C^*–algebras arising from group actions on the boundary of a triangle building

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0. **Introduction**.

A subgroup of an amenable group is amenable. The C^*-algebra version of this fact is false. This was first proved by M.-D. Choi [6] who proved that the non-nuclear C^*-algebra $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is a subalgebra of the nuclear Cuntz algebra $\mathcal{O}_2$. A. Connes provided another example , based on a crossed product construction. More recently J. Spielberg [23] showed that these examples were essentially the same. In fact he proved that certain of the C^*-algebras studied by J. Cuntz and W. Krieger [10] can be constructed naturally as crossed product algebras. For example if the group Γ acts simply transitively on a homogeneous tree of finite degree with boundary Ω then $C(\Omega) \rtimes \Gamma$ is a Cuntz-Krieger algebra.

Such trees may be regarded as affine buildings of type $\widetilde{A}_1$. The present paper is devoted to the study of the analogous situation where a group Γ acts simply transitively on the vertices of an affine building of type $\widetilde{A}_2$ with boundary Ω [7]. The corresponding crossed product algebra $C(\Omega) \rtimes \Gamma$ is then generated by two Cuntz-Krieger algebras . (See §3.) Moreover we show that $C(\Omega) \rtimes \Gamma$ is simple and nuclear. This is a consequence of the facts that the action of Γ on Ω is minimal, topologically free, and amenable. (See §4.)

* *This research was supported by the Australian Research Council*

1. **Background**.

The original motivation was an attempt to show that the reduced C^*–algebra of a discrete group is subnuclear (i.e. a subalgebra of a nuclear C^*–algebra). M. D. Choi in [6] gave the first result in this direction by embedding $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ in the Cuntz algebra $\mathcal{O}_2$. At about the same time A. Connes gave other examples in a talk at Kingston in 1980. The idea of Connes was included in the first version of [14] and was also rediscovered by J. Quigg and J. Spielberg as part (a) of the following result [21: Corollary 4.3]. See also [2: Théorème 4.6], and the remark following it.

Proposition 1.1. Let G be a locally compact group and let P and Γ be closed subgroups of G such that P is amenable. Then

(a) $C_0(G/P) \rtimes \Gamma$ is a nuclear C^*–algebra;

(b) $C_0(G/P) \rtimes \Gamma$ is canonically isomorphic to the reduced crossed product $C_0(G/P) \rtimes_r \Gamma$.

Connes' Example. Take $\Gamma = PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Let $G = PSL(2, \mathbb{R})$ and P the subgroup of upper triangular matrices. Then G acts transitively on $\mathbb{P}_1\mathbb{R} = \mathbb{T}$ and $P =$ Stab (∞). Hence $G/P \cong \mathbb{T}$. By Proposition 1, we have that $C(\mathbb{T}) \rtimes_r \Gamma$ is nuclear. Therefore we have an embedding of $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ into a nuclear C^*–algebra which at first sight appears to be different from Choi's embedding in $\mathcal{O}_2$.

Note that G/P is the Furstenberg boundary of G [12]. It turns out that the approaches of Choi and Connes may be unified by using the concept of a boundary.

2. **Cuntz–Krieger algebras arising from boundary actions of free products of cyclic groups**.

The ideas of this section are due to J. Spielberg [23]. They provide the motivation for our results for groups acting on buildings.

We present the case $\Gamma = \mathbb{Z} * \mathbb{Z}$, the free group with generators a and b. The Cayley graph of Γ is a homogeneous tree of degree 4. The vertices of the tree are elements of Γ i.e. reduced words in the generators and their inverses.

The *boundary* Ω of the tree can be thought of as the set of all infinite reduced words $\omega = x_1x_2x_3\ldots$, where $x_i \in S = \{a, b, a^{-1}, b^{-1}\}$. Ω has a natural compact (totally disconnected) topology in which an open neighbourhood of $\omega \in \Omega$ consists of those $\omega' \in \Omega$ whose corresponding infinite word agrees with that of ω on a finite initial segment [11]. Left multiplication by $x \in \Gamma$ defines a homeomorphism of Ω and so induces an action α of Γ by

$$\alpha(x)f(\omega) = f(x^{-1}\omega).$$

For $x \in \Gamma$ let $\Omega(x)$ be the set of infinite words beginning with x. Then $\Omega(x)$ is open and closed in Ω and the sets $g\Omega(x)$ and $g\left(\Omega\backslash\Omega(x)\right)$, where $g \in \Gamma$ and $x \in S$, form a base for the topology of Ω.

We partition the boundary Ω into four parts as follows.

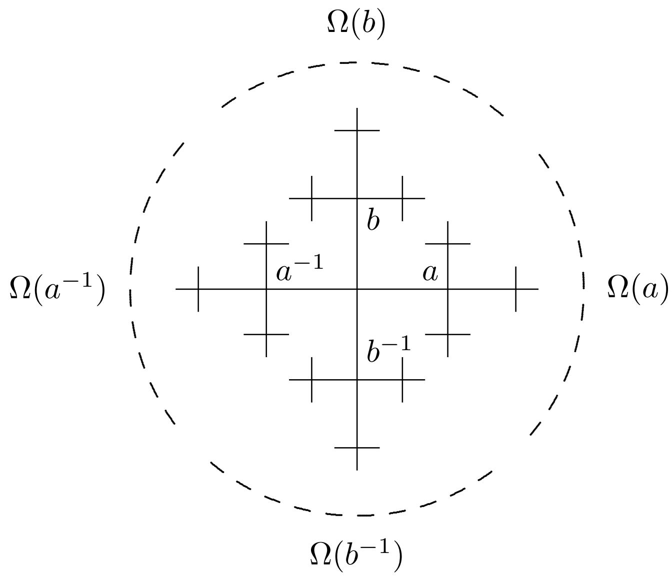


Figure 2.1

Recall that the crossed product $C(\Omega) \rtimes \Gamma$ is the universal covariant representation of $(C(\Omega), \Gamma, \alpha)$. In other words, $C(\Omega) \rtimes \Gamma$ is generated by $C(\Omega)$ and the image of a unitary representation π of Γ such that $\alpha(g)f = \pi(g)f\pi(g)^*$ for $f \in C(\Omega)$ and $g \in \Gamma$, and every such C^*–algebra is a quotient of $C(\Omega) \rtimes \Gamma$. It is convenient to simply write g instead of $\pi(g)$, thereby identifying elements of Γ with unitaries in $C(\Omega) \rtimes \Gamma$. Also for $x \in \Gamma$ let p_x denote the projection defined by the characteristic function $1_{\Omega(x)} \in C(\Omega)$.

For $g \in \Gamma$, we have

$$g\ p_x g^{-1} = \alpha(g)1_{\Omega(x)} = 1_{g\Omega(x)}$$

and therefore, for each $x \in S$,

$$p_x + x\ p_{x^{-1}}\ x^{-1} = 1.$$

Moreover

$$p_a + p_{a^{-1}} + p_b + p_{b^{-1}} = 1$$

and covariant representations of $(C(\Omega), \Gamma, \alpha)$ correspond to representations of these relations.

For $x \in S$, define a partial isometry $s_x \in C(\Omega) \rtimes \Gamma$ by

$$s_x = x(1 - p_{x^{-1}})$$

Then

$$s_x s_x^* = x(1 - p_{x^{-1}})x^{-1} = p_x$$

and

$$s_x^* s_x = 1 - p_{x^{-1}} = \sum_{y \neq x^{-1}} s_y s_y^*$$

These relations show that the partial isometries $s_x, x \in S$, generate the Cuntz–Krieger algebra $\mathcal{O}_A$ of [10], where

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

relative to $\{a, a^{-1}, b, b^{-1}\} \times \{a, a^{-1}, b, b^{-1}\}$.

We can recover the generators for $C(\Omega) \rtimes \Gamma$ from the s_x's by

$$x = s_x + s_{x^{-1}}^*$$

and

$$p_x = s_x s_x^*.$$

Therefore $C(\Omega) \rtimes \Gamma = \mathcal{O}_A$, which is a simple C^*–algebra since A is an irreducible matrix [10]. It follows that the reduced crossed product $C(\Omega) \rtimes_r \Gamma$, being a quotient of $C(\Omega) \rtimes \Gamma$ is also isomorphic to $\mathcal{O}_A$. The algebra $\mathcal{O}_A$ is nuclear [10] and so we obtain in particular that $C_r^*(\Gamma)$ is subnuclear.

In [23], J. Spielberg covers the case where Γ is a free product of cyclic groups of arbitrary order. In particular this method gives an alternative demonstration of Choi's embedding of $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ into $\mathcal{O}_2$.

Remarks.(1) The fact that the algebras $\mathcal{O}_A$ obtained in this way are nuclear may also be seen by using the fact that the action of Γ on Ω is amenable in the sense of [2]. This can be done by constructing a sequence of functions (f_n) on $\Gamma \times \Omega$ satisfying condition (h) of [2: Théorème 4.9].. For the case where Γ is the free group on two generators this is done explicitly in the appendix of [16]. See also [15]. More generally the arguments of [1] show how to do it for any hyperbolic group Γ. It follows from [2: Théorème 4.5 and Proposition 4.8] that the crossed product algebras $C(\Omega) \rtimes \Gamma$ and $C(\Omega) \rtimes_r \Gamma$ are canonically isomorphic and nuclear.

(2) The definition of the operator s_x may be expressed as follows.

$$s_x = x \sum_{y \in A_x} p_y$$

where A_x is the set of elements y of the generating set S such that $xy \neq e$. That is the Cayley graph of Γ contains a semi-infinite geodesic of the form

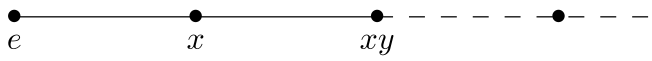


Figure 2.2

In this geometric form, we shall extend the definitions to the case of affine buildings.

3. **Crossed product algebras associated with triangle buildings.**

The construction of the previous section shows that if one has a group Γ acting simply transitively on the vertices of a tree with boundary Ω then, relative to the natural action on Ω, the crossed product algebra $C(\Omega) \rtimes \Gamma$ can be identified as a (simple, nuclear) Cuntz–Krieger algebra.

We now give an analogous construction for groups acting on the vertices of affine buildings of type $\widetilde{A}_2$.

3.1 *Triangle Buildings*

A triangle building is a thick affine building $\triangle$ of type $\widetilde{A}_2$ [22]. This means that $\triangle$ is a chamber system consisting of vertices, edges and triangles (*chambers*). Each edge lies on $q+1$ triangles, where $q \geq 2$ is the *order* of $\triangle$. An *apartment* is a subcomplex of $\triangle$ isomorphic to the Euclidean plane tesselated by equilateral triangles. A *sector* (or *Weyl chamber*) is a sector in some apartment

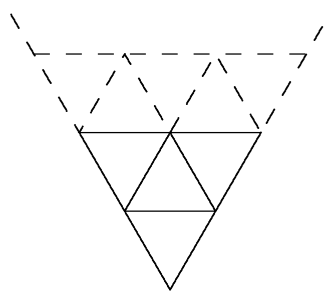


Figure 3.1

Two sectors are *equivalent* (or parallel) if their intersection contains a sector. In the case of a tree, sectors correspond to semi-infinite geodesics and apartments to (doubly infinite) geodesics.

3.2 *The boundary* Ω *of* $\triangle$

The boundary Ω is defined to be the set of equivalence classes of sectors in $\triangle$. Fix a vertex x. For any $\omega \in \Omega$ there is a unique sector $S^x(\omega)$ in the class ω having base vertex x [22: Theorem 9.6].

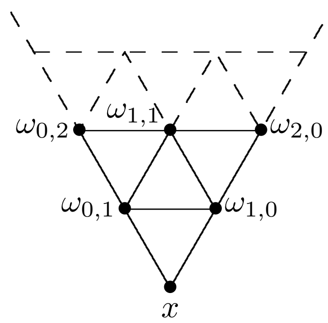


Figure 3.2.1

In the terminology of [5: Chapter VI.9] Ω is the set of chambers of the building at infinity $\triangle^\infty$. Also Ω is a totally disconnected compact Hausdorff space with a base for the topology being given by sets of the form

$$\Omega^x(v) = \{\omega \in \Omega : S^x(\omega) \text{ contains } v\}$$

where v is a vertex of $\triangle$ [7 : §2].

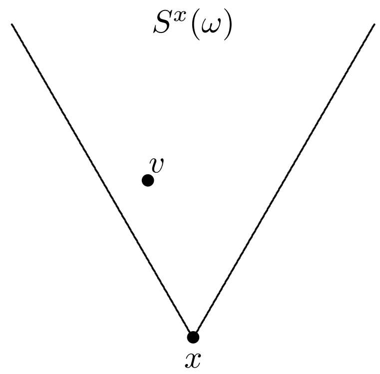


Figure 3.2.2

3.3 *The group* Γ

The buildings we consider are constructed in [8]. They are exactly the triangle buildings on whose vertices a group acts simply transitively and in a "type rotating" way.

Let (P, L) be a projective plane of order q. There should be no confusion between this P and the group P of section 1. There are q^2+q+1 points (elements of P) and q^2+q+1 lines (elements of L). Each point lies on $q+1$ lines and each line contains $q+1$ points. Let $\lambda : P \to L$ be a bijection (a point–line correspondence). Let $\mathcal{T}$ be a set of triples (x, y, z) where $x, y, z \in P$, with the following properties.

(i) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for some $z \in P$ if and only if y and $\lambda(x)$ are incident (i.e. $y \in \lambda(x)$).

(ii) $(x, y, z) \in \mathcal{T} \Rightarrow (y, z, x) \in \mathcal{T}$.

(iii) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for at most one $z \in P$.

$\mathcal{T}$ is called a *triangle presentation* (or *triella*) compatible with λ. A complete list is given in [8] of all triangle presentations for $q = 2$ and $q = 3$.

Let $\{a_x : x \in P\}$ be q^2+q+1 distinct letters and form the group

$$\Gamma = \left\langle a_x, x \in P \mid a_x a_y a_z = 1 \text{ for } (x, y, z) \in \mathcal{T} \right\rangle$$

Then $\mathcal{T}$ gives rise to a triangle building $\triangle_{\mathcal{T}}$ whose vertices and edges form the Cayley graph of Γ with respect to the generators $a_x, x \in P$, and their inverses, and whose chambers are the sets

$$\{g, ga_x^{-1}, ga_y\}$$

where $g \in \Gamma$ and $(x, y, z) \in \mathcal{T}$, for some z.

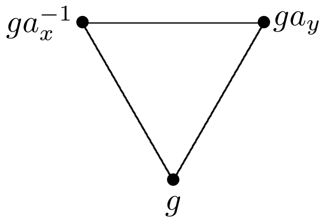


Figure 3.3.1

This chamber can also be represented by the diagram

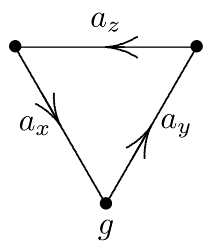


Figure 3.3.2

It is often convenient to identify the point $x \in P$ with the generator $a_x \in \Gamma$ The lines in L correspond to the inverse generators of Γ and the point-line correspondence $\lambda : P \to L$ satisfies $a_{\lambda(x)} = a_x^{-1}$ for $x \in P$ [8]. We may therefore write x^{-1} for a_x^{-1} and identify x^{-1} with $\lambda(x)$.

The set of nearest neighbours of an element $g \in \Gamma$ (the *residue* of g) can be identified with a projective plane of order q, having $q^2 + q + 1$ points $\{gx : x \in P\}$ and $q^2 + q + 1$ lines $\{gx^{-1} : x \in P\}$. According to condition (1) above the line gx^{-1} is incident with the point gy if and only if $xyz = 1$ for some $z \in P$. This is equivalent to $gx^{-1} = gyz$, which means that gx^{-1} and gy are adjacent in the Cayley graph of Γ, as in Figure 3.3.1.

From now on let $\triangle$ be a triangle building arising from a triangle presentation as above and let Γ be the corresponding group which acts simply transitively on the vertices. We may identify Γ with the vertices of its Cayley graph i.e. with the vertices of $\triangle$.

Any element $g \in \Gamma \backslash \{e\}$ can be written uniquely in the left normal form

$$g = x_1^{-1} x_2^{-1} \dots x_n^{-1} y_1 y_2 \dots y_m$$

where there are no obvious cancellations and $x_i, y_j \in P$, $1 \leq i \leq n, 1 \leq j \leq m$. (As usual, the concise notation x, y has been used in place of a_x, a_y.) [9: Lemma 6.2]. The absence of "obvious" cancellations means that $x_i \notin \lambda(x_{i+1}) \quad (1 \leq i < n)$, $y_{j+1} \notin \lambda(y_j) \quad (1 \leq j < m)$, and $x_n \neq y_1$. Also any such word for g is a minimal word for g in the generators $x \in P$ and their inverses [9:Lemma 6.2].

It follows from [9: Lemma 6.2] and the remarks following it, that if a group element g is expressed as above in left normal form then its position in a sector based at e containing it is obtained by moving n steps up the "left hand" wall of the sector then m steps in the direction of the "right hand" wall. (The choice of "left" and "right" is purely conventional in our digrams.) Moreover we can fill out any apartment containing e by starting at e and repeatedly multiplying on the right by a generator or inverse generator in a way consistent with the triangle presentation of the group. An apartment is a union of six sectors based at e.

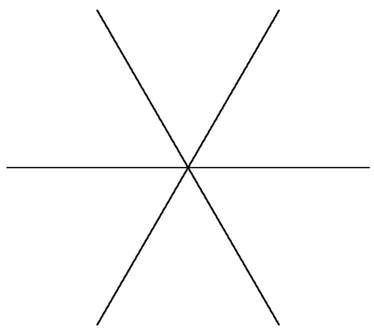


Figure 3.3.3

A simple example arises when $q = 2$ and Γ is taken to have generators $a_0, a_1, \dots, a_6$ and relations

$$a_{[i]}\ a_{[i+1]}\ a_{[i+3]} = 1, \qquad i \in \{0, \dots, 6\}$$

where $[j]$ denotes j mod 7. See [8 : Part II, §4]. Then Figure 3.3.4 represents part of an apartment which contains e.

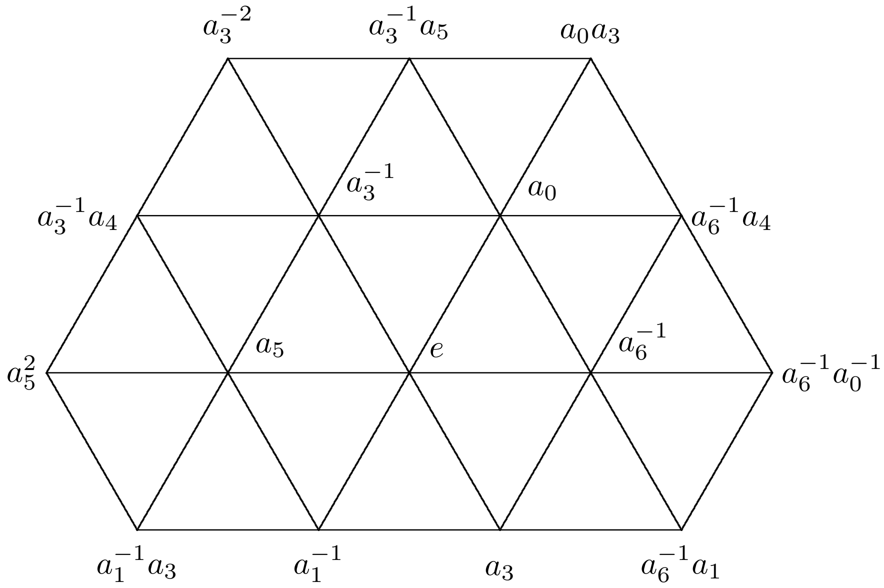


Figure 3.3.4

For the purposes of constructing such apartments it is more convenient to label the edges as directed line segments, where the label corresponds to right multiplication by a generator or an inverse generator. For example,the initial portion of one of the six sectors based at e in this apartment may be represented as in Figure 3.3.5.

Since $q = 2$ in this example, the projective plane of nearest neighbours of e has seven points and seven lines. Thus the residue of e consists of fourteen vertices. A vertex representing a point is adjacent to a vertex representing a line if the point and line are incident in the projective plane. The adjacencies between these vertices are shown in Figure 3.3.6. (Every vertex shown is of course also adjacent to e.)

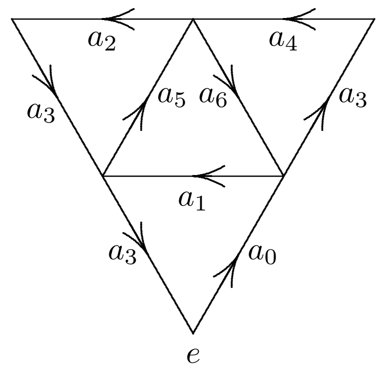


Figure 3.3.5

Each of the 28 hexagons contained in Figure 3.3.6 lies in a different apartment of $\triangle$ containing e.

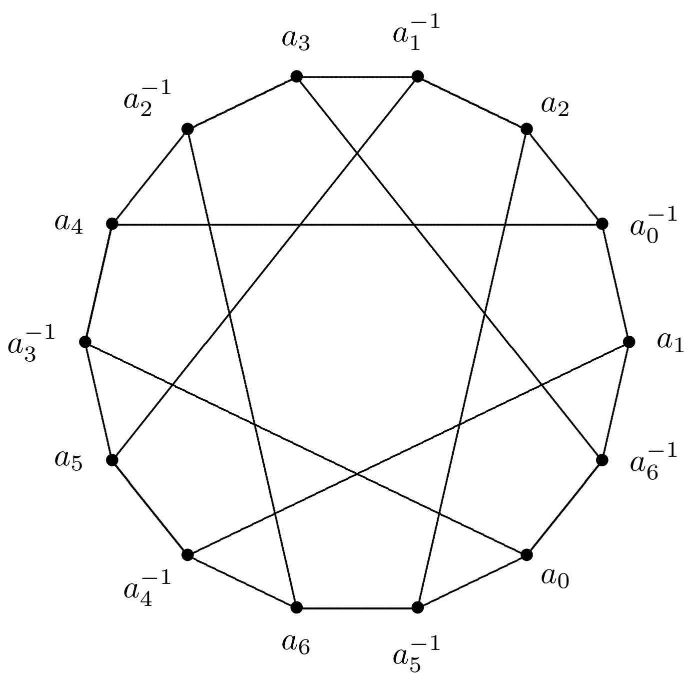


Figure 3.3.6

3.4 *The algebra* $C(\Omega) \rtimes \Gamma$.

As before, $\triangle$ is a triangle building, Γ is a group which acts simply transitively on the vertices of $\triangle$ and we identify Γ with the vertices of its Cayley graph i.e. with the vertices of $\triangle$.

We first partition the boundary Ω of $\triangle$. Given generators $a, b \in P$ with $b \in \lambda(a)$ let $\Omega(a^{-1}, b)$ denote the set of elements $\omega \in \Omega$ whose representative sector $S(\omega)$ has base chamber $\{e, a^{-1}, b\}$.

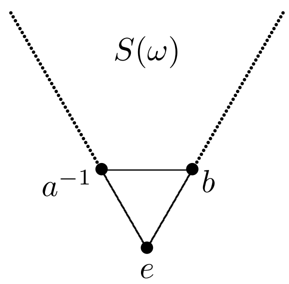


Figure 3.4.1

In the notation of §3.2, $\Omega(a^{-1}, b) = \Omega^e(a^{-1}) \cap \Omega^e(b)$. Since there is a unique sector based at e representing a point $\omega \in \Omega$, it follows that the sets $\Omega(a^{-1}, b)$ are pairwise disjoint. To find the number of possible $\Omega(a^{-1}, b)$ note that there are q^2+q+1 choices for a^{-1} and then $q+1$ choices for b, since each line contains $q+1$ points. It follows that Ω is partitioned into $(q+1)(q^2+q+1)$ open and closed sets of the form $\Omega(a^{-1}, b)$. The following key technical result will be used later.

Lemma 3.1 Given generators $a, b \in P$ with $b \in \lambda(a)$ and any basic open set $\Omega^e(v)$ where v is a vertex of $\triangle$, there exists $k \in \Gamma$ such that $k\Omega(a^{-1}, b) \subset \Omega^e(v)$

Proof. Assume that v has left normal form $v = x_1^{-1}x_2^{-1} \ldots x_n^{-1}y_1y_2 \ldots y_m$, where $m \geq 1$. (If $m = 0$ the argument is simpler.) Since $b \in \lambda(a)$, we have $abc = 1$ for a (unique) $c \in P$.

Consider the projective plane of neighbours of e. In it there are $q+1$ lines containing b. Therefore there are $q+1$ possible $z_1 \in P$ such that $b \in \lambda(z_1)$. Similarly, there are $q+1$

points on the line $\lambda(y_m)$; i.e. there are $q+1$ possible $z_2 \in P$ such that $z_2 \in \lambda(y_m)$. Since $|P| = q^2+q+1 > 2(q+1)$, we can choose $z \in P$ such that $b \notin \lambda(z)$ and $z \notin \lambda(y_m)$. It follows that the word $x_1^{-1}x_2^{-1}\ldots x_n^{-1}y_1y_2\ldots y_m zb$ has no obvious cancellations and so, by [9: Lemma 6.2], $|vzb| = |v|+2$. Also, by the remark following [9: Lemma 6.2], $|vza^{-1}| = |vzbc| \geq |v|+2$, and so $|vza^{-1}| = |v|+2$.

Now let $\omega_0 \in \Omega(a^{-1}, b)$. Then the sector $S^{vz}(vz\omega_0)$ is a subsector of a sector with base vertex e which contains v. (See Figure 3.4.2.)

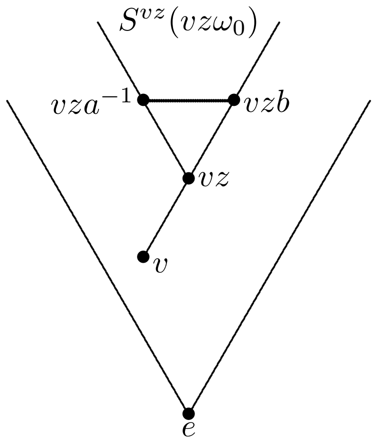


Figure 3.4.2

The sector with base vertex e is parallel to $S^{vz}(vz\omega_0)$ and so , by uniqueness, must be $S^e(vz\omega_0)$. Let $k = vz$. Then $k\omega_0 \in \Omega^e(v)$. Thus $k\Omega(a^{-1}, b) \subset \Omega^e(v)$, as required.

If $m = 0$ then the argument is similar. We need only choose $z \in P$ with $b \notin \lambda(z)$ and $z \neq x_n$.

$\square$

There is a natural well–defined action of Γ on Ω coming from left multiplication of every vertex in a sector by an element $g \in \Gamma$ [7 : §2]. This induces an action α of Γ by automorphisms of $C(\Omega)$ via

$$\alpha(g)\ f(\omega) = f(g^{-1}\omega)$$

By analogy with the tree case of §2 we now define partial isometries $s^+_{a^{-1},b}$, $s^-_{a^{-1},b}$ in $C(\Omega) \rtimes \Gamma$ which generate $C(\Omega) \rtimes \Gamma$ and satisfy Cuntz–Krieger type relations.

First of all note that the characteristic function $1_{\Omega(a^{-1},b)}$ is continuous on Ω and so defines a projection $p_{a^{-1},b}$ in $C(\Omega) \rtimes \Gamma$.

Define

$$s^+_{a^{-1},b} = b \sum p_{c^{-1},d} \tag{1}$$

where the sum is over those (c,d) for which there is a sector of the form shown in Figure 3.4.3.

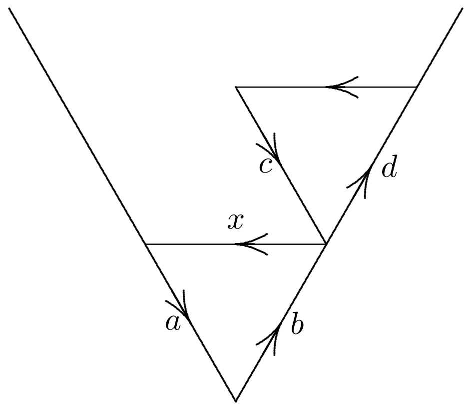


Figure 3.4.3

The possible (c,d) are determined as follows. Firstly $d \notin \lambda(b)$ and so there are (q^2+q+1) $-(q+1) = q^2$ choices for d. Once d is chosen, we have $x \in \lambda(c)$ and $d \in \lambda(c)$, which determines c uniquely. Therefore there are q^2 terms in the sum (1). Denote by $A^+_{a^{-1},b}$ the set of such (c,d). Now $s^+_{a^{-1},b}$ is a partial isometry with

$$s^+_{a^{-1},b}\, s^{+*}_{a^{-1},b} = p_{a^{-1},b}$$

and

$$s^{+*}_{a^{-1},b}\, s^+_{a^{-1},b} = \sum_{(c,d)\in A^+_{a^{-1},b}} s^+_{c^{-1},d}\, s^{+*}_{c^{-1},d}. \tag{2}$$

These partial isometries therefore generate a Cuntz–Krieger algebra [10].

The action of $s^+_{a^{-1},b}$ may be represented pictorially as in Figure 3.4.4.

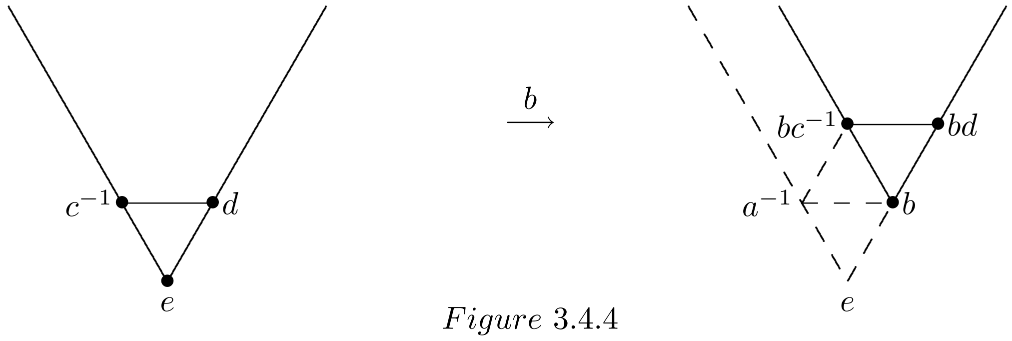


Figure 3.4.4

Note that the unbroken sector on the right with base vertex b is parallel to the whole sector and hence is a representative of an element in $\Omega(a^{-1}, b)$.

The following relations are also clearly satisfied

$$\sum_{(c,d)\in A^+_{a^{-1},b}} b\ p_{c^{-1},d}\ b^{-1} = p_{a^{-1},b}. \tag{3}$$

Similarly we define

$$s^-_{a^{-1},b} = a^{-1} \sum p_{c^{-1},d}, \tag{4}$$

where the sum is over the set $A^-_{a^{-1},b}$ of those (c, d) for which there is a sector as inFigure 3.4.5

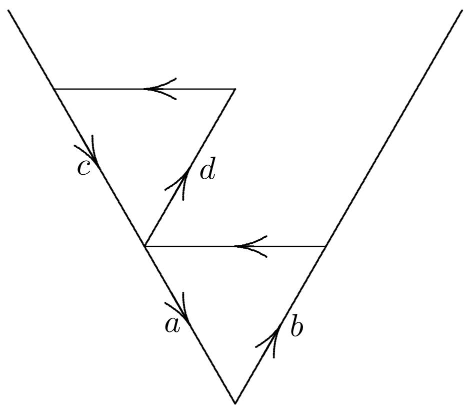


Figure 3.4.5

where $a \notin \lambda(c)$. Again there are q^2 terms in the sum.

Then $s^-_{a^{-1},b}$ is a partial isometry with

$$s^-_{a^{-1},b} \ s^{-^*}_{a^{-1},b} = p_{a^{-1},b}$$

and

$$s^{-^*}_{a^{-1},b} \ s^-_{a^{-1},b} = \sum_{(c,d)\in A^-_{a^{-1},b}} s^-_{c^{-1},d} \ s^{-^*}_{c^{-1},d} \tag{5}$$

so that these partial isometries also generate a Cuntz–Krieger algebra. There are also covariance relations analogous to (3).

The action of $s^-_{a^{-1},b}$ can be represented by Figure 3.4.6.

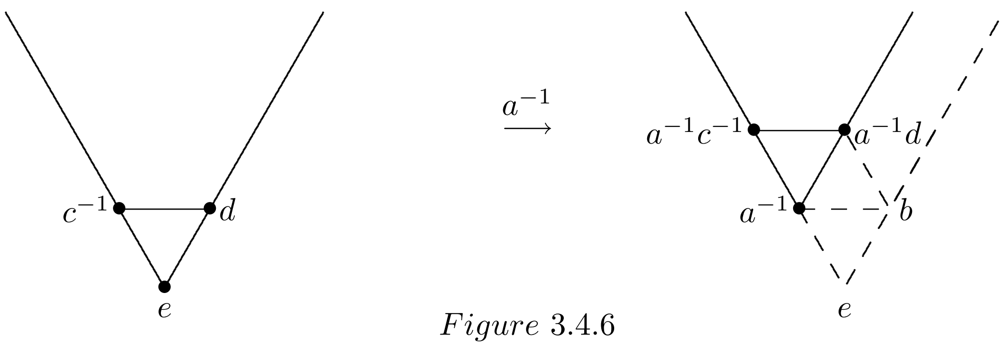


Figure 3.4.6

Denote by $C^*(s^+)$, $C^*(s^-)$ the C^*–subalgebras of $C(\Omega) \rtimes \Gamma$ generated by the operators $s^+_{a^{-1},b}$, $s^-_{a^{-1},b}$ respectively. Clearly

$$C^*(s^+) \cap C^*(s^-) \supset \left\{ p_{a^{-1},b} : b \in \lambda(a) \right\}.$$

We are interested in the C^*–algebra $C^*(s^\pm)$ which is generated by both sets of operators.

Theorem. $C^*(s^\pm) = C(\Omega) \rtimes \Gamma$.

Proof. First of all we show that any generator $b \in \Gamma$ lies in $C^*(s^\pm)$. We do this by demonstrating the following explicit formula for b in terms of elements of $C^*(s^\pm)$.

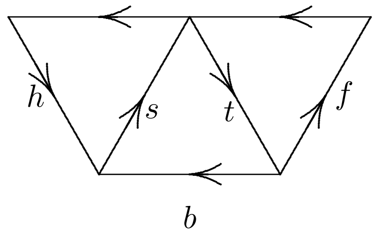
$$b = \sum_a s^+_{a^{-1},b} + \sum_k s^{-^*}_{b^{-1},k} + \sum s^-_{t^{-1},f} s^{+^*}_{h^{-1},s}. \tag{6}$$

We consider the three sums separately.

(A) **The first sum** is over those $a \in P$ such that $b \in \lambda(a)$. Thus the sum contains $q+1$ terms and the initial projection of each term is the sum of the projections of the form $p_{c^{-1},d} = 1_{\Omega(c^{-1},d)}$ satisfying the additional requirement that $(c,d) \in A^{+}_{a^{-1},b}$. [See equation (2).] As previously noted, there are q^2 such projections, and so the initial space of the first sum is the sum of $q^2(q+1)$ projections of the form $p_{c^{-1},d}$.

(B)**The second sum** is over $k \in \lambda(b)$. There are $q+1$ such k's and the initial projection of $s^{-*}_{b^{-1},k}$ is $p_{b^{-1},k}$. Thus the initial space of the second sum is the sum of $q+1$ projections of the form $p_{b^{-1},k}$. Note that $p_{b^{-1},k}$ is different from the projections $p_{c^{-1},d}$ in (A), because $k \in \lambda(b)$ whereas $d \notin \lambda(b)$; so that $k \neq d$.

(C) **The third sum** is over those t, f, h, s for which the following diagram is possible.



$Figure 3.4.7$

That is

$s \in \lambda(b)$ $\quad$ ($q+1$ choices for s)

$b = t^{-1}s^{-1}$

$h \neq b$ and $s \in \lambda(h)$ $\quad$ (q choices for h)

$f \neq b$ and $f \in \lambda(t)$ $\quad$ (q choices for f)

Therefore the third sum has $q^2(q+1)$ terms and yields a partial isometry whose initial projection equals the sum of $q(q+1)$ projections $p_{h^{-1},s}$.

Now $p_{h^{-1},s}$ differs from any projection $p_{c^{-1},d}$ of (A) since $s \in \lambda(b)$ in the first case but $d \notin \lambda(b)$ in the second; and it differs from any $p_{b^{-1},k}$ in (B) since $h \neq b$. The initial projections involved in (A), (B), and (C) are therefore all different.

For all three sums the total number of initial projections of the form $p_{x^{-1},y}$ is

$$q^2(q+1) + (q+1) + q(q+1) = (q+1)(q^2+q+1).$$

Thus the initial projection is the identity operator. Similarly the final projection is the sum of $(q+1) + q^2(q+1) + q(q+1) = (q+1)(q^2+q+1)$ projections which sum to the identity.

On each initial projection the sum on the right of (6) is multiplication by b and hence (6) is established. Since each generator of Γ lies in $C^*(s^\pm)$, the same is true for any element of Γ.

We can now complete the proof. Recall that

$$1_{\Omega(a^{-1},b)} = p_{a^{-1},b} = s^+_{a^{-1},b}\ s^{+*}_{a^{-1},b}$$

and so $1_{\Omega(a^{-1},b)}$ lies in $C^*(s^\pm)$. Therefore $1_{g\Omega(a^{-1},b)} = \alpha(g)\ 1_{\Omega(a^{-1},b)} = g1_{\Omega(a^{-1},b)}\ g^{-1}$ lies in $C^*(s^\pm)$ for each $g \in \Gamma$, since $\Gamma \subset C^*(s^\pm)$. Similarly $1_{g(\Omega\setminus\Omega(a^{-1},b))} \in C^*(s^\pm), g \in \Gamma$. We claim that the family of $g\Omega(a^{-1},b)$ and $g\left(\Omega \setminus \Omega(a^{-1},b)\right)$ forms a base for the topology of Ω. The Stone–Weierstrass theorem then shows that $C(\Omega) \subset C^*(s^\pm)$. Therefore $C(\Omega) \rtimes \Gamma = C^*(s^\pm)$.

In order to prove our claim, let $\Omega^e(v)$ be any basic open set, and let $\omega \in \Omega^e(v)$. Define a and b by $S^v(\omega) \in \Omega^v(va^{-1}) \cap \Omega^v(vb)$. In other words, a and b are defined by the labels on the left and right edges of the base triangle in $S^v(\omega)$. Since $S^v(\omega) \subseteq S^e(\omega)$, we have $|va^{-1}| = |vb| = |v| + 1$. Then, as in the proof of Lemma 3.1, $v\Omega(a^{-1},b) \subseteq \Omega^e(v)$, and by construction, $\omega \in v\Omega(a^{-1},b)$.

$\square$

Remark. Since $C^*(s^+) \cap C^*(s^-) \supset \left\{p_{a^{-1},b} : b \in \lambda(a)\right\}$, the above proof shows that $C(\Omega) \subset \alpha(\Gamma)(C^*(s^+) \cap C^*(s^-))$.

3.5 *Weak Commutativity.*

A natural project is to study the algebras $C^*(s^{\pm})$ by analogy with the Cuntz-Krieger algebras. One would hope, for example that the relations (2) and (5) defining the Cuntz-Krieger algebras $C^*(s^+)$ and $C^*(s^-)$ together with some connecting relations would characterize the generated C^*-algebra up to canonical isomorphism.We show a weak form of commutativity between $C^*(s^+)$ and $C^*(s^-)$.

Suppose that $s^+_{a^{-1},b}s^-_{c^{-1},d} \neq 0$, that is $(c,d) \in A^+_{a^{-1},b}$. Then

$$s^+_{a^{-1},b}s^-_{c^{-1},d} = bc^{-1} \sum_{(e,f)\in A^-_{c^{-1},d}} p_{e^{-1},f}$$

Similarly, if $s^-_{a^{-1},b}s^+_{g^{-1},h} \neq 0$, that is $(g,h) \in A^-_{a^{-1},b}$, then

$$s^-_{a^{-1},b}s^+_{g^{-1},h} = a^{-1}h \sum_{(e,f)\in A^+_{g^{-1},h}} p_{e^{-1},f}$$

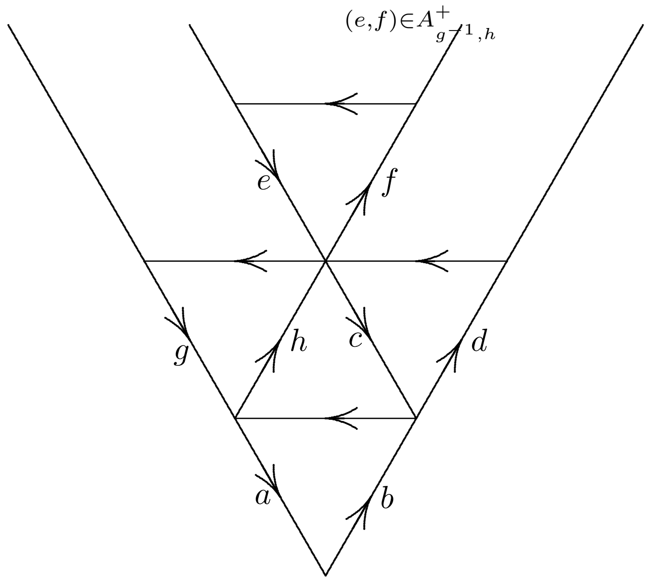


Figure 3.5.1

Now suppose that $bc^{-1} = a^{-1}h$ and that $(e,f) \in A^-_{c^{-1},d} \cap A^+_{g^{-1},h}$ [there are q possible such (e,f)]. This means that there is a diagram as in Figure 3.5.1

Then $s^+_{a^{-1},b} s^-_{c^{-1},d} p_{e^{-1},f} = s^-_{a^{-1},b} s^+_{g^{-1},h} p_{e^{-1},f}$

Summing over all possibilities gives

$$\sum_d s^+_{a^{-1},b} s^-_{c^{-1},d} = \sum_g s^-_{a^{-1},b} s^+_{g^{-1},h}. \tag{7}$$

For fixed a,b,c,h, there are q terms in each sum.

4. **The action of Γ on Ω.**

4.1 **Minimality of the action.**

By definition, the action of Γ is *minimal* if, for all $\omega \in \Omega$, $\Gamma\omega$ is dense in Ω.

Proposition 4.1.1 The action of Γ on Ω is minimal.

Proof. Let $\omega_0 \in \Omega$ and let $\Omega^e(v) = \{\omega \in \Omega : v \in S^e(\omega)\}$ be a basic open set in Ω, where v is a vertex of Δ (i.e. $v \in \Gamma$) and $S^e(\omega)$ denotes the unique sector in the class ω having base vertex e. We must find $k \in \Gamma$ such that $k\omega_0 \in \Omega^e(v)$. Let the sector $S^e(\omega_0)$ have base chamber $\{e, a^{-1}, b\}$, where $b \in \lambda(a)$; that is $\omega_0 \in \Omega(a^{-1}, b)$. By Lemma 3.1, there exists $k \in \Gamma$ such that $k\Omega(a^{-1}, b) \subset \Omega^e(v)$. In particular $k\omega_0 \in \Omega^e(v)$.

$\square$

4.2. **The action of Γ on Ω is amenable**.

By analogy with the Cuntz-Krieger algebras one would hope that the algebras $C^*(s^\pm)$ are nuclear. We approach this problem by using the results of [2]. If we can show that the action of Γ on Ω is amenable in the sense of [2] then $C^*(s^{\pm}) = C(\Omega) \rtimes \Gamma$ is isomorphic to the reduced crossed product $C(\Omega) \rtimes_r \Gamma$ [2: Proposition 4.8] and is nuclear [2: Théorème 4.5].

We verify that the action of Γ on Ω is amenable by constructing a sequence f_i of continuous real valued functions with compact support on $\Gamma \times \Omega$ such that

(a) $\quad \sum_{t\in\Gamma} |f_i(t,\omega)|^2 = 1 \quad$ for all $\omega \in \Omega$ and $i \in \mathbb{N}$

(b) $\quad lim_i \sum_{t\in\Gamma} \overline{f_i(t,\omega)} f_i(s^{-1}t, s^{-1}\omega) = 1 \quad$ uniformly on Ω for each $s \in \Gamma$

(See Théorème 4.9(h) and Remarque 4.10 in [2].)

Denote by $\omega_{m,n}$ the vertices of the sector $S^e(\omega)$.

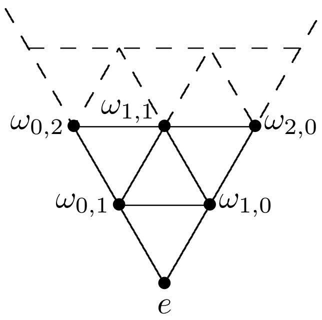


Figure 4.2.1

For $\omega \in \Omega$, define $f_i(\cdot,\omega)$ to be the characteristic function of the triangular region $\{t \in S^e(\omega) : |t| \leq i-1\}$, normalized so that (a) holds. i.e. $\|f_i(\cdot,\omega)\|_2 = 1$.

By definition,

$$f_i(t,\omega) = \begin{cases} (i(i+1)/2)^{-1/2} & \text{if } t \in S^e(\omega) \text{ and } |t| \leq i-1 \\ 0 & \text{otherwise.} \end{cases}$$

and there are exactly $i(i+1)/2$ elements $t \in \Gamma$ for which $f_i(t,\omega) \neq 0$. Also, since $s^{-1}t \in S^e(s^{-1}\omega)$ if and only if $t \in S^s(\omega)$, we have

$$f_i(s^{-1}t, s^{-1}\omega) = \begin{cases} (i(i+1)/2)^{-1/2} & \text{if } t \in S^s(\omega) \text{ and } |s^{-1}t| \leq i-1 \\ 0 & \text{otherwise.} \end{cases}$$

We estimate the number N_i of vertices $t \in S^e(\omega) \cap S^s(\omega)$ such that $|t| \leq i-1$ and $|s^{-1}t| \leq i-1$. Assume that $i \geq |s|$ and note that by [7: Corollary 2.3], $\omega_{m,n} \in S^e(\omega) \cap S^s(\omega)$ for $m, n \geq |s|$.

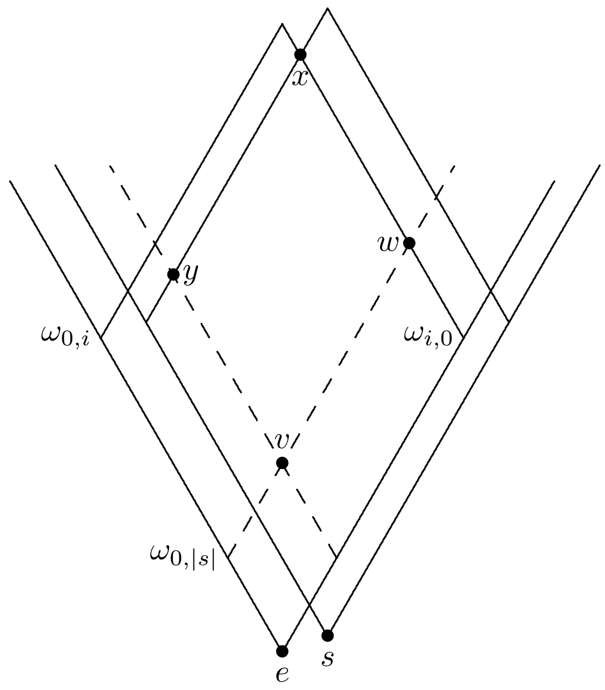


Figure 4.2.2

Let $v = \omega_{|s|,|s|}$ and $w = \omega_{i,|s|}$. In Figure 4.2.2, the convex hull of $\{x, y, v, w\}$ consists of vertices which lie in $S^e(\omega) \cap S^s(\omega)$.

Now $|\omega_{m,n}| \leq i-1$ whenever $m+n \leq i-1$, and $|s^{-1}\omega_{m,n}| \leq i-1$ whenever $m+n \leq i-|s|-1$, so clearly $N_i \geq (i-3|s|)(i-3|s|+1)/2$. Therefore $\overline{f_i(t,\omega)} f_i(s^{-1}t, s^{-1}\omega) = (i(i+1)/2)^{-1}$ for at least $(i-3|s|)(i-3|s|+1)/2$ values of t. It follows that

$$1 \geq \sum_{t \in \Gamma} \overline{f_i(t,\omega)} f_i(s^{-1}t, s^{-1}\omega) \geq (i(i+1)/2)^{-1}(i-3|s|)(i-3|s|+1)/2 \to 1$$

as $i \to \infty$. Condition (b) above is then satisfied and therefore the action of Γ on Ω is amenable. We have therefore proved

Proposition 4.2.1. The algebra $C^*(s^{\pm})$ is nuclear.

4.3. **The action of Γ on Ω is topologically free**

The proof of the stated assertion requires a preliminary result , which is of independent interest. We refer to the appendix for further information on the concepts used in its proof.

Proposition 4.3.1 Let V be an open subset of Ω. There exists an apartment each of whose six boundary points lies in V.

Proof. Identify the boundary Ω with the set of $\pi/3$-angled sectors based at e. That is, identify $\omega \in \Omega$ with $S^e(\omega)$. Any open set $V \subseteq \Omega$ contains a set of the form

$$\Omega(T) = \{\omega \in \Omega : T \subset S^e(\omega)\}$$

for some (big) equilateral trangle T based at e.

We shall find an apartment each of whose 6 boundary points lies in $\Omega(T)$. To start, for reasons that become clear only later, we add one layer to T, choosing it arbitrarily. Then we add a big "upside down" equilateral triangle to the extended T, calling the whole resulting diamond shaped figure T'. See Figure 4.3.1. The tip vertex of T' we label v_1.

By an element of $\triangle$ we mean any vertex, edge, or triangle (chamber) of $\triangle$. For any such element E there exist sectors based at e which contain E. Moreover, although there are uncountably many such sectors, in each of them E lies in the same relative location. This gives a retraction, r_e, of the entire building onto an abstract sector. By abstract, we mean a sector which is specially intended as the image of r_e, instead of being some particular sector of the building. To say that E "lies" in a given position is to say that $r_e(E)$ takes some particular value. The particular value is easily indicated on a figure.

Each edge is contained in exactly $q + 1$ triangles. If we retract an edge, which does not lie on a ray through e, and its $q + 1$ containing triangles, then one of the triangles will lie below the edge and the other q will lie above it. This applies whether the edge lies horizontally or at a slant, and it is so because there is a unique stretched gallery of a given

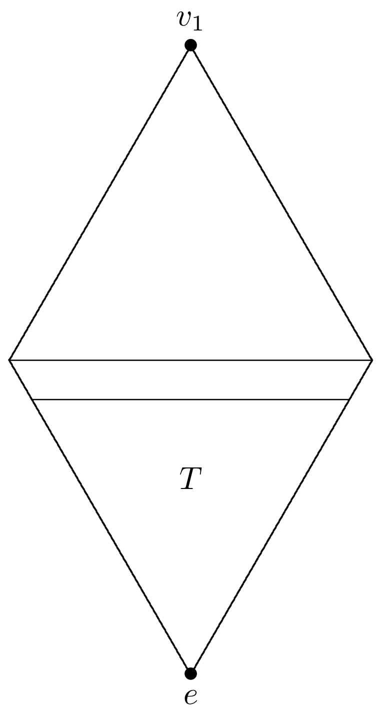


Figure 4.3.1

type from e to the edge. The last triangle of that stretched gallery is the one which lies below the edge.

Let A be a triangle emanating from v_1 which lies as indicated in Figure 4.3.2.

Any sector W_1 based at v_1 with first triangle A will lie as indicated in Figure 4.3.2 . (Proof: one may construct a gallery of stretched type from e to any element E of W_1 out of a stretched gallery from e to A and another from A to E. Any gallery of stretched type (i.e. having the same type as a stretched gallery) is in fact stretched, and therefore lies in some apartment. This establishes the value of $r_e(E)$, that is, it establishes where E lies.)

One can fill in the figure consisting of T' and W_1 to a unique sector based at e. To see this, just fill in one layer at a time starting from the boundary of the given figure and work towards the boundary of the desired figure. For example, starting at the vertex v_1, the chambers A and T_1 in Figure 4.3.2(a) are determined. Each vertex of the building which

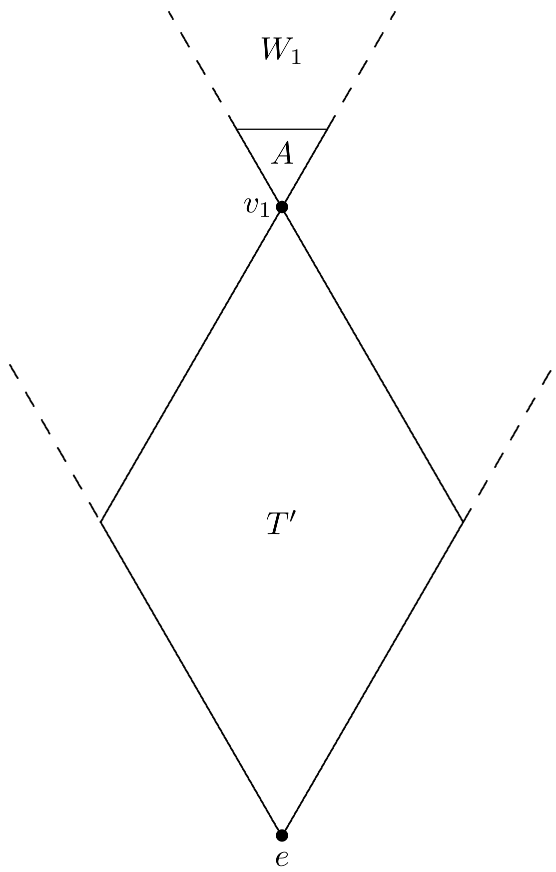


Figure 4.3.2

is adjacent to v_1 corresponds to a point or line of a finite projective plane. It follows that z is uniquely determined, since it is incident with both x and y. Similarly w is uniquely determined. All the chambers in Figure 4.3.2(a) are now determined. Continue in this way, working next from the the vertex x instead of v_1 .

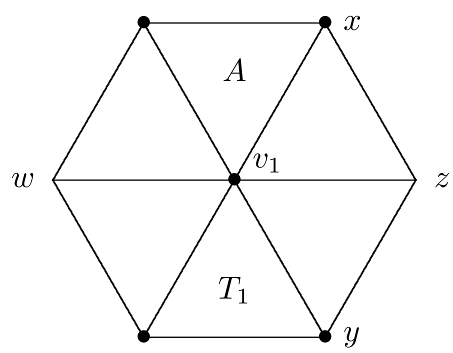


Figure 4.3.2(*a*)

It follows that the point of Ω represented by W_1 belongs to $\Omega(T)$, indeed to

$$\Omega(T') = \{\omega \in \Omega : T' \subset S^e(\omega)\}$$

Consider the *star* of the vertex v_1. That is the union of all the triangles of the building which contain v_1. In this star find a hexagon which has four triangles A_1 , ..., A_4 which all lie as A does. Choose A_1 from among the q^3 possibilities. There are then q triangles which share with A_1 the edge that would make them possible A_2's. Choose one of the $q-1$ of these which lies as A rather than as D in Figure 4.3.3.

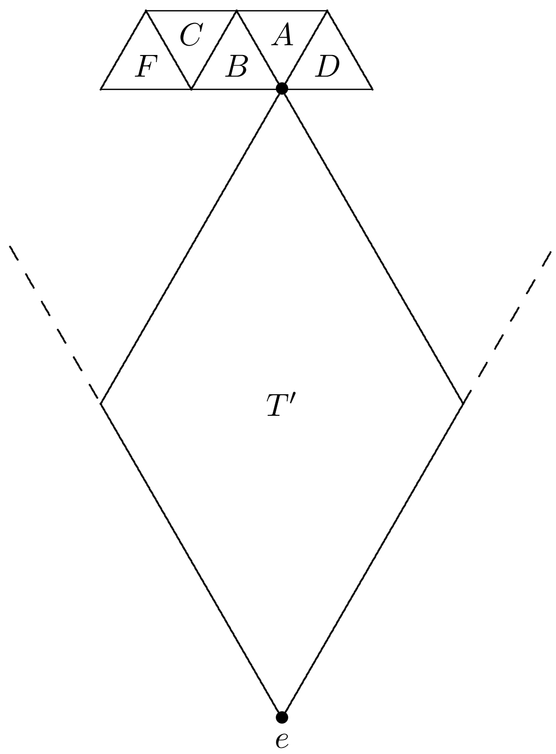


Figure 4.3.3

Similarly, there are again $q-1$ choices for A_3, and again for A_4. The A_1 , ..., A_4 so chosen form a stretched gallery, so they lie together in some apartment; in particular they lie in some hexagon (Figure 4.3.4).

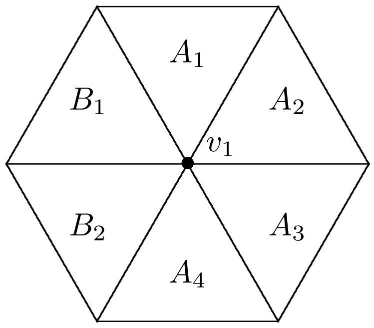


Figure 4.3.4

The remaining chambers B_1 , B_2 of the hexagon might lie as A or they might lie as B . It is not possible for one to lie one way and the other to lie the other. If they both lie as A, then we can construct the desired apartment by expanding out from the hexagon. For then each of the six sectors in this apartment based at v_1 lies as W_1 (Figure 4.3.2) and so represents a point of Ω belonging to $\Omega(T)$. So we assume that B_1 and B_2 both lie as B .

Let us expand the hexagon in Figure 4.3.4 as follows to make an asymmetric hexagon of ten triangles (Figure 4.3.5) which we will take as part of our apartment. First choose C_1 in any of q ways. All of these lie as C in Figure 4.3.3 , because B_1 lies below the edge it shares with the potential C_1's. Next choose as C_1' any of the $q-1$ triangles bordering C_1 and lying as C rather than as F. The choices of C_1 and C_1' determine C_2' and C_2 uniquely. Just like C_1, C_2 necessarily lies as C. This then implies that C_2' also lies as C, because we know where two of its edges lie.

Extend the asymmetric hexagon to an apartment $\mathcal{A}$, and consider the sector W_2 in that apartment. Let v_2 be the base vertex of W_2. Then W_2 necessarily lies as shown in Figure 4.3.6

Indeed, the arguments earlier applied to W_1 based at v_1 apply just as well to W_2 based at v_2.

We know we can extend W_2 in a unique way to a sector ω_2 based at e. We have no

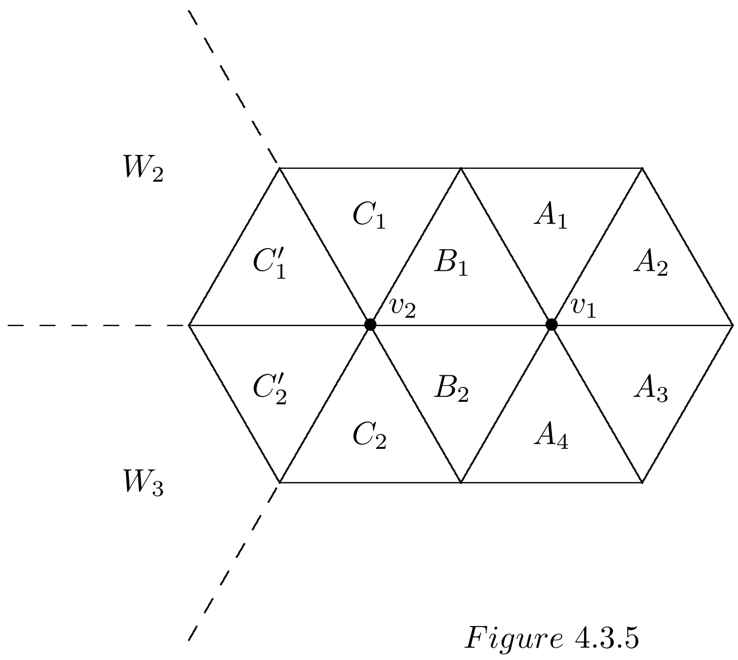


Figure 4.3.5

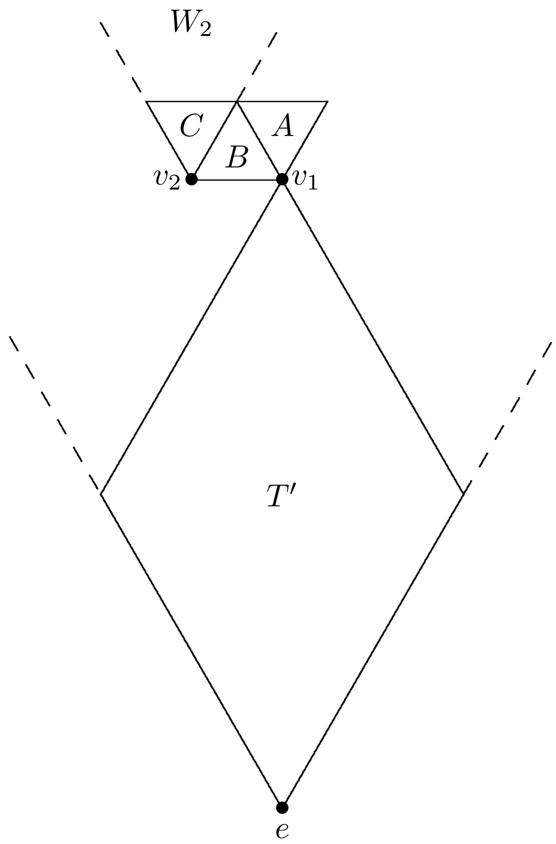


Figure 4.3.6

reason to suppose that v_1 will lie in ω_2. However consider the vertex v_3 of T' in Figure 4.3.7.

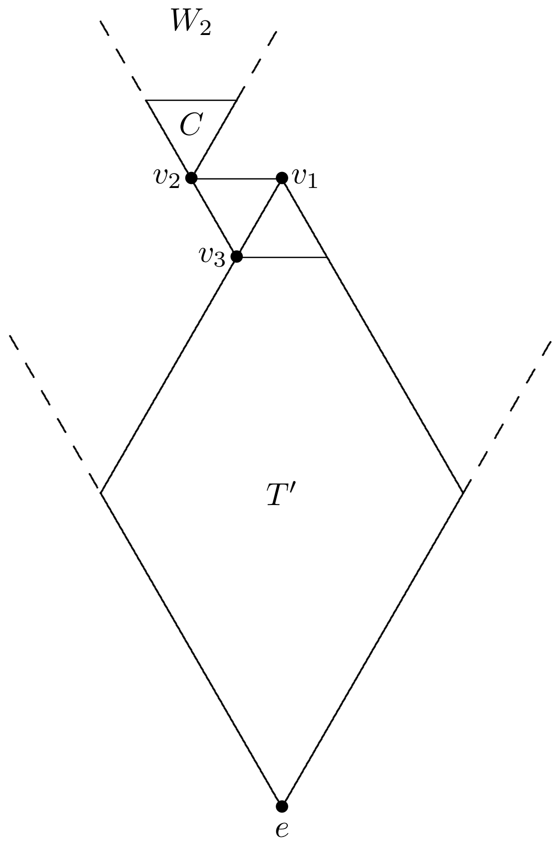


Figure 4.3.7

Since v_3 is the unique vertex connected to v_1 and lying below it and to the left, it must be the third vertex of the unique triangle containing and lying below the edge from v_1 to v_2. It is now clear that v_3 lies on a stretched gallery from e to v_2, so we can be sure that v_3 belongs to ω_2. It follows that ω_2 contains all of T'' (Figure 4.3.8).

Because we foresightedly added a layer to our original T, it further follows that ω_2 contains T, i.e. $\omega_2 \in \Omega(T)$. One of the six boundary points of the apartment $\mathcal{A}$ is represented by W_2, and so is equal to $\omega_2 \in \Omega(T)$. Similarly the boundary point represented by W_3 lies in $\Omega(T)$. Since the other 4 boundary points were already known to lie in $\Omega(T)$, we have found our desired apartment.

$\square$

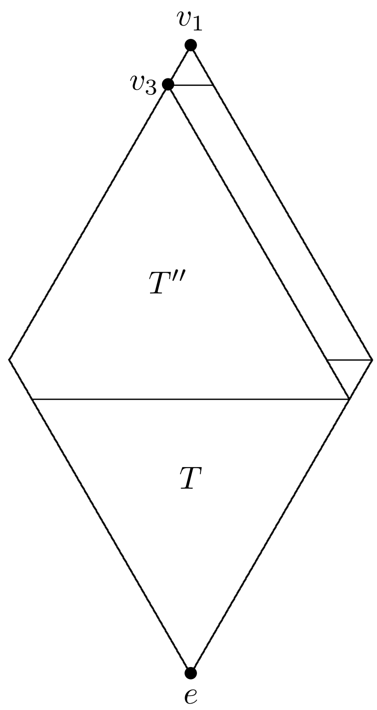


Figure 4.3.8

Theorem 4.3.2 The action of Γ on Ω is topologically free. That is, for all $g \in \Gamma \backslash \{e\}$,

$$int\{\omega \in \Omega : g\omega = \omega\} = \emptyset.$$

Proof. Suppose that g fixes all points of some nonempty open set $V \subset \Omega$. By Proposition 4.3.1 , there is an apartment $\mathcal{A}$ all of whose boundary points $\omega_1, \ldots, \omega_6$ are contained in V. Now since the boundary points $\omega_1, \ldots, \omega_6$ of $\mathcal{A}$ are fixed by g it follows that $\mathcal{A}$ is stabilized by g. To see this, fix an arbitrary vertex x_0 in $\mathcal{A}$. For $i = 1, \ldots, 6$, the sectors $S^{x_0}(\omega_i)$ and $S^{gx_0}(g\omega_i)$ are equivalent and hence contain a common subsector S_i. Thus $S_i \subset \mathcal{A} \cap g\mathcal{A}$, $i = 1, \ldots, 6$. In particular $\mathcal{A} \cap g\mathcal{A} \neq \emptyset$. We may therefore suppose from the start that the base vertex x_0 lies in $\mathcal{A} \cap g\mathcal{A}$. Now $g\mathcal{A}$ contains $S^{gx_0}(g\omega_i) = S^{gx_0}(\omega_i), \quad i = 1, \ldots, 6$. But $\mathcal{A} = \cup_{i=1}^{6} S^{gx_0}(\omega_i)$. Therefore $g\mathcal{A} = \mathcal{A}$ and g acts on $\mathcal{A}$ by some translation rather than some rotation or glide reflection.

Moreover all nearby apartments (i.e. all apartments having sufficiently many chambers in common with $\mathcal{A}$) must be stabilized by g, since their boundary points will also be in

V and hence fixed by g. Translating by g shows that if $\mathcal{A}$ and a nearby apartment have one chamber in common, then they have infinitely many chambers in common. But this is impossible, since there are apartments arbitrarily close to $\mathcal{A}$ having only finitely many chambers in common with $\mathcal{A}$. To see this, simply choose a large convex subset K of $\mathcal{A}$. Add a layer of triangles to the boundary of K, none of which belongs to $\mathcal{A}$, and extend the resulting region K' to an apartment $\mathcal{A}'$. The contractibility of the building implies that $\mathcal{A} \cap \mathcal{A}' = K$.

□

5. The algebraic structure of $C^*(s^{\pm})$.

We can now show our main result.

Theorem 5.1. *The C^*-algebra $C^*(s^{\pm}) = C(\Omega) \rtimes \Gamma$ is simple and nuclear and canonically isomorphic to the reduced crossed product $C(\Omega) \rtimes_r \Gamma$.*

Proof. Since the action of Γ on Ω is minimal (Proposition 4.1.1) and topologically free (Proposition 4.3.2), it follows from [3: Corollary to Theorem 1] that $C(\Omega) \rtimes_r \Gamma$ is simple. The remaining assertions follow from amenability of the action (§4.2).

□

Corollary. *Under the same hypotheses, the C^*-algebra $C_r^*(\Gamma)$ is subnuclear.*

Proof. $C_r^*(\Gamma)$ embeds in $C(\Omega) \rtimes_r \Gamma$.

□

Remarks.1. It was shown in [7] that the group Γ has Kazhdan's property (T) and so is very far from being amenable.

2. A.M. Mantero and A. Zappa [18] have shown that, under the general assumptions of section 3.3, the action of Γ on Ω satisfies a certain geometric condition of [4] which

implies that $C_r^*(\Gamma)$ is simple. The minimality of the action (Proposition 4.1.1) and [4: Theorem 5] give another proof that $C(\Omega) \rtimes_r \Gamma$ is simple .

5.2 **The linear case :** $\widetilde{A}_2(F, v)$

Consider the class of affine buildings $\widetilde{A}_2(F, v)$ that arise in a natural way from linear groups [22: §9.2]. Not all the $\widetilde{A}_2$ buildings considered above can be constructed in this way. However for this class we can obtain simplicity and nuclearity of the crossed product using results of [21].

Let F be a local field whose residual field has order q. There is a triangle building $\triangle_F$ associated with $SL(3, F)$ [5:VI.9F],[22:§9.2]. Let $\mathcal{O}$ be the valuation ring of F with respect to its valuation v and let $\pi \in F^\times$ be a uniformizer. i.e. $v(\pi) = 1$. A *lattice* in F^3 is a free $\mathcal{O}$-submodule of F^3 of rank 3. Two such lattices L and L' are *equivalent* if $L' = aL$ for some $a \in K^\times$. The vertices of $\triangle_F$ are the lattice classes. Its edges are pairs of vertices x and y such that if L is in the class of x there is an L' in the class of y for which $\pi L \subset L' \subset L$. The chambers consist of triples of distinct vertices $[L]$,$[L']$,$[L'']$ whose representatives can be chosen such that $L \supset L' \supset L'' \supset \pi L$.

The group $G = PGL(3, F)$ acts on $\triangle_F$ by defining $g \cdot [L] = [gL]$, for $g \in G$ and each lattice L. A subgroup Γ which acts simply transitively on the vertices of $\triangle_F$ in the case $F = F_q((X))$ is constructed explicitly in [8:I §4]. G acts transitively on the boundary Ω and there is a point $\omega_0 \in \Omega$ whose stabilizer is the group P of upper triangular matrices in G [5:Proposition VI.9F]. Hence Ω is isomorphic to G/P as a topological G-space.

Assume that $\triangle = \triangle_F$ and G,P,Γ are as above. The group P is solvable, hence amenable, and so the fact that $C(\Omega) \rtimes \Gamma$ is nuclear and canonically isomorphic to $C(\Omega) \rtimes_r \Gamma$ follows from [21: Corollary 4.3].

Moreover one can show that the action of G on Ω is topologically free directly as follows.

Identify Ω with G/P as above. Let $g \in \Gamma\backslash\{e\}$ and let $\omega = hP \in \Omega$ with $g\omega = \omega$ i.e. $h^{-1}gh = p \in P$. We must find $\omega_0 \in \Omega$ close to ω such that $g\omega_0 \neq \omega_0$. That is $h_0 \in G$ close to h such that $h_0^{-1}gh_0 \notin P$. Equivalently, we must find $k \in G$ close to e such that $k^{-1}pk \notin P$.

Now p is upper triangular and $p \neq 1$, so we may suppose that either $p_{12} \neq 0$ or $p_{11} \neq p_{22}$. Let $\varepsilon \in F$ and define $k \in G$ by

$$(1-\varepsilon^2)k = \begin{bmatrix} 1 & \varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Direct calculation shows that the (2,1) entry of $k^{-1}pk$ is $\varepsilon(1-\varepsilon^2)^{-1}(p_{22} - p_{11} - p_{12}\varepsilon)$, which is nonzero for suitable small $\varepsilon \in F$. This proves the result.

Finally, the fact that the action of Γ on G/P is minimal (§3.7) is analogous to the following result [19: Section8]: Let G be a connected semisimple Lie group, P a parabolic subgroup of G and Γ_0 a lattice in G. Then Γ_0 acts minimally on G/P.

Appendix:*Apartments and Galleries in Triangle Buildings*

In this appendix we gather together various additional definitions and results about triangle buildings. The aim is to make it easier for someone unfamiliar with buildings to read the later parts of the paper, particularly section 4.3. For further information we refer to [5] and [22]

We deal exclusively with affine buildings of type $\widetilde{A}_2$. Such a building is a simplicial complex $\triangle$ which is a union of certain subcomplexes called *apartments*. Each apartment Σ is a Coxeter complex realized as a tiling of the Euclidean plane by equilateral triangles. Any two simplices A,B in the building are contained in a common apartment. Furthermore, if Σ, Σ' are apartments containing A,B, then there exists an isomorphism between Σ and Σ'

fixing A and B pointwise [5: Chapter IV]. We assume throughout that $\triangle$ has a *complete* system of apartments: that is, the unique largest system of apartments which contains every other system of apartments [5: Chapter IV.4]. The maximal simplices in $\triangle$ are all equilateral triangles, and are called *chambers*. Two chambers C,C' are *adjacent* if they have a common edge. A *gallery* is a sequence of chambers

$$(C_0, \ldots, C_d)$$

such that any two consecutive chambers C_{i-1} and C_i are adjacent. We say that such a gallery connects C_0 to C_d. The integer d is called the length of the gallery. Any two chambers are certainly connected by a gallery since they lie in a common apartment. The distance between chambers C and D in the building is the minimal length of a gallery $(C_0, \ldots, C_d)$ connecting C to D. A gallery which achieves this minimum is called a *minimal gallery* from C to D. More generally, the distance between a chamber C and a simplex A in the building is the minimal length of a gallery $(C_0, \ldots, C_d)$ with $C_0 = C$ and C_d containing A. A gallery which achieves this minimum is called a *stretched gallery* from C to A.

Each vertex of $\triangle$ is labelled with a *type* which is an integer in $\{0, 1, 2\}$, and each chamber has exactly one vertex of each type. The case we are interested in is when the 1-skeleton of $\triangle$ is the Cayley graph of a group Γ with generating set P. Recall from section 3.3 that the elements of P correspond to the points in a finite projective plane of order q. In this case let $\tau : \Gamma \to \mathbb{Z}/3\mathbb{Z}$ be the homomorphism determined by $\tau(a_x) = 1$ for each $x \in P$. Then $\tau(g)$ is the type of g for each $g \in \Gamma$. Thus e has type 0, each generator a_x has type 1, and each inverse generator a_x^{-1} has type 2.

Two adjacent chambers C,C' are said to be *i-adjacent* if the vertex of each which does not lie on their common edge has type i. The type of a gallery $(C_0, \ldots, C_d)$ is the sequence of labels $(s_1, \ldots, s_d)$ such that C_{i-1} and C_i are i-adjacent for $i = 1, \ldots, d$. For example, in Figure A.1 the gallery $(C_0, C_1, C_2, C_3, C_4, C_5)$ has type $(2, 0, 2, 1, 2)$.

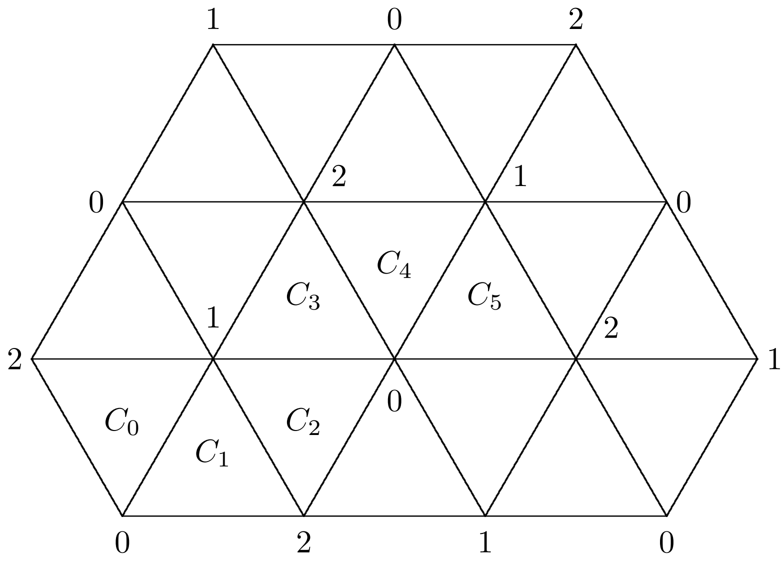


Figure A.1

It follows from [5: Chapter IV.3, Proposition 2] that a gallery is stretched if it has the same type as a stretched gallery.

Each apartment in the building $\triangle$ can be identified with a copy of the Euclidean plane and hence inherits a natural metric which gives rise to a well-defined metric on the whole building [5: Chapter IV.3]. A subcomplex of $\triangle$ is called *convex* if it contains every minimal gallery connecting any two chambers in it. The following result is very useful. (See [5: Chapter VI.7 Theorems 1 and 2] for details.)

Theorem A.1. A subcomplex Y of an affine building of type $\widetilde{A}_2$ which is either convex or has nonempty interior and is isometric to a subset of the Euclidean plane $\mathbb{R}^2$ is contained in an apartment. Consequently Y is an apartment if and only if Y is isometric to $\mathbb{R}^2$.

It is also important to note that $\triangle$ is topologically rather trivial.

Theorem A.2. ([22: Appendix 4,page 185].) An affine building is contractible.

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