}_2(\mathbb{Q}_p)$

A lattice $L \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-submodule:

$$L = \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2,$$

where $(v_1, v_2)$ is a basis of $\mathbb{Q}_p^2$.

Equivalence relation:

$$L_1 \sim L_2 \iff L_1 = aL_2, \quad a \in \mathbb{Q}_p^\times$$

An equivalence class $[L]$ is a vertex of $T_{p+1}$.

An edge of $T_{p+1}$ is $([L_1], [L_2])$ where

$$L_1 \supset L_2 \supset pL_1.$$
A Combinatorial Construction

The free group of rank 2
\[ \Gamma = \langle a, b \rangle \]
has a Cayley graph \( T \).

Vertex set: \( \Gamma \)

Edges: \( s \in S = \{ a, a^{-1}, b, b^{-1} \} \).

\( T \) is a tree.

\( \Gamma \) acts on \( T \) by left multiplication.
The quotient graph $X = \Gamma \backslash T$ has
- fundamental group $\Gamma$;
- universal cover $T$.

Can embed $\Gamma < \text{PGL}_2(\mathbb{Q}_3)$ as an arithmetic subgroup...
A Result of Jacobi

If $p$ is an odd prime, the number of

$$(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$$

such that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$$

is

$$8(p + 1).$$

Suppose $p \equiv 1 \pmod{4}$. Then exactly one $a_j$ is odd and the number of representations with $a_0$ odd, $a_0 > 0$, is

$$p + 1.$$  

Consequence:

Let $S_p$ be the set of integer quaternions

$$a = a_0 + a_1i + a_2j + a_3k \in \mathbb{H}(\mathbb{Z})$$

with $a_0 > 0$, $a_0$ odd, and $|a|^2 = p$. Then

$$|S_p| = p + 1.$$
An Arithmetic Construction

If $p \equiv 1 \pmod{4}$ is prime, then

$$x^2 \equiv -1 \pmod{p}$$

has a solution in $\mathbb{Z}$, so by Hensel’s Lemma,

$$x^2 = -1$$

has a solution $i_p$ in $\mathbb{Q}_p$.

Define

$$\psi_p : \mathbb{H}(\mathbb{Z}) \to PGL_2(\mathbb{Q}_p)$$

by

$$\psi_p(a) = \begin{pmatrix} a_0 + a_1 i_p & a_2 + a_3 i_p \\ -a_2 + a_3 i_p & a_0 - a_1 i_p \end{pmatrix}.$$ 

**Theorem.** $\psi_p(S_p)$ contains $p+1$ elements and generates a free group $\Gamma_p$ of rank $\frac{p+1}{2}$.


*Uses “unique factorisation” for integer quaternions.*
This construction extends to primes

\[ p \equiv 3 \pmod{4} \]

using solutions in \( \mathbb{Z} \) of

\[ x^2 + y^2 \equiv -1 \pmod{p}. \]

[D. Rattaggi (2003)]

\( \Gamma_p \) acts freely and transitively on the vertices of the \((p + 1)\)-regular tree \( T_{p+1} \).

i.e. \( X = \Gamma_p \backslash T_{p+1} \) has one vertex.
Products of Trees

$T_m \times T_n$ is a 2-dimensional complex which is a union of flat subcomplexes

Each vertex is a corner of $m \times n$ squares:
Neighbourhood of an edge:

Neighbourhood of a vertex:

The link of a vertex:
If $p \neq l$ are odd primes then

$$G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

acts on a product of trees

$$\Delta = T_{p+1} \times T_{l+1}.$$ 

Define

$$\psi : \mathbb{H}(\mathbb{Z}) \to PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

by

$$\psi(a) = (\psi_p(a), \psi_l(a)).$$

Let $\Gamma_{p,l}$ be the subgroup of

$$PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

generated by $\psi(S_p \cup S_l)$. 

[S. Mozes (1995)]

$\Gamma_{p,l}$ is generated by two free subgroups:

$$\langle \psi(S_p) \rangle \cong \Gamma_p$$

$$\langle \psi(S_l) \rangle \cong \Gamma_l$$

but

$$\Gamma_{p,l} \not\cong \Gamma_p \times \Gamma_l$$
\( \Gamma_{p,l} \) acts freely on \( \Delta \).

The quotient complex \( X_{p,l} \) has one vertex and is obtained by glueing together \( \frac{(p+1)(l+1)}{4} \) squares.

**Example.** \( \Gamma_{3,5} \) has generators

\[
\begin{align*}
a_1 &= \psi(1 + j + k), & a_1^{-1} &= \psi(1 - j - k), \\
a_2 &= \psi(1 + j - k), & a_2^{-1} &= \psi(1 - j + k), \\
b_1 &= \psi(1 + 2i), & b_1^{-1} &= \psi(1 - 2i), \\
b_2 &= \psi(1 + 2j), & b_2^{-1} &= \psi(1 - 2j), \\
b_3 &= \psi(1 + 2k), & b_3^{-1} &= \psi(1 - 2k).
\end{align*}
\]

and relators

\[
\begin{align*}
a_1b_1a_2b_2, & \quad a_1b_2a_2b_1^{-1}, & \quad a_1b_3a_2^{-1}b_1, & \quad a_1b_3^{-1}a_1b_2^{-1}, & \quad a_1b_1^{-1}a_2^{-1}b_3, \\
a_2b_3a_2b_2^{-1}.
\end{align*}
\]
$X_{3,5}$ is obtained by gluing together 6 squares

and has fundamental group $\Gamma_{3,5}$. 
Properties of Quaternions

1. If \( x, y \in \mathbb{H}(\mathbb{Q}) \), with \( \Re(x) \neq 0, \Re(y) \neq 0 \),
   \[
   x = x_0 + x_1i + x_2j + x_3k, \\
   y = y_0 + y_1i + y_2j + y_3k,
   \]
   then
   \[
   xy = yx
   \]
   if and only if
   \[
   (x_1, x_2, x_3) = \lambda(y_1, y_2, y_3)
   \]
   where \( \lambda \in \mathbb{Q} \).

2. There is a homomorphism
   \[
   \theta : \mathbb{H}(\mathbb{Q}) - \{0\} \rightarrow \text{SO}_3(\mathbb{Q})
   \]
   defined by
   \[
   \theta(y)x = yxy^{-1}
   \]
   for \( x = x_1i + x_2j + x_3k \in \mathbb{Q}^3 \).
Properties of $\Gamma_{p,l}$

[D. Rattaggi, G. Robertson]

1. $\Gamma_{p,l}$ is **commutative transitive**.

\[ x \leftrightarrow y \text{ and } y \leftrightarrow z \implies x \leftrightarrow z \quad \text{if } x, y, z \neq 1. \]

2. The only nontrivial direct product subgroup of $\Gamma_{p,l}$ is

\[ \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2. \]

[Answers a question of D. Wise.]

3. If $A_1$ and $A_2$ are distinct maximal abelian subgroups of $\Gamma_{p,l}$ then

\[ A_1 \cap A_2 = \{1\}. \]

4. $\Gamma_{p,l}$ is a **CSA-group**: for each maximal abelian subgroup $A$,

\[ gAg^{-1} \cap A = \{1\} \]

if $g \in \Gamma - A$. 

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Maximal Abelian Subgroups

If \( x = x_0 + z_0(c_1i + c_2j + c_3k) \in \mathbb{H}(\mathbb{Z}) \) with
\[
z_0 \neq 0 \quad \text{and} \quad \gcd(c_1, c_2, c_3) = 1
\]
define
\[
n = n(x) = c_1^2 + c_2^2 + c_3^2 > 0.
\]
If \( \psi(x) \in \Gamma_{p,l} \), the centralizer
\[
A = \{ g \in \Gamma_{p,l} : g\psi(x) = \psi(x)g \}
\]
is a maximal abelian subgroup of \( \Gamma_{p,l} \),
\[
A \cong \mathbb{Z} \quad \text{or} \quad A \cong \mathbb{Z}^2
\]
and \( n \) depends only on \( A \).

**Theorem.** If
\[
\left( \frac{-n}{p} \right) = \left( \frac{-n}{l} \right) = 1 \tag{1}
\]
then \( A \cong \mathbb{Z}^2 \).

Every maximal abelian subgroup \( A \cong \mathbb{Z}^2 \) is conjugate to one satisfying (1).
There exist maximal abelian subgroups

\[ A \cong \mathbb{Z} . \]

**Example.** \( A = \langle \psi(1 + j + k) \rangle \in \Gamma_{3,5} \)

Here \( n = 1^2 + 1^2 = 2 \) and

\[
\left( \frac{-2}{3} \right) = 1, \quad \left( \frac{-2}{5} \right) = -1 .
\]