Free Groups
Products of Trees
Arithmetic of Quaternions

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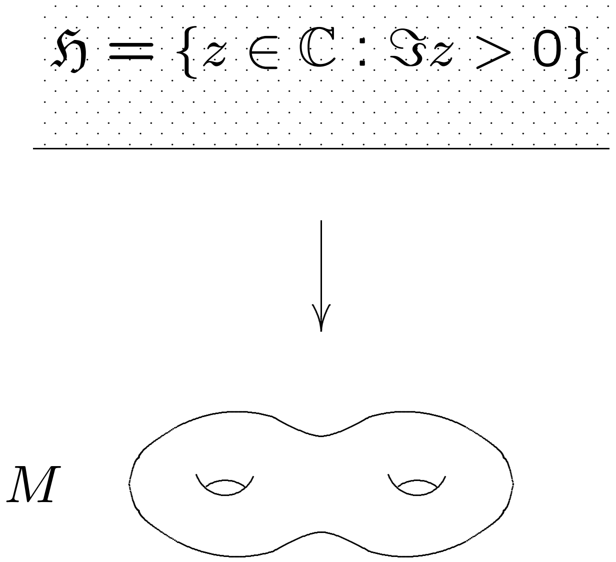
Motivation

A Closed Surface M has Universal Cover



$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}$$

The **fundamental group** Γ of M acts on $\mathfrak{H}$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

Γ can be realized as a subgroup of

$$\mathsf{PSL}_2(\mathbb{R}) = \mathsf{SL}_2(\mathbb{R})/\{-I, I\}$$

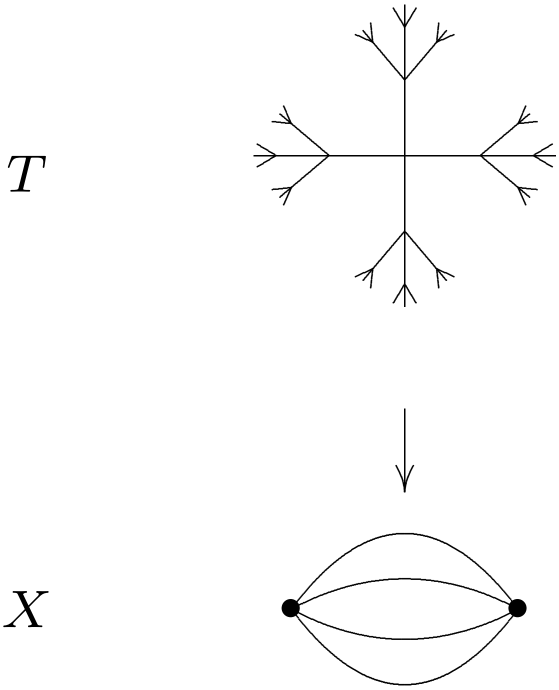
and

$$\Gamma \backslash \mathfrak{H} = M.$$

Discrete Analogue

$X:$ A finite connected graph.

$T:$ The universal covering space (a tree).



The fundamental group Γ of X is a **free group** and

$$\Gamma \backslash T = X.$$

What is the analogue of $\mathrm{PSL}_2(\mathbb{R})$**?**

Replace $\mathbb{R}$ by a **local field** ...

For $p \geq 2$ prime, $\mathbb{Q}_p$ is the field of formal sums :

$$x = a_j p^j + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots ,$$

where each $a_i \in \{0, 1, \cdots , p-1\}$ and $a_j \neq 0$.

$$|x| = p^{-j} \qquad \textit{discrete valuation}$$

The p-adic integers

$$\begin{aligned} \mathbb{Z}_p &= \{x \in \mathbb{Q}_p : |x| \leq 1\} \\ &= \text{ set of sums with } j \geq 0 \\ &= \overline{\mathbb{Z}} \qquad \textit{a compact open subring of } \mathbb{Q}_p \end{aligned}$$

Balls centre 0 :

$$\cdots \supset p^{-1}\mathbb{Z}_p \supset \mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \cdots$$

The group

$$\mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^{\times}$$

acts on a $(p+1)$-regular tree T_{p+1}.

The tree T_{p+1} of $\mathsf{PGL}_2(\mathbb{Q}_p)$

A lattice $L \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-submodule :

$$L = \mathbb{Z}_p \mathbf{v}_1 + \mathbb{Z}_p \mathbf{v}_2,$$

where $(\mathbf{v}_1, \mathbf{v}_2)$ is a basis of $\mathbb{Q}_p^2$.

Equivalence relation :

$$L_1 \sim L_2 \quad \Longleftrightarrow \quad L_1 = aL_2, \qquad a \in \mathbb{Q}_p^\times$$

An equivalence class $[L]$ is a **vertex** of T_{p+1}.

An **edge** of T_{p+1} is $([L_1], [L_2])$ where

$$L_1 \supset L_2 \supset pL_1.$$

$\mathsf{GL}_2(\mathbb{Q}_p)$ acts transitively on the set of bases $(\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{Q}_p^2$.

$\therefore G = \mathsf{PGL}_2(\mathbb{Q}_p)$ acts transitively on vertices.

A Combinatorial Construction

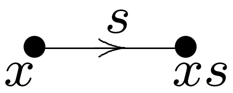
The free group of rank 2

$$\Gamma = \langle a, b \rangle$$

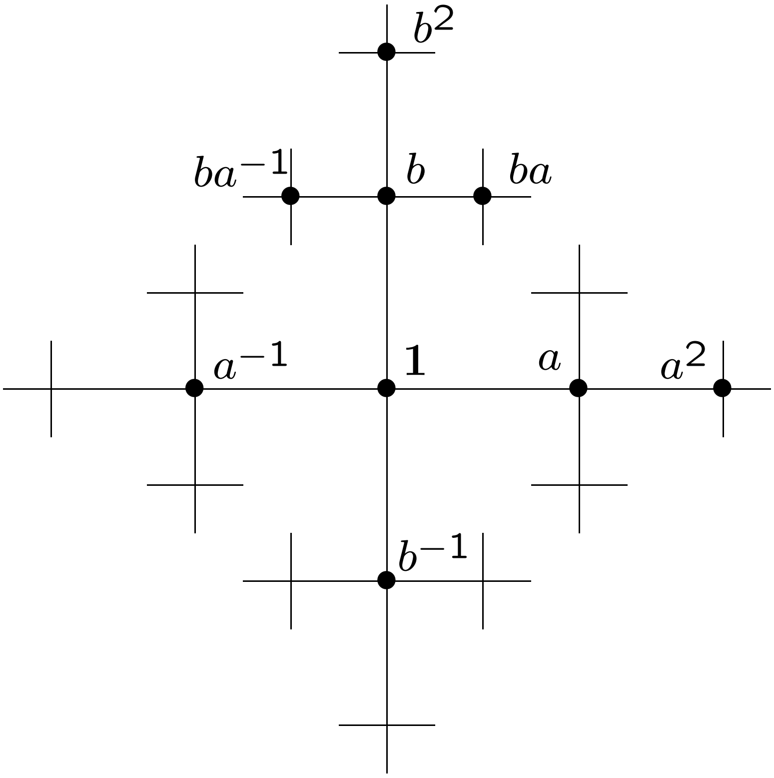
has a **Cayley graph** T.

Vertex set : Γ

Edges :



$s \in S = \left\{a, a^{-1}, b, b^{-1}\right\}$.

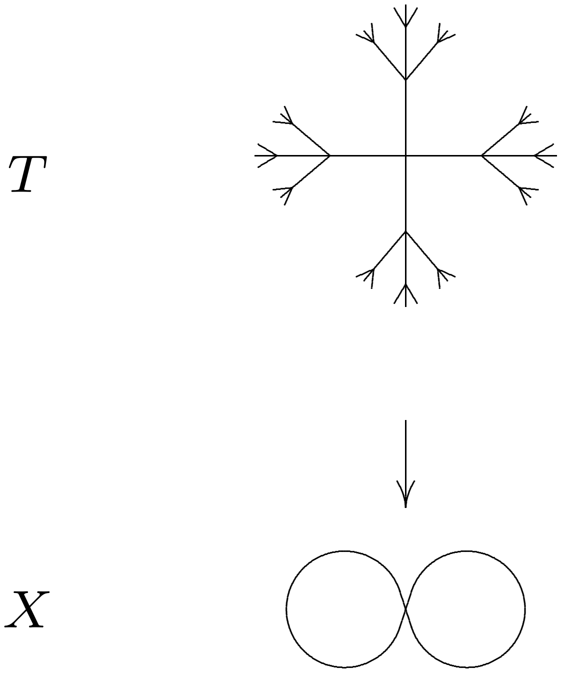


T is a **tree**.

Γ acts on T by left multiplication.

The quotient graph $X = \Gamma\backslash T$ has

- fundamental group Γ ;
- universal cover T.



Can embed $\Gamma < \mathrm{PGL}_2(\mathbb{Q}_3)$ as an **arithmetic subgroup**. . .

A Result of Jacobi

If p is an odd prime, the number of

$$(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$$

such that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$$

is

$$8(p+1).$$

Suppose $p \equiv 1 \pmod 4$. Then exactly **one** a_j is odd and the number of representations with a_0 odd, $a_0 > 0$, is

$$p + 1\,.$$

Consequence:
Let S_p be the set of integer quaternions

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}(\mathbb{Z})$$

with $a_0 > 0$, a_0 odd, and $|a|^2 = p$.
Then

$$|S_p| = p + 1\,.$$

An Arithmetic Construction

If $p \equiv 1 \pmod 4$ is prime, then

$$x^2 \equiv -1 \pmod p$$

has a solution in $\mathbb{Z}$, so by **Hensel's Lemma**,

$$x^2 = -1$$

has a solution i_p in $\mathbb{Q}_p$.

Define

$$\psi_p : \mathbb{H}(\mathbb{Z}) \to PGL_2(\mathbb{Q}_p)$$

by

$$\psi_p(a) = \begin{pmatrix} a_0 + a_1 i_p & a_2 + a_3 i_p \\ -a_2 + a_3 i_p & a_0 - a_1 i_p \end{pmatrix} .$$

Theorem. $\psi_p(S_p)$ contains $p+1$ elements and generates a free group Γ_p of rank $\frac{p+1}{2}$.

Lubotzky, Phillips, Sarnak (1988)

Uses "unique factorisation" for integer quaternions.

This construction extends to primes

$$p \equiv 3 \pmod 4$$

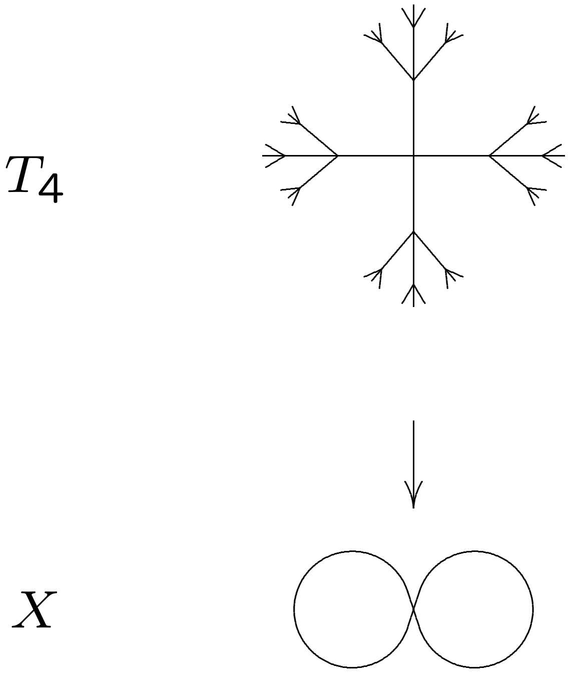
using solutions in $\mathbb{Z}$ of

$$x^2 + y^2 \equiv -1 \pmod p\,.$$

[D. Rattaggi (2003)]

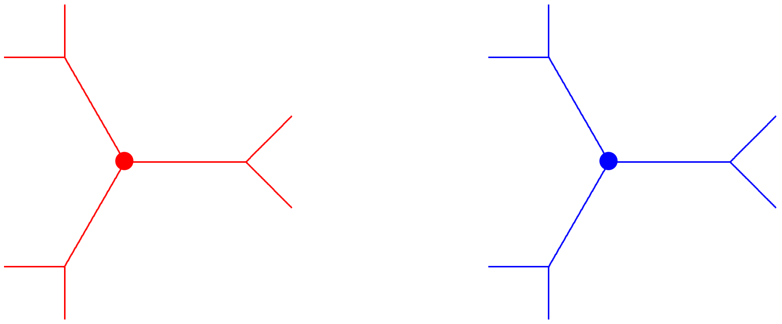
Γ_p acts freely and transitively on the vertices of the $(p+1)$-regular tree T_{p+1}.

i.e. $X = \Gamma_p \backslash T_{p+1}$ has one vertex.

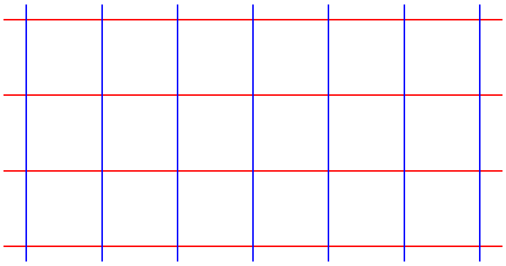


$$\boxed{p = 3}$$

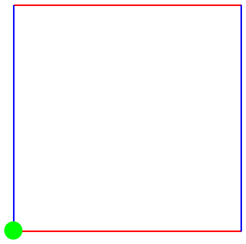
Products of Trees



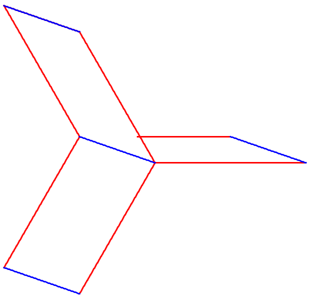
$T_m \times T_n$ is a 2-dimensional complex which is a union of flat subcomplexes



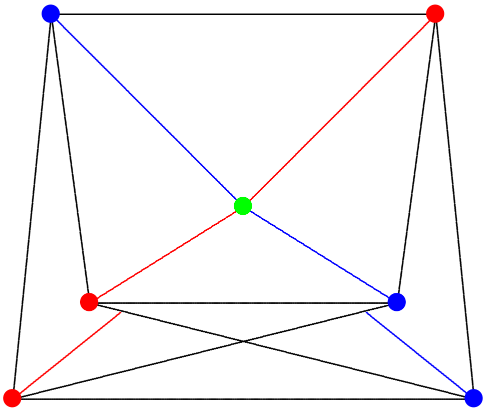
Each vertex is a corner of $m \times n$ squares :



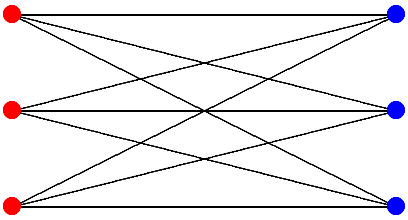
Neighbourhood of an edge :



Neighbourhood of a vertex :



The **link** of a vertex:



If $p \neq l$ are odd primes then

$$G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

acts on a product of trees

$$\Delta = T_{p+1} \times T_{l+1} \, .$$

Define

$$\psi : \mathbb{H}(\mathbb{Z}) \to PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

by

$$\psi(a) = (\psi_p(a), \psi_l(a)) \ .$$

Let $\Gamma_{p,l}$ be the subgroup of

$$PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

generated by $\psi(S_p \cup S_l)$.

[S. Mozes (1995)]

$\Gamma_{p,l}$ is generated by two free subgroups :

$$\langle \psi(S_p) \rangle \cong \Gamma_p$$
$$\langle \psi(S_l) \rangle \cong \Gamma_l$$

but

$$\boxed{\Gamma_{p,l} \ncong \Gamma_p \times \Gamma_l}$$

$\Gamma_{p,l}$ acts freely on Δ.

The quotient complex $X_{p,l}$ has one vertex and is obtained by glueing together $\frac{(p+1)(l+1)}{4}$ squares.

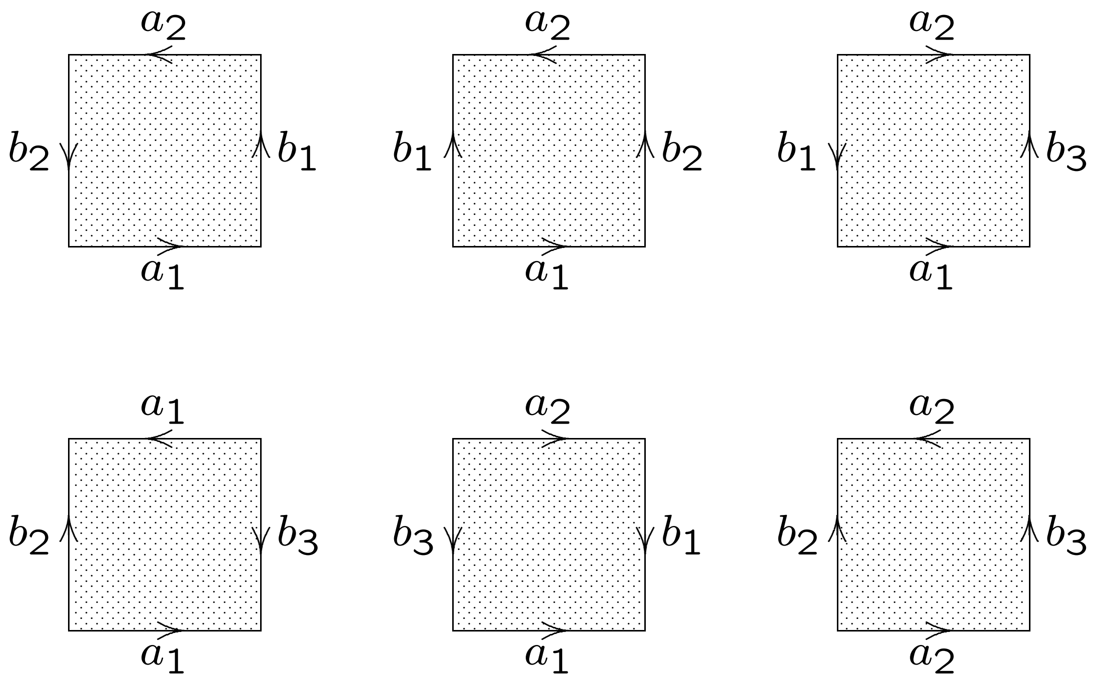
Example. $\Gamma_{3,5}$ has generators

$$\begin{aligned}
a_1 &= \psi(1+\mathbf{j}+\mathbf{k}), & a_1^{-1} &= \psi(1-\mathbf{j}-\mathbf{k}),\\
a_2 &= \psi(1+\mathbf{j}-\mathbf{k}), & a_2^{-1} &= \psi(1-\mathbf{j}+\mathbf{k}),\\
b_1 &= \psi(1+2\mathbf{i}), & b_1^{-1} &= \psi(1-2\mathbf{i}),\\
b_2 &= \psi(1+2\mathbf{j}), & b_2^{-1} &= \psi(1-2\mathbf{j}),\\
b_3 &= \psi(1+2\mathbf{k}), & b_3^{-1} &= \psi(1-2\mathbf{k}).
\end{aligned}$$

and relators

$a_1b_1a_2b_2$, $a_1b_2a_2b_1^{-1}$, $a_1b_3a_2^{-1}b_1$, $a_1b_3^{-1}a_1b_2^{-1}$, $a_1b_1^{-1}a_2^{-1}b_3$, $a_2b_3a_2b_2^{-1}$.

$X_{3,5}$ is obtained by glueing together 6 squares



and has fundamental group $\Gamma_{3,5}$.

Properties of Quaternions

1. If $x, y \in \mathbb{H}(\mathbb{Q})$, with $\Im(x) \neq 0$, $\Im(y) \neq 0$,

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},$$
$$y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k},$$

then

$$xy = yx$$

if and only if

$$(x_1, x_2, x_3) = \lambda(y_1, y_2, y_3)$$

where $\lambda \in \mathbb{Q}$.

2. There is a homomorphism

$$\theta : \mathbb{H}(\mathbb{Q}) - \{0\} \to \mathrm{SO}_3(\mathbb{Q})$$

defined by

$$\theta(y)x = yxy^{-1}$$

for $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{Q}^3$.

Properties of $\Gamma_{p,l}$

[D. Rattaggi, G. Robertson]

1. $\Gamma_{p,l}$ is **commutative transitive**.

$x \leftrightarrow y$ and $y \leftrightarrow z \quad \Rrightarrow \quad x \leftrightarrow z \qquad$ if $x, y, z \neq 1$.

2. The only nontrivial direct product subgroup of $\Gamma_{p,l}$ is

$$\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 .$$

[Answers a question of D. Wise.]

3. If A_1 and A_2 are distinct maximal abelian subgroups of $\Gamma_{p,l}$ then

$$A_1 \cap A_2 = \{1\} .$$

4. $\Gamma_{p,l}$ is a **CSA-group** :
for each maximal abelian subgroup A,

$$gAg^{-1} \cap A = \{1\}$$

if $g \in \Gamma - A$.

Maximal Abelian Subgroups

If $x = x_0 + z_0(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) \in \mathbb{H}(\mathbb{Z})$ with

$$z_0 \neq 0 \quad \text{and} \quad \gcd(c_1, c_2, c_3) = 1$$

define

$$n = n(x) = c_1^2 + c_2^2 + c_3^2 > 0\,.$$

If $\psi(x) \in \Gamma_{p,l}$, the centralizer

$$A = \{g \in \Gamma_{p,l} \ : \ g\psi(x) = \psi(x)g\}$$

is a maximal abelian subgroup of $\Gamma_{p,l}$,

$$A \cong \mathbb{Z} \quad \text{or} \quad A \cong \mathbb{Z}^2$$

and n depends only on A.

Theorem. If

$$\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1 \tag{1}$$

then $A \cong \mathbb{Z}^2$.

Every maximal abelian subgroup $A \cong \mathbb{Z}^2$ is conjugate to one satisfying (1).

There exist maximal abelian subgroups

$$A \cong \mathbb{Z}\,.$$

Example. $A = \langle \psi(1+j+k) \rangle \in \Gamma_{3,5}$

Here $n = 1^2 + 1^2 = 2$ and

$$\left(\frac{-2}{3}\right) = 1, \quad \left(\frac{-2}{5}\right) = -1\,.$$