# TORSION IN $K$-THEORY FOR BOUNDARY ACTIONS ON AFFINE BUILDINGS OF TYPE $\widetilde{A}_n$

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ABSTRACT. Let $\Gamma$ be a torsion free lattice in $G = \mathrm{PGL}(n+1, \mathbb{F})$, where $n \geq 1$ and $\mathbb{F}$ is a non-archimedean local field. Then $\Gamma$ acts on the Furstenberg boundary $G/P$, where $P$ is a minimal parabolic subgroup of $G$. The identity element $\mathbf{I}$ in the crossed product $C^*$-algebra $C(G/P) \rtimes \Gamma$ generates a class $[\mathbf{I}]$ in the $K_0$ group of $C(G/P) \rtimes \Gamma$. It is shown that $[\mathbf{I}]$ is a torsion element of $K_0$ and there is an explicit bound for the order of $[\mathbf{I}]$. The result is proved more generally for groups acting on affine buildings of type $\widetilde{A}_n$. For $n = 1, 2$ the Euler-Poincaré characteristic $\chi(\Gamma)$ annihilates the class $[\mathbf{I}]$.

## INTRODUCTION

A. Connes [Co1, Corollary 6.7] proved that if $\Gamma$ is a torsion free cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$ then the class of the identity $[\mathbf{I}]$ in $K_0(C(\mathbb{P}_1(\mathbb{R})) \rtimes \Gamma)$ has torsion. See also [N, Co2, HN, A-D]. More precisely, the order of $[\mathbf{I}]$ is $-\chi(\Gamma)$, where $\chi(\Gamma)$ is the Euler-Poincaré characteristic of the corresponding Riemann surface [A-D, Proposition 2.9], [HN, Section 10]. If $\Gamma$ is a torsion free non-cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$, which means that $\Gamma$ is a free group, then $[\mathbf{I}] = 0$ in $K_0(C(\mathbb{P}_1(\mathbb{R})) \rtimes \Gamma)$ [A-D, Theorem 2.1]. Moreover, Connes proved [Co1, Corollary 5.9] that if $\Gamma$ is any countable subgroup of $\mathrm{PSL}(2, \mathbb{C})$ then the canonical map $K_0(C(\mathbb{P}_1(\mathbb{C}))) \to K_0(C(\mathbb{P}_1(\mathbb{C})) \rtimes \Gamma)$ is injective and so in this case $[\mathbf{I}]$ is a non-torsion element in $K_0$. There is therefore a dramatic difference between the Fuchsian and the Kleinian cases.

Now let $\mathbb{F}$ be a non-archimedean local field and let $\Gamma$ be a torsion free (cocompact) lattice in $G = \mathrm{PGL}(2, \mathbb{F})$. As in the Fuchsian case, $\chi(\Gamma).[\mathbf{I}] = 0$ in $K_0(C(\mathbb{P}_1(\mathbb{F})) \rtimes \Gamma)$. This assertion has a simple geometric proof, which is presented in Section 1 below. The Bruhat-Tits building associated with $\mathrm{PGL}(2, \mathbb{F})$ is a regular tree $\Delta$ whose boundary $\partial\Delta$ may be identified with the projective line $\mathbb{P}_1(\mathbb{F})$. The result is proved in the more general situation where $\Gamma$ is a discrete group acting freely with

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compact quotient on a tree $\Delta$. Such a group $\Gamma$ is necessarily a free group [S2, Chapitre I.3.3].

A tree is an affine building of type $\widetilde{A}_1$. The principal object of this paper is to prove corresponding results for group actions on buildings of type $\widetilde{A}_n$, where $n \geq 2$. These buildings are in some respects higher dimensional analogues of trees, but have a very much more rigid structure.

Let $\Delta$ be a building of type $\widetilde{A}_n$. Then the link of any fixed vertex in $\Delta$ is the flag complex of a finite projective geometry. The order $q$ of this projective geometry is independent of the vertex, and is referred to as the order of the building $\Delta$. The case $n = 2$ is particularly interesting because then the link of a vertex is a projective plane, which is possibly non-Desarguesian, and $\Delta$ need not be the Bruhat-Tits building of a linear group. The boundary $\partial\Delta$ is the set of chambers of the spherical building at infinity [Ron, Chapter 9], endowed with a natural totally disconnected compact Hausdorff topology [Ca2, Section 4]. Denote by $\mathrm{Aut}(\Delta)$ the automorphism group of $\Delta$, equipped with the compact open topology. Then $\mathrm{Aut}(\Delta)$ acts in a natural way on $\partial\Delta$.

Given an $\widetilde{A}_n$ building $\Delta$ with vertex set $\Delta^0$, there is a type map $\tau : \Delta^0 \to \mathbb{Z}/(n+1)\mathbb{Z}$ such that each maximal simplex (*chamber*) has exactly one vertex of each type. An automorphism $\alpha$ of $\Delta$ is said to be *type-rotating* if there exists $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ such that $\tau(\alpha v) = \tau(v) + i$ for all vertices $v \in \Delta^0$.

The main result is the following.

**Theorem 1.** Let $\Gamma$ be a torsion free discrete group of type rotating automorphisms of a locally finite affine building $\Delta$ of type $\widetilde{A}_n$. Suppose that $\Gamma$ acts cocompactly on $\Delta$ with a finite number $n_0$ of vertex orbits. In the group $K_0(C(\partial\Delta) \rtimes \Gamma)$, let $[\mathbf{I}]$ denote the class of the identity element.

   (1) If $n = 1$ or 2, then $\chi(\Gamma).[\mathbf{I}] = 0$, where $\chi(\Gamma)$ is the Euler-Poincaré characteristic of the group $\Gamma$.
   (2) If $n \geq 2$, then the order of $\Delta$ is a prime power $q$ and $m.[\mathbf{I}] = 0$, where
$$m = \begin{cases} n_0(q-1) & \text{if } n \text{ is odd,} \\ n_0(q^2-1) & \text{if } n \text{ is even.} \end{cases}$$

If $n = 2$ then both parts of Theorem 1 apply and can be combined to sharpen the result. This is done in Section 4.

The condition in Theorem 1 that the building $\Delta$ is locally finite is an easy consequence of the other hypotheses, by Lemma 2.1 below. Also the finite cell complex $\Gamma\backslash\Delta$ is a $K(\Gamma, 1)$ space and the group $\Gamma$ has finite cohomological dimension [S1, I.1.5]. It follows that $\chi(\Gamma)$ coincides with the usual Euler-Poincaré characteristic of the cell complex $\Gamma\backslash\Delta$ [S1, I.1.5, Proposition 9]. Moreover $n_0$ is the number of vertices of $\Gamma\backslash\Delta$.

Let $\mathbb{F}$ be a non-archimedean local field with residue field of order $q$. The Bruhat-Tits building $\Delta$ of $G = \mathrm{PGL}(n+1, \mathbb{F})$ is a building of type $\widetilde{A}_n$ and the action of $G$ on $\Delta$ is type rotating [St]. The vertex set of $\Delta$ may be identified with the homogeneous space $G/K$, where $K$ is an open maximal compact subgroup of $G$. Then $n_0$ is the number of double cosets in $\Gamma \backslash G / K$. The boundary $\partial \Delta$ may be identified with the Furstenberg boundary $G/P$, where $P$ is a minimal parabolic subgroup of $G$. A torsion free lattice $\Gamma$ in $G = \mathrm{PGL}(n+1, \mathbb{F})$ is automatically cocompact [S2, Chapitre II.1.5, p. 116]. The result therefore applies to such lattices, acting on the Furstenberg boundary and shows that the class $[\mathbf{I}]$ in $K_0(C(G/P) \rtimes \Gamma)$ has torsion. If the Haar measure $\mu$ on $G$ is normalized so that $\mu(K) = 1$, then $n_0$ is the covolume of $\Gamma$ in $G$. It follows from Theorem 1 that there is a bound for the order of $[\mathbf{I}]$, which depends only on the covolume of $\Gamma$ in $G$.

The tree result is proved in Section 1, followed by the higher dimensional cases in Sections 2 and 3. Section 4 is concerned with lattices in $\mathrm{PGL}(n+1, \mathbb{F})$ and computation of the Euler-Poincaré characteristic.

It is natural to ask if there is a similar result for lattices in the group $\mathrm{PGL}(n+1, \mathbb{R})$. The methods used here do not throw any light on this question.

It is worth remarking that the crossed product $C^*$-algebras considered here are purely infinite, simple, separable, unital, nuclear and satisfy the Universal Coefficient Theorem. This was proved in [RS1] for buildings of type $\widetilde{A}_2$, and follows from [JR] in higher dimensions. These algebras are therefore classified up to isomorphism by their two K-groups together with the class of $[\mathbf{I}]$ in $K_0$ [K1]. Properties of the class $[\mathbf{I}]$ are therefore particularly significant.

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## 1. The $\widetilde{A}_1$ result

The tree case is treated first, since the proof is relatively simple and motivates the harder $\widetilde{A}_2$ case. Two vertices of a tree have the same type if and only if the graph distance between them is even. Any automorphism of a tree is type rotating, so the type rotating hypothesis in Theorem 1 is vacuous in the $\widetilde{A}_1$ case.

Let $\Delta$ be a locally finite tree whose vertices all have degree at least three. The terminology of [S2] will be used where possible. In particular the edges of $\Delta$ are directed edges. Each geometric edge of $\Delta$ corresponds to two directed edges $d$ and $\overline{d}$. Let $\Delta^0$ denote the set of vertices and $\Delta^1$ the set of directed edges of $\Delta$.

Let $\Gamma$ be a torsion free discrete subgroup of $\mathrm{Aut}(\Delta)$ which acts cocompactly on $\Delta$. Then $\Gamma$ acts freely (c.f. Lemma 2.1 below), and so is a free group [S2, I.3.3]. In particular $\Gamma$ acts without inversion, which

means that no element $\gamma \in \Gamma \setminus \{1\}$ stabilizes a geometric edge of $\Delta$. It follows that there is an orientation on the edges which is invariant under $\Gamma$ [S2, I.3.1]. Choose such an orientation. This orientation consists of a partition $\Delta^1 = \Delta^1_+ \sqcup \overline{\Delta^1_+}$ and a bijective involution $d \mapsto \overline{d} : \Delta^1 \to \Delta^1$ which interchanges the two components of $\Delta^1$. Each directed edge $d$ has an origin $o(d) \in \Delta^0$ and a terminal vertex $t(d) \in \Delta^0$ such that $o(\overline{d}) = t(d)$. The quotient $X = \Gamma \backslash \Delta$ is a finite connected graph with vertex set $V = \Gamma \backslash \Delta^0$ and directed edge set $E = E_+ \sqcup \overline{E_+}$, where $E_+ = \Gamma \backslash \Delta^1_+$ and $\overline{E_+} = \Gamma \backslash \overline{\Delta^1_+}$. The graph $X$ has an induced involution $e \mapsto \overline{e} : E \to E$ and there are maps $o, t : E \to V$ with $o(\overline{e}) = t(e)$. The Euler-Poincaré characteristic of the graph is $\chi(X) = n_0 - n_1$ where $n_0 = \#(V)$ and $n_1 = \#(E_+)$.

The boundary $\partial \Delta$ of the tree $\Delta$ can be identified with the set of equivalence classes of infinite semi-geodesics in $\Delta$, where equivalent semi-geodesics contain a common sub-semi-geodesic. Also $\partial \Delta$ has a natural compact totally disconnected topology [S2, I.2.2].

The group $\Gamma$ acts on $\partial \Delta$ and hence on $C(\partial \Delta)$ via $\gamma \mapsto \alpha_\gamma$, where $\alpha_\gamma f(\omega) = f(\gamma^{-1} \omega)$, for $f \in C(\partial \Delta)$, $\gamma \in \Gamma$. The algebraic crossed product relative to this action is the $*$-algebra $k(\Gamma, C(\partial \Delta))$ of functions $\phi : \Gamma \to C(\partial \Delta)$ of finite support, with multiplication and involution given by

$$\phi * \psi(\gamma_0) = \sum_{\gamma \in \Gamma} \phi(\gamma) \alpha_\gamma(\psi(\gamma^{-1} \gamma_0)) \quad \text{and} \quad \phi^*(\gamma) = \alpha_\gamma(\phi(\gamma^{-1})^*).$$

The full crossed product algebra $C(\partial \Delta) \rtimes \Gamma$ is the completion of the algebraic crossed product in an appropriate norm [Ped]. There is a natural embedding of $C(\partial \Delta)$ into $C(\partial \Delta) \rtimes \Gamma$ which maps $f \in C(\partial \Delta)$ to the function taking the value $f$ at the identity of $\Gamma$ and 0 elsewhere. The identity element $\mathbf{I}$ of $C(\partial \Delta) \rtimes \Gamma$ is then identified with the constant function $\mathbf{I}(\omega) = 1$, $\omega \in \partial \Delta$. There is a natural unitary representation $\pi : \Gamma \to C(\partial \Delta) \rtimes \Gamma$, where $\pi(\gamma)$ is the function taking the value $\mathbf{I}$ at $\gamma$ and 0 otherwise. It is convenient to denote $\pi(\gamma)$ simply by $\gamma$. Thus a typical element of the dense $*$-algebra $k(\Gamma, C(\partial \Delta))$ can be written as a finite sum $\sum_\gamma f_\gamma \gamma$, where $f_\gamma \in C(\partial \Delta)$, $\gamma \in \Gamma$. The definition of the multiplication implies the covariance relation

$$\alpha_\gamma(f) = \gamma f \gamma^{-1} \quad \text{for} \quad f \in C(\partial \Delta), \gamma \in \Gamma.$$

If $d \in \Delta^1$, let $\Omega(d)$ denote the clopen subset of $\partial \Delta$ corresponding to the set of all semi-geodesics with initial edge $d$ and initial vertex $o(d)$. The indicator function $p_d$ of the set $\Omega(d)$ is continuous and so lies in $C(\partial \Delta) \subset C(\partial \Delta) \rtimes \Gamma$.

For each $d \in \Delta^1$, the element $p_d$ is a projection in $C(\partial \Delta) \rtimes \Gamma$ and therefore defines an equivalence class $[p_d]$ in $K_0(C(\partial \Delta) \rtimes \Gamma)$. (The standard reference for the K-theory of $C^*$-algebras is [Bl].) It is important to observe that edges $d_1, d_2$ lying in the same $\Gamma$-orbit give rise

to equivalent projections $p_{d_1}, p_{d_2}$, because of the covariance relations in the crossed product algebra (c.f. Lemma 2.4 below). Therefore the equivalence class $[p_d]$ depends only on the directed edge $e = \Gamma d \in E$. Write $[e] = [p_d]$. In this way, each edge $e \in E$ gives rise to an element $[e] \in K_0(C(\partial\Delta) \rtimes \Gamma)$.

The projections $p_d$ satisfy the following relations as functions in $C(\partial\Delta)$, as illustrated in Figure 1.

(1.1a)
$$\sum_{\substack{d \in \Delta^1 \\ o(d)=a}} p_d = \mathbf{I}, \qquad \text{for } a \in \Delta^0;$$

(1.1b)
$$p_d + p_{\overline{d}} = \mathbf{I}, \qquad \text{for } d \in \Delta^1.$$

The relations (1.1) project to the following relations in $K_0(C(\partial\Delta) \rtimes \Gamma)$.
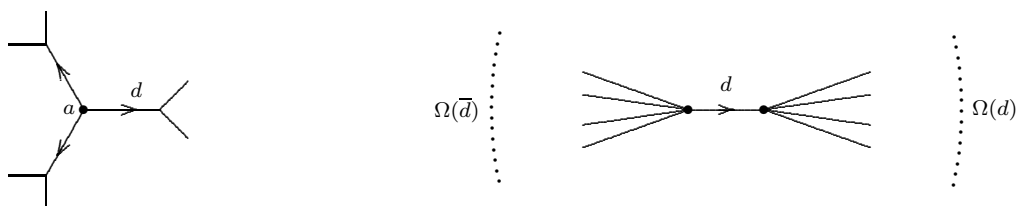


FIGURE 1. The geometric meaning of relations (1.1).

(1.2a)
$$\sum_{\substack{e \in E \\ o(e)=v}} [e] = [\mathbf{I}], \qquad \text{for } v \in V;$$

(1.2b)
$$[e] + [\overline{e}] = [\mathbf{I}], \qquad \text{for } e \in E.$$

Since the map $e \mapsto o(e) : E \to V$ is surjective, the relations (1.2) imply that

$$n_0[\mathbf{I}] = \sum_{v \in V} \sum_{\substack{e \in E \\ o(e)=v}} [e] = \sum_{e \in E} [e]$$

$$= \sum_{e \in E_+} ([e] + [\overline{e}]) = \sum_{e \in E_+} [\mathbf{I}]$$

$$= n_1[\mathbf{I}].$$

Therefore $(n_0 - n_1).[\mathbf{I}] = 0$. This proves Theorem 1(1) in the $\widetilde{A}_1$ case.

The basic idea of this proof will be generalized to buildings of type $\widetilde{A}_2$ in Section 3.

**Remark 1.1.** If the tree $\Delta$ is homogeneous of degree $q+1$, with $q \geq 2$, then $\chi(\Gamma) = \frac{n_0}{2}(1-q) \neq 0$ and so the element $[\mathbf{I}]$ is torsion.

## 2. THE $\widetilde{A}_n$ RESULT

Let $\Delta$ be a locally finite affine building of type $\widetilde{A}_n$, where $n \geq 2$. Thus $\Delta$ is an $n$-dimensional simplicial complex whose $n$-simplices are called *chambers*. Let $\Delta^k$ denote the set of $k$-simplices in $\Delta$ for $k = 0, 1, \ldots, n$. Recall that each vertex $\delta^0 \in \Delta^0$ is labelled with a type $\tau(\delta^0) \in \mathbb{Z}/(n+1)\mathbb{Z}$, and any chamber $\delta^n \in \Delta^n$ has precisely one vertex of each type. An *apartment* in $\Delta$ is a subcomplex which is isomorphic to a Coxeter complex of type $\widetilde{A}_n$. The building $\Delta$ is the union of its apartments.
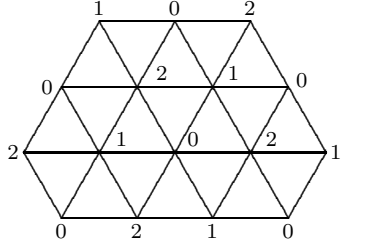


FIGURE 2. Part of an apartment in a building of type $\widetilde{A}_2$, showing vertex types.

A *sector* is a simplicial cone made up of chambers in some apartment and containing a unique base chamber [Ron, Chapter 9]. Two sectors are *equivalent* if their intersection contains a sector. The boundary $\partial\Delta$ of $\Delta$ is by definition the set of equivalence classes of sectors in $\Delta$. For any $\omega \in \partial\Delta$ and $\delta^0 \in \Delta^0$ there is a unique sector $[\delta^0, \omega)$ in the class $\omega$ having base vertex $\delta^0$ [Ron, Theorem 9.6]. The boundary is a totally disconnected compact Hausdorff space [CMS, Section 2], [Ca2, Section 4].

Let $\Gamma$ be a torsion free discrete group acting cocompactly on $\Delta$ by type rotating automorphisms. Then $X = \Gamma \backslash \Delta$ is a cell complex with universal covering $\Delta$. Let $X^k$ denote the set of $k$-cells in $X$ for $k = 0, 1, \ldots, n$. There is a natural induced action of $\Gamma$ on $\partial\Delta$, and we can form the universal crossed product $C^*$-algebra $C(\partial\Delta) \rtimes \Gamma$.

**Lemma 2.1.** *The group $\Gamma$ acts freely without inversion on $\Delta$. That is, no element $\gamma \in \Gamma \setminus \{1\}$ stabilizes a nonempty simplex of $\Delta$.*

*Proof.* Let $\delta$ be a nonempty simplex of $\Delta$ and define

$$G_\delta = \{g \in Aut(\Delta) \, ; \, g\delta = \delta\}.$$

Then $G_\delta$ is a compact subgroup of the locally compact group $Aut(\Delta)$. (See, for example, the proof of Theorem 4.1 of [GS].) Since $\Gamma$ is a discrete subgroup of the automorphism group of $\Delta$, it follows that $\Gamma \cap G_\delta$ is a finite subgroup of $\Gamma$. Since $\Gamma$ is torsion free, $\Gamma \cap G_\delta = \{1\}$. □

The first step is to formalize the notion of a directed cell in $X$. Let $\sigma^n$ be a model typed $n$-simplex with vertices $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}$. Assume that the vertex $\mathbf{p}$ of $\sigma^n$ has type $\tau(\mathbf{p}) = p \in \mathbb{Z}/(n+1)\mathbb{Z}$.
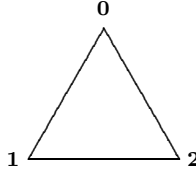


FIGURE 3. The case $n = 2$: the model simplex $\sigma^2$.

An isometry $r : \sigma^n \to \Delta$ is said to be *type rotating* if there exists $j \in \mathbb{Z}/(n+1)\mathbb{Z}$ such that, for each vertex $\mathbf{p}$ of $\sigma^n$, $\tau(r(\mathbf{p})) = \tau(\mathbf{p}) + j$ (mod $n+1$) . That is, $\tau(r(\mathbf{p})) = p + \tau(r(\mathbf{0}))$ (mod $n+1$). Let $D^n$ denote the set of type rotating isometries $r : \sigma^n \to \Delta$. Recall that any chamber $\delta^n \in \Delta^n$ has precisely one vertex of each type. Therefore, for each $j \in \mathbb{Z}/(n+1)\mathbb{Z}$ there is a unique $r \in D^n$ such that $r(\sigma^n) = \delta^n$ and $r(\mathbf{0})$ has type $j$. An element $r \in D^n$ is therefore uniquely determined by the pair $(r(\sigma^n), r(\mathbf{0}))$. We therefore identify $D^n$ with the set of directed chambers of $\Delta$. Each geometric chamber $\delta^n \in \Delta^n$ is the image of each of the (n+1) directed chambers $\{r \in D^n \; ; \; r(\sigma^n) = \delta^n\}$ under the map $r \mapsto r(\sigma^n)$.
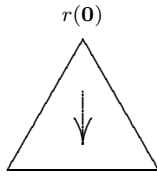


FIGURE 4. The image of a directed chamber $r$.

The next lemma records this observation.

**Lemma 2.2.** *The map $r \mapsto r(\sigma^n)$ from $D^n$ to $\Delta^n$ is $(n+1)$-to-1.*

Let $\mathfrak{D}^n = \Gamma \backslash D^n$. Call $\mathfrak{D}^n$ the set of *directed $n$-cells* of $X = \Gamma \backslash \Delta$. There is a commutative diagram

$$
\begin{array}{ccc}
D^n & \xrightarrow{\ r \mapsto r(\sigma^n)\ } & \Delta^n \\
\downarrow & & \downarrow \\
\mathfrak{D}^n & \xrightarrow{\ \varepsilon_n\ } & X^n
\end{array}
$$

where the vertical arrows represent quotient maps and $\varepsilon_n$ is defined by
$\varepsilon_n(\Gamma r) = \Gamma.r(\sigma^n)$.

The next result makes precise the fact that each geometric $n$-cell in
$X^n$ corresponds to exactly $n+1$ directed $n$-cells.

**Lemma 2.3.** *The map $\varepsilon_n : \mathfrak{D}^n \to X^n$ is surjective and $(n+1)$-to-1.*

*Proof.* Fix $\delta^n \in \Delta^n$. By Lemma 2.2, $\{r \in D^n \; ; \; r(\sigma^n) = \delta^n\} = \{r_0, r_1, \ldots, r_n\}$, a set containing precisely $n+1$ elements. Since $\Gamma$ acts
freely without inversion (Lemma 2.1), the set $\{\Gamma r_0, \Gamma r_1, \ldots, \Gamma r_n\} \subset \mathfrak{D}^n$
also contains precisely $n+1$ elements, each of which maps to $\Gamma\delta^n$ under
$\varepsilon_n$.

Now suppose that $\varepsilon_n(\Gamma r) = \Gamma\delta^n$ for some $r \in D^n$. Then $\gamma r(\sigma^n) = \delta^n$
for some $\gamma \in \Gamma$. Thus $\gamma r \in \{r_0, r_1, \ldots, r_n\}$ and so
$$\Gamma r \in \{\Gamma r_0, \Gamma r_1, \ldots, \Gamma r_n\}.$$
This proves that $\varepsilon_n$ is $(n+1)$-to-1. $\qquad\qquad\square$

Let $r \in D^n$; that is, suppose that $r : \sigma^n \to \Delta$ is a type rotating
isometry. Let
$$\Omega(r) = \{\omega \in \partial\Delta \; ; \; r(\sigma^n) \subset [r(\mathbf{0}), \omega)\},$$
the set of boundary points represented by sectors which originate at
$r(\mathbf{0})$ and contain $r(\sigma^n)$. (Figure 5 illustrates the case $n = 2$.)
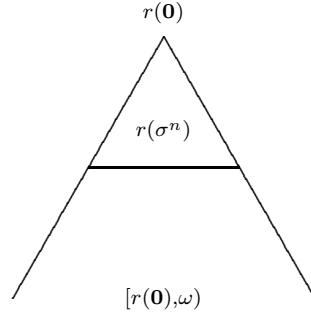


FIGURE 5. The sector $[r(\mathbf{0}), \omega)$, for $\omega \in \Omega(r)$.

Clearly $\Omega(\gamma r) = \gamma\Omega(r)$. Let $p_r \in C(\partial\Delta)$ be the characteristic func-
tion of $\Omega(r)$ and let $\mathbf{I}$ be the constant function $\mathbf{I}(\omega) = 1$, for $\omega \in \partial\Delta$.

The discussion in [CMS, Section 2], [Ca2, Section 4] shows that for
each vertex $\delta^0 \in \Delta^0$,

$$(2.1) \qquad\qquad \mathbf{I} = \sum_{\substack{r \in D^n \\ r(\mathbf{0}) = \delta^0}} p_r.$$

The essential point is that each $\omega \in \partial\Delta$ corresponds to a unique sector
$[\delta^0, \omega)$. It follows from (2.1) that in $K_0(C(\partial\Delta) \rtimes \Gamma)$,

$$[\mathbf{I}] = \sum_{\substack{r \in D^n \\ r(\mathbf{0}) = \delta^0}} [p_r].$$

As in the $\widetilde{A}_1$ case, it is important that the class $[p_r]$ in $K_0$ depends only on the orbit $\Gamma r$.

**Lemma 2.4.** *If $r, s \in D^n$ with $\Gamma r = \Gamma s$ then $[p_r] = [p_s]$.*

*Proof.* If $r = \gamma s$ with $\gamma \in \Gamma$ then the covariance condition for the action of $\Gamma$ on $C(\Omega)$ implies that $p_r = \gamma p_s \gamma^{-1}$. The result now follows because equivalent idempotents belong to the same class in $K_0$. $\qquad\square$

In view of Lemma 2.4, the following notation makes sense.

**Definition 2.5.** If $d = \Gamma r \in \mathfrak{D}^n$, let $[d] = [p_r] \in K_0(C(\partial\Delta) \rtimes \Gamma)$. Also let $d(\mathbf{p}) = \Gamma r(\mathbf{p}) \in X^0$ for $\mathbf{p} \in \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{n}\}$, and let $d(\sigma^n) = \Gamma r(\sigma^n) \in X^n$.

Using this notation, for $x^0 = \Gamma\delta^0 \in X^0$,

$$\{[p_r] \ ; \ r \in D^n, r(\mathbf{0}) = \delta^0\} = \{[d] \ ; \ d \in \mathfrak{D}^n, d(\mathbf{0}) = x^0\}.$$

Therefore in $K_0(C(\partial\Delta) \rtimes \Gamma)$,

$$(2.2) \qquad\qquad [\mathbf{I}] = \sum_{\substack{d \in \mathfrak{D}^n \\ d(\mathbf{0}) = x^0}} [d], \qquad \text{for } \delta^0 \in \Delta^0.$$

Since the building $\Delta$ is of type $\widetilde{A}_n$, where $n \geq 2$, it has additional structure. The link of any fixed vertex $\delta^0 \in \Delta$ is the flag complex of a finite projective geometry $\Pi(\delta^0)$ of dimension $n$ and order $q \geq 2$ [Ron, Theorem 3.5 and p. 95]. Denote by $\text{star}(\delta^0)$ the set of vertices at distance 1 from $\delta^0 \in \Delta$. There is a bijective correspondence between the vertices in $\text{star}(\delta^0)$ and subspaces of the projective geometry $\Pi(\delta^0)$. Two elements in $\text{star}(\delta^0)$ are incident in $\Pi(\delta^0)$ if they lie in a common chamber of $\Delta$. If $n \geq 3$ then $\Pi(\delta^0)$ is a Desarguesian projective geometry $PG(n, q)$ [St, (4.9)] and $q$ is a prime power. On the other hand if $n = 2$, then $\Pi(\delta^0)$ is a (possibly non-Desarguesian) projective plane. In either case, the order $q$ of $\Pi(\delta^0)$ is independent of $\delta^0$ and each $(n-1)$-dimensional simplex of $\Delta$ lies in $q+1$ chambers. The dimension $\dim(x)$ of a subspace $x \in \Pi(\delta^0)$ is the maximum $k$ for which there is a flag of subspaces $x_1 \subset x_2 \subset \cdots \subset x_k = x$. Thus a point of $\Pi(\delta^0)$ has dimension one and a line has dimension 2. This terminology has the unfortunate consequence that a maximal proper subspace of $n$-dimensional projective space has dimension $n$. Note that if $x \in \text{star}(\delta^0)$ then $\tau(x) = \tau(\delta^0) + \dim(x)$.

Let $r \in D^n$ and fix $k \in \{1, 2, \dots, n\}$. Define, for each $r \in D^n$,

$$\mathfrak{S}_k(r) = \{s \in D^n \ ; \ \Omega(s) \subset \Omega(r) \text{ and } s(\mathbf{0}) = r(\mathbf{k})\}.$$

The set $\mathfrak{S}_k(r)$ contains $q^{k(n+1-k)}$ elements $s \in D^n$. In fact, a more general result is contained in [Ca2]. See in particular Lemma 2.1 and the proof of Lemma 2.4 of that article. We sketch the proof.

Consider the set of chambers $C$ such that some sector based at $r(\mathbf{0})$ contains a subsector, based at $r(\mathbf{k})$, whose initial chamber is $C$. The
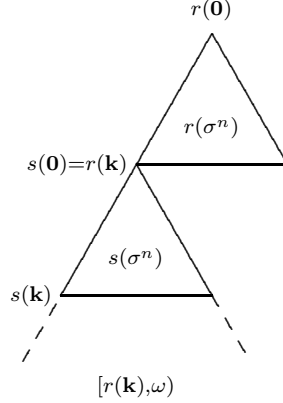
FIGURE 6. The condition $s \in \mathfrak{S}_k(r)$.

set of such chambers $C$ is exactly the preimage of a certain chamber with respect to the retraction from $r(\sigma^n)$. As such, the number of possible $C$ is $q^l$ where $l$ is the number of transitions in any stretched gallery between $r(\sigma^n)$ and any such chamber $C$.

As both of those chambers contain the vertex $r(\mathbf{k})$, one can calculate the number of transitions working inside the spherical building $\mathrm{star}(r(\mathbf{k}))$. In that spherical building, the chambers of an apartment can be parameterized by the elements of the Weyl group, which is just the symmetric group on $\{1, 2, \ldots, n+1\}$. Fixing specific coordinates one can identify $r(\sigma^n)$ with the identity permutation and $C$ with the permutation

$$(k+1 \quad k+2 \quad \ldots \quad n+1 \quad 1 \quad 2 \quad \ldots \quad k)$$

The number of transitions in the stretched gallery we are considering is equal to the length of this element in the Weyl group (relative to the standard Coxeter basis, $(12)$, $(23)$, etc.), which in turn is equal to the number of pairs whose order is inverted. This last is easily seen to be $k(n+1-k)$.

Thus $\mathfrak{S}_k(r)$ contains $q^{k(n+1-k)}$ elements $s \in D^n$ and $\Omega(r)$ may be expressed as a disjoint union

$$\Omega(r) = \bigsqcup_{s \in \mathfrak{S}_k(r)} \Omega(s).$$

Therefore

(2.3) $$p_r = \sum_{s \in \mathfrak{S}_k(r)} p_s.$$

Moreover a dual argument shows that each $s \in D^n$ lies in precisely $q^{k(n+1-k)}$ of the sets $\mathfrak{S}_k(r)$, with $r \in D^n$.

Define as follows a $\{0,1\}$-matrix $M_k$, with entries indexed by $\mathfrak{D}^n \times \mathfrak{D}^n$. If $c, d \in \mathfrak{D}^n$, then $M_k(d,c) = 1$ if and only if there are representative isometries $r, s \in D^n$, with $c = \Gamma r$, $d = \Gamma s$ and $s \in \mathfrak{S}_k(r)$; otherwise $M_k(d,c) = 0$. Each row or column of $M_k$ has precisely $q^{k(n+1-k)}$ nonzero entries and equation (2.3) implies that, for each $c \in \mathfrak{D}$,

$$(2.4) \qquad [c] = \sum_{d \in \mathfrak{D}^n} M_k(d,c)[d].$$

Recall that $n_0$ denotes the number of vertices of $X = \Gamma\backslash\Delta$, that is the number of $\Gamma$-orbits of vertices of $\Delta$. It follows from equations (2.2), (2.4) that

$$n_0[\mathbf{I}] = \sum_{c \in \mathfrak{D}^n} [c] = \sum_{c \in \mathfrak{D}^n} \sum_{d \in \mathfrak{D}^n} M_k(d,c)[d]$$

$$= \sum_{d \in \mathfrak{D}^n} \sum_{c \in \mathfrak{D}^n} M_k(d,c)[d] = \sum_{d \in \mathfrak{D}^n} q^{k(n+1-k)}[d]$$

$$= n_0 q^{k(n+1-k)}[\mathbf{I}].$$

Therefore, for each $k \in \{1, 2, \ldots, n\}$,

$$(2.5) \qquad n_0(q^{k(n+1-k)} - 1)[\mathbf{I}] = 0.$$

In particular, setting $k = 1, 2$ gives

$$n_0(q^n - 1)[\mathbf{I}] = 0 \qquad \text{and} \qquad n_0(q^{2(n-1)} - 1)[\mathbf{I}] = 0.$$

If $n$ is odd then $\gcd(n, 2(n-1)) = 1$ and so $n_0(q-1)[\mathbf{I}] = 0$.
If $n$ is even then $\gcd(n, 2(n-1)) = 2$ and so $n_0(q^2 - 1)[\mathbf{I}] = 0$.

This completes the proof of Theorem 1(2). $\qquad\qquad\square$

**Remark 2.6.** The same argument, greatly simplified, applies in the case of a homogeneous tree $\Delta$ of degree $q + 1$, where $q \geq 2$, and shows that $n_0(q-1)[\mathbf{I}] = 0$. This is weaker than the result obtained in Section 1, Remark 1.1.

## 3. The $\widetilde{A}_2$ result

This section will be concerned with buildings of type $\widetilde{A}_2$. The previous results all apply, with $n = 2$. The terminology and results from Section 2 will therefore be used without further comment.

Let $\sigma^e$ denote the face $\{\mathbf{1}, \mathbf{2}\}$ of $\sigma^2 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. We regard $\sigma^e$ as a model 1-simplex. An isometry $r : \sigma^e \to \Delta$ is said to be *type rotating* if there exists $j \in \mathbb{Z}/3\mathbb{Z}$ such that, for each vertex $\mathbf{p}$ of $\sigma^2$, $\tau(r(\mathbf{p})) = \tau(\mathbf{p}) + j \pmod 3$ for each vertex $\mathbf{p}$ of $\sigma^2$. That is, $\tau(r(\mathbf{p})) = p + \tau(r(\mathbf{0})) \pmod 3$. Let $D^e$ denote the set of type rotating isometries $r : \sigma^e \to \Delta$.

Any 1-simplex $\delta^1 \in \Delta^1$ has two vertices of two different types. The type rotating condition implies that there is a unique $r \in D^e$ such that $r(\sigma^e) = \delta^1$. That is, $r \in D^e$ is uniquely determined by its image $r(\sigma^e)$. We use $D^e$ to choose a preferred orientation for each edge of

$\Delta$. By a *directed edge*, we mean an edge with its preferred orientation. We therefore identify $D^e$ with the set of directed edges of $\Delta$. Each geometric edge $\delta^1 \in \Delta^1$ is the image of a *unique* directed edge $r \in D^e$ under the map $r \mapsto r(\sigma^e)$. That is, each edge of $\Delta$ has a unique direction determined by the typing of the vertex set of $\Delta$. See Figure 7. This is only superficially different from the $\widetilde{A}_1$ case in Section 1, where each geometric edge corresponds to two directed edges. In the tree case an edge is a maximal simplex.
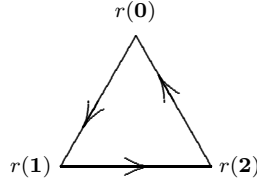


$$r(\mathbf{0})$$

$$r(\mathbf{1}) \qquad\qquad r(\mathbf{2})$$

FIGURE 7. Directed edges of a 2-simplex in $\Delta$.

**Lemma 3.1.** *The map $r \mapsto r(\sigma^e)$ from $D^e$ to $\Delta^1$ is bijective.*

Let $\mathfrak{D}^e = \Gamma\backslash D^e$. Call $\mathfrak{D}^e$ the set of *directed edges* of $X = \Gamma\backslash\Delta$. There is a commutative diagram

$$
\begin{array}{ccc}
D^e & \xrightarrow{r \mapsto r(\sigma^e)} & \Delta^1 \\
\downarrow & & \downarrow \\
\mathfrak{D}^e & \xrightarrow{\varepsilon_1} & X^1
\end{array}
$$

where the vertical arrows represent quotient maps and $\varepsilon_1$ is defined by $\varepsilon_1(\Gamma r) = \Gamma.r(\sigma^e)$.

The next result makes precise the fact that each geometric edge in $X^1$ corresponds to exactly one directed edge (i.e. has a unique direction). It is an easy consequence of the previous lemma.

**Lemma 3.2.** *The map $\varepsilon_1 : \mathfrak{D}^e \to X^1$ is bijective.*

If $r \in D^2$, let $\overline{\Omega}(r)$ denote the set of $\omega \in \partial\Delta$ such that $r(\mathbf{0})$ is in the interior of the convex hull $\mathrm{conv}(r(\sigma^n) \cup [r(\mathbf{0}),\omega))$ as illustrated in Figure 8. It is easy to see that $\overline{\Omega}(r)$ is a clopen set.
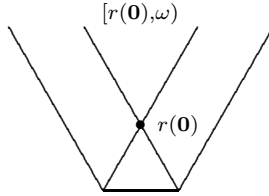


$$[r(\mathbf{0}),\omega)$$

$$r(\mathbf{0})$$

FIGURE 8

Let $\overline{p}_r \in C(\partial\Delta)$ be the characteristic function of $\overline{\Omega}(r)$. We have the following analogue of Lemma 2.4, which has the same proof.

**Lemma 3.3.** *If $r, s \in D^2$ with $\Gamma r = \Gamma s$ then $[\bar{p}_r] = [\bar{p}_s]$.*

In view of Lemma 3.3, the following notation makes sense.

**Definition 3.4.** *If $d = \Gamma r \in \mathfrak{D}^2$, let $[\bar{d}] = [\bar{p}_r] \in K_0(C(\partial\Delta) \rtimes \Gamma)$.*

Recall now the following result.

**Lemma 3.5.** [Ron, Lemma 9.4] *Given any chamber $\delta^2 \in \Delta^2$ and any sector $S$ in $\Delta$, there exists a sector $S_1 \subset S$ such that $S_1$ and $\delta^2$ lie in a common apartment.*

It follows by considering parallel sectors in an appropriate apartment that if $\delta^2 \in \Delta^2$ and if $\omega \in \partial\Delta$, then $\omega$ has a representative sector $S$ that lies relative to $\delta^2$ in one of the two positions in Figure 9, in some apartment containing them both.
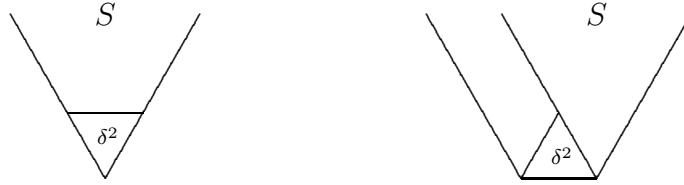


FIGURE 9. Relative positions of a chamber and a representative sector.

Therefore

$$\mathbf{I} = \sum_{\substack{r \in D^2 \\ r(\sigma^2) = \delta^2}} p_r + \sum_{\substack{r \in D^2 \\ r(\sigma^2) = \delta^2}} \bar{p}_r$$

where there are three terms in each sum.

Now for $x^2 = \Gamma\delta^2 \in X^2$,

$$\{[p_r] \; ; \; r \in D^2, r(\sigma^2) = \delta^2\} = \{[d] \; ; \; d \in \mathfrak{D}^2, d(\sigma^2) = x^2\}.$$

Therefore in $K_0(C(\partial\Delta) \rtimes \Gamma)$,

$$(3.1) \qquad [\mathbf{I}] = \sum_{\substack{d \in \mathfrak{D}^2 \\ d(\sigma^2) = x^2}} [d] + \sum_{\substack{d \in \mathfrak{D}^2 \\ d(\sigma^2) = x^2}} [\bar{d}].$$

If $r \in D^e$ then any $\omega \in \partial\Delta$ has a representative sector $S$ which lies relative to $r(\sigma^e)$ in one of the three positions in Figure 10, in some apartment containing them both. This is an easy consequence of Lemma 3.5.

Let $\Omega(r), \overline{\Omega}(r), \hat{\Omega}(r)$ denote the corresponding clopen subsets of $\partial\Delta$, and let $q_r, \bar{q}_r, \hat{q}_r$ respectively denote the corresponding characteristic functions. These functions are idempotents in $C(\partial\Delta) \rtimes \Gamma$ and the equivalence classes $[q_r], [\bar{q}_r], [\hat{q}_r]$ in $K_0$ depend only on $\Gamma r$. If $\delta^1 \in \Delta^1$ then there is a unique $r \in D^e$ such that $r(\sigma^e) = \delta^1$ and we have $\mathbf{I} = q_r + \bar{q}_r + \hat{q}_r$.
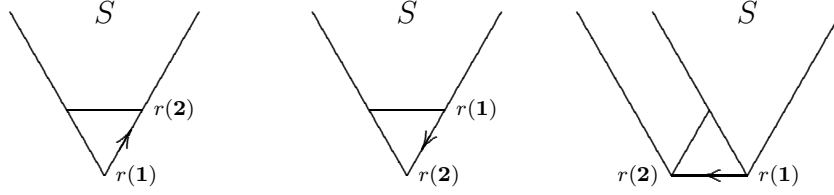
FIGURE 10. Relative positions of an edge and a representative sector.

**Definition 3.6.** If $e = \Gamma r \in \mathfrak{D}^e$, where $r \in D^e$, let $[e] = [q_r]$, $[\overline{e}] = [\overline{q}_r]$ and $[\hat{e}] = [\hat{q}_r]$

We now have

$$(3.2) \qquad\qquad [\mathbf{I}] = [e] + [\overline{e}] + [\hat{e}]$$

**Definition 3.7.** If $e = \Gamma r \in \mathfrak{D}^e$ and $\mathbf{p} \in \{\mathbf{1}, \mathbf{2}\}$, let $e(\mathbf{p}) = \Gamma r(\mathbf{p}) \in X^0$.

Think of $e(\mathbf{1})$ as the initial vertex and $e(\mathbf{2})$ as the final vertex of the directed edge $e$.

For each vertex $\delta^0 \in \Delta^0$,

$$\mathbf{I} = \sum_{\substack{r \in D^e \\ r(\mathbf{1})=\delta^0}} q_r = \sum_{\substack{r \in D^e \\ r(\mathbf{2})=\delta^0}} \overline{q}_r.$$

It follows that if $x^0 = \Gamma \delta^0 \in X^0$ then

$$(3.3) \qquad\qquad [\mathbf{I}] = \sum_{\substack{e \in \mathfrak{D}^e \\ e(\mathbf{1})=x^0}} [e] = \sum_{\substack{e \in \mathfrak{D}^e \\ e(\mathbf{2})=x^0}} [\overline{e}].$$

Given $r \in D^e$ and any chamber $\delta^2 \in \Delta^2$ containing $r(\sigma^e)$, $r$ can be extended uniquely to an element $s \in D^2$ such that $s(\sigma^2) = \delta^2$. Therefore $\hat{\Omega}(r)$ may be expressed as a disjoint union

$$\hat{\Omega}(r) = \bigsqcup \{\overline{\Omega}(s) \; ; \; s \in D^2, \; s|_{\sigma^e} = r\}.$$

It follows that $\hat{q}_r = \sum \overline{p}_s$, where the sum is over $s \in D^2$ such that $s|_{\sigma^e} = r$. Choose one $r$ from each $\Gamma$-orbit in $D^e$. Summing over these $r$ gives

$$\sum_r \hat{q}_r = \sum_r \sum_{s|_{\sigma^e}=r} \overline{p}_s$$

where the double sum on the right contains precisely one $s$ from each $\Gamma$-orbit in $D^2$, because $\Gamma$ acts freely without inversion on $\Delta$. Taking equivalence classes in $K_0$ gives

$$(3.4) \qquad\qquad \sum_{e \in \mathfrak{D}^e} [\hat{e}] = \sum_{d \in \mathfrak{D}^2} [\overline{d}].$$

*Proof of Theorem 1 in the $\widetilde{A}_2$ case.*

For $k = 0, 1, 2$, let $n_k$ denote the number of $k$-cells in $X^k$. By Lemma 3.2, $n_1 = \#(\mathfrak{D}^e)$ and by Lemma 2.3, $3n_2 = \#(\mathfrak{D}^2)$. It follows from equation (2.2) with $n = 2$, that

$$n_0[\mathbf{I}] = \sum_{x^0 \in X^0} \sum_{\substack{d \in \mathfrak{D}^2 \\ d(\mathbf{0}) = x^0}} [d] = \sum_{d \in \mathfrak{D}^2} [d].$$

Equation (3.1) implies that

$$(3.5) \quad n_2[\mathbf{I}] = \sum_{x^2 \in X^2} \left( \sum_{\substack{d \in \mathfrak{D}^2 \\ d(\sigma^2) = x^2}} [d] + \sum_{\substack{d \in \mathfrak{D}^2 \\ d(\sigma^2) = x^2}} [\overline{d}] \right) = \sum_{d \in \mathfrak{D}^2} [d] + \sum_{d \in \mathfrak{D}^2} [\overline{d}].$$

Therefore

$$(3.6) \qquad\qquad (n_2 - n_0)[\mathbf{I}] = \sum_{d \in \mathfrak{D}^2} [\overline{d}].$$

Now by (3.3),

$$(3.7) \qquad\qquad n_0[\mathbf{I}] = \sum_{e \in \mathfrak{D}^e} [e] = \sum_{e \in \mathfrak{D}^e} [\overline{e}].$$

Using equations (3.2),(3.7),(3.4) and (3.6) successively yields

$$\begin{aligned}
n_1[\mathbf{I}] &= \sum_{e \in \mathfrak{D}^e} [e] + \sum_{e \in \mathfrak{D}^e} [\overline{e}] + \sum_{e \in \mathfrak{D}^e} [\hat{e}] \\
&= 2n_0[\mathbf{I}] + \sum_{e \in \mathfrak{D}^e} [\hat{e}] \\
&= 2n_0[\mathbf{I}] + \sum_{d \in \mathfrak{D}^2} [\overline{d}] \\
&= 2n_0[\mathbf{I}] + (n_2 - n_0)[\mathbf{I}] \\
&= n_0[\mathbf{I}] + n_2[\mathbf{I}].
\end{aligned}$$

This proves Theorem 1(1) in the $\widetilde{A}_2$ case. $\qquad\qquad\square$

Note that the above proof has nowhere used the fact that the $\widetilde{A}_2$ building $\Delta$ has order $q$.

## 4. EULER-POINCARÉ CHARACTERISTIC AND THE $\widetilde{A}_2$ CASE

In this section we sharpen the result of Theorem 1 in the $\widetilde{A}_2$ case. We assume throughout that $\Delta$ is a building of type $\widetilde{A}_2$ and order $q$, and that $\Gamma$ is a torsion free discrete subgroup of $\mathrm{Aut}(\Delta)$ which acts cocompactly on $\Delta$.

The first step is to obtain an explicit expression for $\chi(\Gamma)$. Recall that $\chi(\Gamma)$ coincides with the usual Euler-Poincaré characteristic of cell

complex $X = \Gamma\backslash\Delta$ [S1, I.1.5, Proposition 9]. That is $\chi(\Gamma) = n_0 - n_1 + n_2$, where $n_k$ denotes the number of $k$-cells in $X^k$.

Since the building $\Delta$ is of type $\widetilde{A}_2$, the link of any fixed vertex $\delta^0 \in \Delta$ is the flag complex of a finite projective plane $\Pi(\delta^0)$ of order $q$. The vertices in $\mathrm{star}(\delta^0)$ are the points and lines of $\Pi(\delta^0)$. A point and a line in $\mathrm{star}(\delta^0)$ are incident in $\Pi(\delta^0)$ if they lie in a common chamber of $\Delta$. Now $\Pi(\delta^0)$ contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Moreover, each point is incident with $q + 1$ lines. Therefore the vertex $\delta^0 \in \Delta$ lies on $2(q^2 + q + 1)$ edges in $\Delta^1$ and on $(q + 1)(q^2 + q + 1)$ chambers in $\Delta^2$. Since there are $n_0$ vertex orbits in $\Delta^0$, $\#(\mathfrak{D}^1) = 2n_0(q^2 + q + 1)$ and $\#(\mathfrak{D}^2) = n_0(q + 1)(q^2 + q + 1)$. It follows from Lemma 2.3 that $n_1 = n_0(q^2 + q + 1)$ and $n_2 = n_0(q + 1)(q^2 + q + 1)/3$. Therefore

$$(4.1) \qquad \chi(\Gamma) = \frac{n_0}{3}(q - 1)(q^2 - 1).$$

**Remark 4.1.** Let $\mathbb{F}$ be a non-archimedean local field with residue field of order $q$. Let $\Gamma$ be a torsion free lattice in $G = \mathrm{PGL}(3, \mathbb{F})$. Recall that such a lattice is necessarily cocompact. The group $G$ acts in a type rotating manner on its Bruhat-Tits building $\Delta$ [St]. The building $\Delta$ is of type $\widetilde{A}_2$ and Theorem 1 therefore applies to any torsion free lattice $\Gamma$ in $G$. By [S1, 3.3], $G$ has a unique invariant measure $\mu_G$ (called the *canonical measure* of $G$) which is an *Euler-Poincaré measure*. That is $\mu_G(G/\Gamma) = \chi(\Gamma)$, for each torsion free lattice $\Gamma$ in $G$.

In the present situation it is not hard to determine $\mu_G$ directly. Let $\mathcal{O}$ denote the valuation ring of $\mathbb{F}$. Then $K = \mathrm{PGL}(3, \mathcal{O})$ is an open maximal compact subgroup of $G$ and the vertex set of $\Delta$ may be identified with the homogeneous space $G/K$. Then arguing as in [S1, p. 150, Théorème 7] shows that $\mu_G$ is the Haar measure on $G$ normalized so that $\mu_G(K) = \frac{1}{3}(q - 1)(q^2 - 1)$, in agreement with (4.1).

The statement of Theorem 1 can now be improved in the $\widetilde{A}_2$ case.

**Corollary 4.2.** *Let $\Gamma$ be a torsion free discrete group of type rotating automorphisms of a locally finite affine building $\Delta$ of type $\widetilde{A}_2$ and order $q$. Suppose that $\Gamma$ acts cocompactly on $\Delta$ with a finite number $n_0$ of vertex orbits. Then $m.[\mathbf{I}] = 0$ in the group $K_0(C(\partial\Delta) \rtimes \Gamma)$, where*

$$m = \begin{cases} \frac{n_0}{3}(q^2 - 1) & \text{if } q \not\equiv 1 \pmod 3, \\ n_0(q^2 - 1) & \text{if } q \equiv 1 \pmod 3. \end{cases}$$

*Moreover, if $q \equiv 0 \pmod 3$, then $n_0$ is a multiple of 3.*

*Proof.* Theorem 1 implies that $m.[\mathbf{I}] = 0$, where $m$ is the greatest common divisor of $n_0(q^2 - 1)$ and $\chi(\Gamma)$. There are different outcomes, depending on the value of $q \pmod 3$.

(1) If $q \equiv 0 \pmod 3$, then $n_0$ is a multiple of 3, since $\chi(\Gamma)$ is an integer. Using the fact that $(3, q - 1) = 1$ gives

$$m = \left(3\frac{n_0}{3}(q^2 - 1), (q - 1)\frac{n_0}{3}(q^2 - 1)\right) = \frac{n_0}{3}(q^2 - 1).$$

(2) If $q \equiv 1 \pmod 3$ then $\chi(\Gamma) = \frac{q-1}{3}n_0(q^2 - 1)$ and so $m = n_0(q^2 - 1)$. This does not improve Theorem 1(2).

(3) If $q \equiv 2 \pmod 3$ then again the fact that $(3, q - 1) = 1$ implies that

$$m = \left(3n_0\frac{(q^2 - 1)}{3}, (q - 1)n_0\frac{(q^2 - 1)}{3}\right) = \frac{n_0}{3}(q^2 - 1).$$

$\square$

**Example 4.3.** Suppose that $\Gamma$ is a discrete subgroup of $\mathrm{Aut}(\Delta)$ which acts freely and *transitively* in a type rotating manner on the vertex set of $\Delta$. That is, $n_0 = 1$. We refer to such a group $\Gamma$ as an $\widetilde{A}_2$ group [CMSZ]. An $\widetilde{A}_2$ group may have 3-torsion and stabilize a chamber of $\Delta$. In fact this always happens if $q \equiv 0 \pmod 3$.

Let $\Gamma$ is a *torsion free* $\widetilde{A}_2$ group. If $q = 2$ then there are five possible such groups $\Gamma$, denoted A1, A2, B1, B2, C1 in [CMSZ]. These groups are lattices in $\mathrm{PGL}(3, \mathbb{F})$, where $\mathbb{F} = \mathbb{F}_2((X))$ (the groups A1, A2) or $\mathbb{F} = \mathbb{Q}_2$ (the groups B1, B2, C1). Corollary 4.2 implies that $[\mathbf{I}] = 0$ in these cases, a fact confirmed by the computations in [RS2].

Other examples of torsion free $\widetilde{A}_2$ groups are the *regular* groups for $q = 2, 4, 5, 7, 8, 11$, constructed in [CMSZ, I, Section 4]. These groups are lattices in $\mathrm{PGL}(3, \mathbb{F})$, where $\mathbb{F} = \mathbb{F}_q((X))$. In particular, if $\Gamma$ is the regular $\widetilde{A}_2$ group with $q = 5$, then $\frac{(q^2-1)}{3} = 8$ whereas direct calculation [RS2] gives $4.[\mathbf{I}] = 0$. Therefore the estimates for the order of $[\mathbf{I}]$ given by Corollary 4.2 are not best possible.

**Remark 4.4.** In [RS2, RS3], T. Steger and the author performed extensive computations which determined the order of $[\mathbf{I}]$ for many $\widetilde{A}_2$ groups with $q \le 11$. The computations were done for all the $\widetilde{A}_2$ groups in the cases $q = 2, 3$ and for several representative groups for each of the other values of $q \le 11$. If $q = 2$ there are precisely eight $\widetilde{A}_2$ groups $\Gamma$, all of which embed as lattices in a linear group $\mathrm{PGL}(3, \mathbb{F})$ where $\mathbb{F} = \mathbb{F}_2((X))$ or $\mathbb{F} = \mathbb{Q}_2$. If $q = 3$ there are 89 possible $\widetilde{A}_2$ groups, of which 65 do not embed naturally in linear groups.

The results of the computations are indicated in the table below.

| $q$ | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|---|
| order of $[\mathbf{I}]$ | 1 | 2 | 1 | 4 | 2 | 7 | 8 | 10 |

This experimental evidence suggests that for boundary crossed product algebras associated with $\widetilde{A}_2$ groups it is always true that $[\mathbf{I}]$ has order $q - 1$ for $q \not\equiv 1 \pmod 3$ and has order $(q - 1)/3$ for $q \equiv 1 \pmod 3$.

It was proved in [RS2, Proposition 8.3] that the order of [**I**] cannot be smaller than these conjectured values.

It is striking that the order of [**I**] appears to depend *only* on the parameter $q$. Moreover the order is the same whether or not the group is torsion free. It is tempting to conjecture that under the general hypotheses of Theorem 1, where $\Gamma$ need not act transitively on the vertices of $\Delta$, the order of [**I**] depends only on $q$ and the covolume $n_0$.

**Remark 4.5.** If the building $\Delta$ is of type $\widetilde{A}_n$, where $n \geq 3$, then according to a well known Theorem of J. Tits [Ron, p. 137], $\Delta$ is the building of $\mathrm{PGL}(n+1, \mathbb{F})$, where $\mathbb{F}$ is a (possibly non-commutative) non-archimedean local field.

Let $\Gamma$ be a torsion free lattice in $\mathrm{PGL}(n+1, \mathbb{F})$, where $n \geq 1$ and $\mathbb{F}$ is a non-archimedean local field with residue field of order $q$. Then arguing as in [S1, p. 150, Théorème 7] gives

$$(4.2) \qquad \chi(\Gamma) = \frac{(-1)^n}{n+1} n_0 (q-1)(q^2-1)\dots(q^n-1),$$

where, as usual, $n_0$ is the number of vertices of $X = \Gamma \backslash \Delta$. If $n \geq 3$, the second part of Theorem 1 does not imply that $\chi(\Gamma).[\mathbf{I}] = 0$. For example, if $q = n = 4$ then $\chi(\Gamma)$ is not divisible by $(q^2 - 1)$. It seems likely however that in general, $\chi(\Gamma).[\mathbf{I}] = 0$. Note that it follows from [BW, XIII Theorem 2.6] that the Betti numbers of $\Gamma$ satisfy $b_i(\Gamma) = 0$ for $i \neq 0, n$ and therefore $\chi(\Gamma) = 1 + (-1)^n b_n(\Gamma)$.

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