

BOUNDARY C^* -ALGEBRAS OF TRIANGLE GEOMETRIES

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ABSTRACT. Let Δ be a building of type \tilde{A}_2 and order q , with maximal boundary Ω . Let Γ be a group of type preserving automorphisms of Δ which acts regularly on the chambers of Δ . Then the crossed product C^* -algebra $C(\Omega) \rtimes \Gamma$ is isomorphic to $M_{3(q+1)} \otimes \mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$, where \mathcal{O}_n denotes the Cuntz algebra generated by n isometries whose range projections sum to the identity operator.

1. INTRODUCTION

The boundary at infinity plays an important role in studying the action of groups on euclidean buildings and other spaces of non-positive curvature. The motivation of this article is to determine what information about a group may be recovered from the K-theory of the boundary crossed product C^* -algebra. The focus is on groups of automorphisms of euclidean buildings. Groups of automorphisms of \tilde{A}_2 buildings which act regularly on the set of chambers have been studied by several authors [7, 8, 13, 15]. The existence of groups which act regularly on the set of chambers (i.e. the existence of “tight triangle geometries”) is proved in [13, Theorem 3.3].

The main result is the following, where M_n denotes the algebra of complex $n \times n$ matrices and \mathcal{O}_n is the Cuntz C^* -algebra which is generated by n isometries on a Hilbert space whose range projections sum to I .

Theorem 1.1. *Let Δ be a building of type \tilde{A}_2 and order q , with maximal boundary Ω . Let Γ be a group of type preserving automorphisms of Δ which acts regularly on the chambers of Δ . Then the crossed product C^* -algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$ is isomorphic to $M_{3(q+1)} \otimes \mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$.*

The theorem is proved by giving an explicit representation of \mathfrak{A}_Γ as a rank two Cuntz-Krieger algebra in the sense of [11]. The K-theory of \mathfrak{A}_Γ is then determined explicitly as $K_0(\mathfrak{A}_\Gamma) \cong K_1(\mathfrak{A}_\Gamma) \cong \mathbb{Z}_{q^2-1}$ with the class $[1]$ in $K_0(\mathfrak{A}_\Gamma)$ corresponding to $3(q+1) \in \mathbb{Z}_{q^2-1}$. The classification theorem of E. Kirchberg and N.C. Phillips for p.i.s.u.n C^* algebras [1] completes the proof. It is notable that, for these groups, $C(\Omega) \rtimes \Gamma$ depends only on the order q of the building and not on the group Γ . This is the first class of higher rank groups for which the boundary C^* -algebra has been computed exactly. There is a sharp contrast with the class of groups which act regularly on the vertex set of an \tilde{A}_2 building, where the boundary C^* -algebra \mathfrak{A}_Γ frequently appears to determine the group Γ , according to explicit computations derived from [12]. For example, the three torsion-free groups $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$ which act regularly on the vertex set of the corresponding building Δ are distinguished from each other by $K_0(\mathfrak{A}_\Gamma)$. On the other hand, for $q = 2$, there are four different groups satisfying the hypotheses of Theorem 1.1, and hence having the same boundary C^* -algebra.

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2. TRIANGLE GEOMETRIES

A locally finite euclidean building whose boundary at infinity is a spherical building of rank 2 is of type \tilde{A}_2 , \tilde{B}_2 or \tilde{G}_2 . This article is concerned with buildings of type \tilde{A}_2 , also called triangle buildings. There is a close connection to projective geometry: the link of each vertex of an \tilde{A}_2 building Δ is the incidence graph of a finite projective plane of order q . The maximal simplices of Δ are triangles, which are called *chambers*. Each vertex of Δ has a *type* $j \in \mathbb{Z}_3$, and each chamber has exactly one vertex of each type. Each edge of Δ lies on $q + 1$ chambers. Orient each edge of Δ from its vertex of type i to its vertex of type $i + 1$. In the link

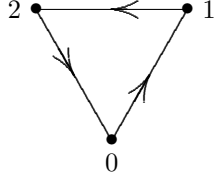


FIGURE 1. A chamber showing vertex types and edge orientations.

of the vertex v of type i , the vertices of type $i + 1$ [type $i + 2$] correspond to the points [lines] of a projective plane of order q . The chambers of Δ which contain v correspond to incident point-line pairs. The number of such chambers is therefore $(q + 1)(q^2 + q + 1)$.

Suppose that Δ is a building of type \tilde{A}_2 and that Γ is a group of type preserving automorphisms of Δ which acts regularly (i.e. freely and transitively) on the chambers of Δ . Then Δ/Γ is called a *tight triangle geometry*. M. Ronan [13, Theorem 3.3] showed how to construct examples for all values of the prime power q . If $q = 2$ then interesting examples arise from groups generated by three elements of order 3 such that any pair generate a Frobenius group of order 21 [13, 7, 8]. These give rise to four non-isomorphic groups, the simplest of which has presentation

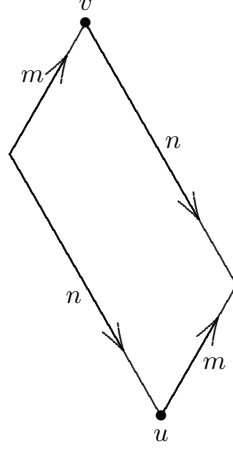
$$(1) \quad \langle s_i, i \in \mathbb{Z}_3 \mid s_i^3 = 1, s_i s_{i+1} = (s_{i+1} s_i)^2 \rangle$$

and acts on the euclidean building Δ of $\mathrm{SL}_3(\mathbb{Q}_2)$. If $q = 8$ there are 44 non-isomorphic groups of this type [15].

From now on Δ will denote a building of type \tilde{A}_2 . Any two vertices $u, v \in \Delta$ belong to a common apartment. The convex hull, in the sense of buildings, between two vertices u and v is illustrated in Figure 2. Fundamental properties of buildings imply that any apartment containing u and v must also contain their convex hull. Except in degenerate cases, the convex hull $\mathrm{conv}\{u, v\}$ is a parallelogram, with base vertex u . Define the *distance*, $d(u, v)$, between vertices u and v to be the graph theoretic distance on the one-skeleton of Δ . Any path from u to v of length $d(u, v)$ lies in their convex hull, and the union of the vertices in such paths is exactly the set of vertices in the convex hull.

Define the *shape* $\sigma(u, v)$ of an ordered pair of vertices (u, v) to be the pair $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ as indicated in Figure 2. Note that $d(u, v) = m + n$. The arrows in Figure 2 point in the direction of cyclically increasing vertex type, i.e. $\{\dots, 0, 1, 2, 0, 1, \dots\}$. If the parallelogram π is the convex hull $\mathrm{conv}\{u, v\}$ of an *ordered* pair of vertices then the vertex $v_{ij} = v_{ij}(\pi)$ is the vertex of π such that $\sigma(u, v_{ij}) = (i, j)$. Parallelograms will always be parametrized in this way relative the initial vertex $u = v_{00}(\pi)$. Denote by $\Pi_{m,n}$ the set of parametrized parallelograms of shape (m, n) .

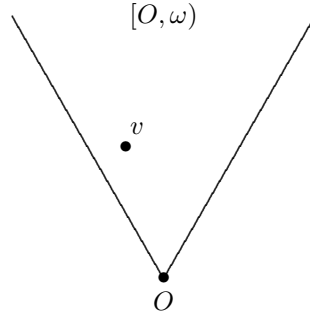
A *sector* is a $\frac{\pi}{3}$ -angled sector made up of chambers in some apartment (Figure 3). Two sectors are *parallel* if their intersection contains a sector. Parallelism is an equivalence relation and the (maximal) boundary Ω of Δ is defined to be the set of

FIGURE 2. Convex hull of two vertices, with $\sigma(u, v) = (m, n)$.

equivalence classes of sectors in Δ . For any vertex v of Δ and any $\omega \in \Omega$ there is a unique sector $[v, \omega)$ in the class ω having base vertex v [14, Section 9.3]. Fix some vertex O in Δ . The boundary of Δ is a totally disconnected compact Hausdorff space with a base for the topology given by sets of the form

$$\Omega(v) = \{\omega \in \Omega : [O, \omega) \text{ contains } v\}$$

where v is a vertex of Δ .

FIGURE 3. The sector $[O, \omega)$, where $\omega \in \Omega(v)$.

A *tile* τ in Δ is defined to be the convex hull $\text{conv}\{u, v\}$ of an ordered pair of vertices with shape $\sigma(u, v) = (2, 2)$, as illustrated in Figure 4. The initial vertex is $u = v_{00}(\tau)$ and the final vertex is $v = v_{22}(\tau)$.

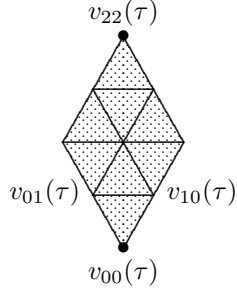
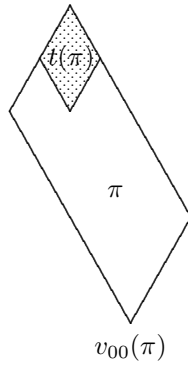
Let π be a parallelogram with shape (m, n) , where $m, n \geq 2$. The terminal tile $t(\pi)$ is the tile τ contained in the parallelogram π and containing its terminal vertex $v_{m,n}(\pi)$ (Figure 5). Let

$$\Omega(\pi) = \{\omega \in \Omega : \pi \subset [v_{00}(\pi), \omega)\},$$

the set of boundary points whose representative sectors based at $v_{00}(\pi)$ contain π .

3. ALGEBRAS ARISING FROM BOUNDARY ACTIONS ON \tilde{A}_2 BUILDINGS

Let Δ be a building of type \tilde{A}_2 and order q , with boundary Ω . Let Γ be a group of type preserving automorphisms of Δ which acts regularly on the chambers of Δ . The group Γ acts on Ω , and one can form the crossed product C^* -algebra $C(\Omega) \rtimes \Gamma$.

FIGURE 4. A tile τ in an apartment.FIGURE 5. A parallelogram π and its terminal tile $t(\pi)$.

This is the universal C^* -algebra generated by the commutative C^* -algebra $C(\Omega)$ and the image of a unitary representation π of Γ , satisfying the covariance relation

$$(2) \quad f(\gamma^{-1}\omega) = \pi(\gamma) \cdot f \cdot \pi(\gamma)^{-1}(\omega)$$

for $f \in C(\Omega)$, $\gamma \in \Gamma$ and $\omega \in \Omega$. It is convenient to denote $\pi(\gamma)$ simply by γ . We show that the crossed product algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$ is generated by an explicit set of partial isometries, described as follows. Fix a vertex O with type 0 in Δ . Denote by Π_O the set of all (parametrized) parallelograms in Δ with base vertex O . If $\pi_1, \pi_2 \in \Pi_O$ with $\Gamma t(\pi_1) = \Gamma t(\pi_2)$, then there is a unique element $\gamma \in \Gamma$ such that $\gamma t(\pi_1) = t(\pi_2)$, since Γ acts freely on the chambers of Δ . Let

$$(3) \quad S_{\pi_2, \pi_1} = \gamma \mathbf{1}_{\Omega(\pi_1)} = \mathbf{1}_{\Omega(\pi_2)} \gamma.$$

Then S_{π_2, π_1} is a partial isometry with initial projection $\mathbf{1}_{\Omega(\pi_1)}$ and final projection $\mathbf{1}_{\Omega(\pi_2)}$.

Lemma 3.1. *The set of partial isometries*

$$\{S_{\pi_2, \pi_1} : \pi_1, \pi_2 \in \Pi_O, \Gamma t(\pi_1) = \Gamma t(\pi_2)\}$$

generates $C(\Omega) \rtimes \Gamma$.

Proof. Let \mathfrak{S} denote the C^* -subalgebra of \mathfrak{A}_Γ generated by $\{S_{\pi_2, \pi_1} : \pi_1, \pi_2 \in \Pi_O, \Gamma t(\pi_1) = \Gamma t(\pi_2)\}$. Since the sets of the form $\Omega(\pi)$, $\pi \in \Pi_O$, form a basis for the topology of Ω , the linear span of the $\Omega(\pi)$ is dense in $C(\Omega)$. It follows that \mathfrak{S} contains $C(\Omega)$. To show that \mathfrak{S} contains \mathfrak{A}_Γ , it suffices to show that it contains Γ .

Fix $\gamma \in \Gamma$. Choose $m_1, m_2 \geq 2$ such that $d(O, \gamma^{-1}O) \leq m_1, m_2$. Note that

$$(4) \quad \gamma = \sum_{\pi} \gamma \mathbf{1}_{\Omega(\pi)}$$

where the sum is over all parallelograms π based at O with $\sigma(\pi) = (m_1, m_2)$. Fix such a parallelogram π . Let $\pi'' = \text{conv}\{\gamma^{-1}O, x\}$. Since $\sigma(t(\pi)) = (2, 2)$, it follows from [11, Lemma 7.5] that $t(\pi'') = t(\pi)$.

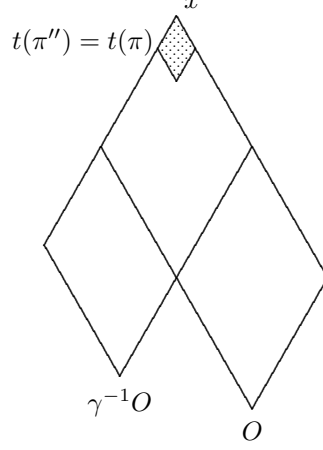


FIGURE 6

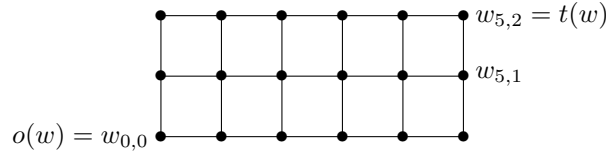
Let $\pi' = \gamma\pi''$. Then $\pi' = \text{conv}\{O, \gamma x\}$ and $t(\pi') = \gamma t(\pi)$. Therefore $\gamma \mathbf{1}_{\Omega(\pi)} = S_{\pi, \pi'} \in \mathfrak{A}_\Gamma$. The result follows from equation (4). \square

4. CUNTZ-KRIEGER ALGEBRAS OF RANK 2

This section recalls the principal facts about these algebras from [11]. Fix a finite set A which we refer to as an “alphabet”. A $\{0, 1\}$ -matrix is a matrix with entries in $\{0, 1\}$. Choose nonzero $\{0, 1\}$ -matrices M_1, M_2 and denote their elements by $M_j(b, a) \in \{0, 1\}$ for $a, b \in A$.

If $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, a word of shape $\sigma(w) = (m, n)$ in the alphabet A is a mapping $(i, j) \mapsto w_{i,j}$ from $\{0, 1, \dots, m\} \times \{0, 1, \dots, n\}$ to A such that $M_1(w_{i+1,j}, w_{i,j}) = 1$ and $M_2(w_{i,j+1}, w_{i,j}) = 1$. Denote by $W_{m,n}$ the set of words w of shape (m, n) . The set $W_{0,0}$ is identified with A .

Define initial and final maps $o : W_{m,n} \rightarrow A$ and $t : W_{m,n} \rightarrow A$ by $o(w) = w_{0,0}$ and $t(w) = w_{m,n}$.

FIGURE 7. Representation of a two dimensional word of shape $(5, 2)$.

Fix a nonempty finite or countable set D (whose elements are *decorations*), and a map $\delta : D \rightarrow A$. Let $\overline{W}_{m,n} = \{(d, w) \in D \times W_{m,n} : o(w) = \delta(d)\}$, the set of *decorated words* of shape m, n , and identify D with \overline{W}_0 via the map $d \mapsto (d, \delta(d))$. Let $W = \bigcup_{m,n} W_{m,n}$ and $\overline{W} = \bigcup_{m,n} \overline{W}_{m,n}$, the sets of all words and all decorated words respectively. Define $o : \overline{W}_{m,n} \rightarrow D$ and $t : \overline{W}_{m,n} \rightarrow A$ by $o(d, w) = d$ and $t(d, w) = t(w)$. Likewise extend the definition of shape to \overline{W} by setting $\sigma((d, w)) = \sigma(w)$.

If $M_1 M_2 = M_2 M_1$ and $M_1 M_2$ is a $\{0, 1\}$ -matrix then by [12, Lemma 1.4] the following condition holds.

(H1): Let $u \in W_{m,n}$ and $v \in W_{r,s}$. If $t(u) = o(v)$ then there exists a unique word $w \in W_{m+r,n+s}$ such that

$$\begin{aligned} w_{i,j} &= u_{i,j} & 0 \leq i \leq m, 0 \leq j \leq n; \\ w_{i,j} &= v_{m+i,n+j} & 0 \leq i \leq r, 0 \leq j \leq s. \end{aligned}$$

We write $w = uv$ and say that the product uv exists.

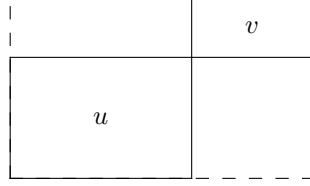


FIGURE 8. The word uv

Assume that condition **(H1)** is satisfied, as well as the following two conditions.

(H2): Consider the directed graph which has a vertex for each $a \in A$ and a directed edge from a to b for each i such that $M_i(b, a) = 1$. This graph is irreducible.

(H3): If $(p_1, p_2) \in \mathbb{Z}^2 - \{(0, 0)\}$, then there exists some $w \in W$ which is not (p_1, p_2) -periodic, in the sense of [11].

Under the above assumptions, define a C^* -algebra, which depends on D , as follows.

Definition 4.1. The C^* -algebra \mathcal{A}_D is the universal C^* -algebra generated by a family of partial isometries $\{s_{u,v} : u, v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfying the relations

$$(5a) \quad s_{u,v}^* = s_{v,u}$$

$$(5b) \quad s_{u,v} s_{v,w} = s_{u,w}$$

$$(5c) \quad s_{u,v} = \sum_{\substack{w \in W; \sigma(w) = \varepsilon, \\ o(w) = t(u) = t(v)}} s_{uw, vw}, \text{ for } \varepsilon \in \{(1, 0), (0, 1)\}$$

$$(5d) \quad s_{u,u} s_{v,v} = 0, \text{ for } u, v \in \overline{W}_0, u \neq v.$$

Theorem 4.2. [11] *The C^* -algebra \mathcal{A}_D is purely infinite, simple and nuclear. Any nontrivial C^* -algebra with generators $S_{u,v}$ satisfying relations (5) is isomorphic to \mathcal{A}_D .*

Remark 4.3. Two decorations $\delta_1 : D_1 \rightarrow A$ and $\delta_2 : D_2 \rightarrow A$ are said to be *equivalent* [10, Section 5] if there is a bijection $\eta : D_1 \rightarrow D_2$ such that $\delta_1 = \delta_2 \eta$. Equivalent decorations δ_1, δ_2 give rise to isomorphic algebras $\mathcal{A}_{D_1}, \mathcal{A}_{D_2}$.

Lemma 4.4. *Given a decoration $\delta : D \rightarrow A$, and $r \in \mathbb{N}$, define another decoration $\delta' : D \times \{1, 2, \dots, r\} \rightarrow A$ by $\delta'((d, i)) = \delta(d)$. The C^* -algebra $\mathcal{A}_{D \times \{1, 2, \dots, r\}}$ is isomorphic to $M_r \otimes \mathcal{A}_D$.*

Proof. (c.f. [11, Lemma 5.12].) If $u, v \in W$, the isomorphism is given by

$$s_{((d,i),u),((d',j),v)} \mapsto s_{(d,u),(d',v)} \otimes e_{ij},$$

where the e_{ij} are matrix units for M_r . The fact that this is an isomorphism follows from [11, Corollary 5.10]. \square

5. $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$ AS A RANK 2 CUNTZ-KRIEGER ALGEBRA

We now show how the set of partial isometries $\{S_{\pi_2, \pi_1} : \pi_1, \pi_2 \in \Pi_O, \Gamma t(\pi_1) = \Gamma t(\pi_2)\}$ defined by (3) allows one to express \mathfrak{A}_Γ as a rank 2 Cuntz-Krieger algebra. The alphabet A is the set of Γ -orbits of tiles. That is $A = \{\Gamma\tau : \tau \in \Pi_{2,2}\}$. Since Γ is type preserving, $A = A_0 \sqcup A_1 \sqcup A_2$, where A_i denotes the set of $\Gamma\tau$ such that τ has base vertex of type i . The decorating set D is the set of tiles τ with fixed base vertex O , and the decorating map $\delta : D \rightarrow A$ is defined by $\delta(\tau) = \Gamma\tau$. From now on, D will always denote this particular decorating set.

The matrices M_1, M_2 with entries in $\{0, 1\}$ are defined as follows. If $a, b \in A$, say that $M_1(b, a) = 1$ if and only if $a = \Gamma\tau_1, b = \Gamma\tau_2$, for some tiles τ_1, τ_2 , where the union of the representative tiles $\tau_1 \cup \tau_2$ is a parallelogram of shape $(3, 2)$ as illustrated on the right of Figure 9, with τ_1 shaded. The definition of M_2 is illustrated on the left of Figure 9, where $a' = \Gamma\tau'_1, b' = \Gamma\tau'_2$ and $\tau'_1 \cup \tau'_2$ is a parallelogram of shape $(2, 3)$.

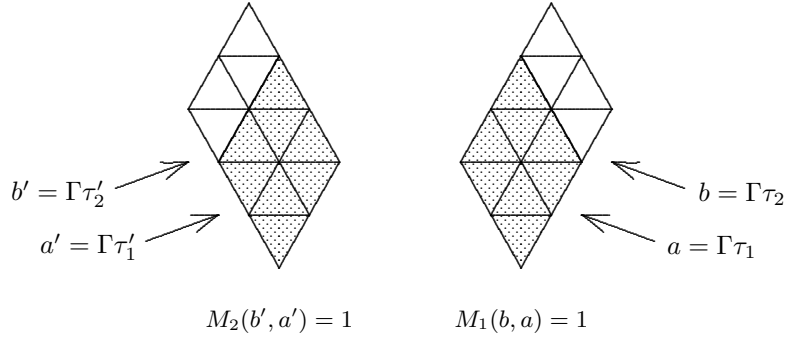


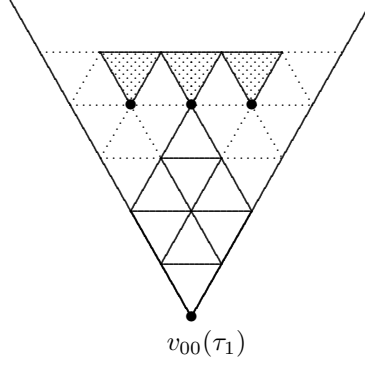
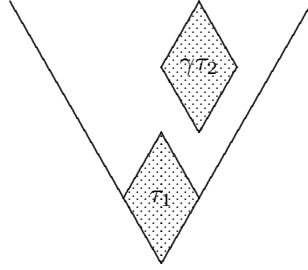
FIGURE 9. Transition matrices.

Proposition 5.1. *The matrices M_1, M_2 satisfy conditions (H1), (H2), and (H3) of Section 4.*

Proof. The verification of conditions (H1) and (H2) is a minor modification of [11, Proposition 7.9], since Γ acts freely on the chambers of Δ , but the verification of (H2) requires a new argument.

Consider the directed graph \mathcal{G} which has vertex set A and a directed edge from $a \in A$ to $b \in A$ for each $i = 1, 2$ such that $M_i(b, a) = 1$. Condition (H2) of [11] is the assertion that this graph is irreducible. In order to prove this, fix tiles τ_1, τ_2 such that $a = \Gamma\tau_1, b = \Gamma\tau_2$. Fix a sector \mathfrak{S} containing τ_1 , with base point $v_{00}(\tau_1)$, as in Figure 10. Choose from among the shaded chambers in Figure 10 the chamber c whose initial vertex has the same type as the initial vertex of τ_2 . Since Γ acts transitively on the chambers of Δ , we may choose $\gamma \in \Gamma$ such that $\gamma^{-1}c$ is the initial chamber of τ_2 . Then $\text{conv}\{\tau_1 \cup \gamma\tau_2\}$ is contained in a sector \mathfrak{S}' with initial tile τ_1 . Figure 11 illustrates one of the three possible relative positions. This shows that there is a directed path of length 3 in the graph \mathcal{G} from $\Gamma\tau_1$ to $\Gamma\tau_2$. \square

The proof that $\{S_{\pi_2, \pi_1} : \pi_1, \pi_2 \in \Pi_O, \Gamma t(\pi_1) = \Gamma t(\pi_2)\}$ is the set of generators for a rank 2 Cuntz-Krieger algebra \mathcal{A}_D , associated with the alphabet A and decorating map $\delta : D \rightarrow A$ defined above, is a minor modification of the argument in Section 7 of [11]. Let $\mathfrak{W}_{m,n} = \Gamma\Pi_{m+2,n+2}$, the set of Γ -orbits of parallelograms of shape $(m+2, n+2)$. The elements of $\mathfrak{W}_{m,n}$ are in bijective correspondence with the words of shape (m, n) in the alphabet A . See Section 4, and compare with [11, Lemma 7.1]. Thus the large parallelograms on the right and left of Figure 9 represent words ab and $a'b'$ of shape $(1, 0)$ and $(0, 1)$ respectively.

FIGURE 10. The sector \mathfrak{S} containing τ_1 .FIGURE 11. The sector \mathfrak{S}' containing τ_1 and $\gamma\tau_2$.

Remark 5.2. The overlap in the tiles used to define the transition matrices M_1 and M_2 is needed in order to establish the bijection from $W_{m,n}$ onto $\mathfrak{W}_{m,n}$. It is not possible to use the more straightforward definition of the transition matrices, in terms of contiguous, but not overlapping, tiles.

In view of Lemma 3.1, we now have the following result.

Proposition 5.3. *If $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, then there is an isomorphism $\mathfrak{A}_\Gamma \cong \mathcal{A}_D$.*

The next result will be used later.

Lemma 5.4. *Let $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. If $a \in A_i$ then $M_j(b, a) = 1$ only if $b \in A_{i+j}$. Moreover, and each row or column of the matrix M_j has precisely q^2 nonzero entries.*

Proof. The first statement is clear from the definition of the transition matrices. We prove the second statement in the case $j = 1$. Choose $b = \Gamma\tau \in A$ and refer to Figure 12, where τ is shaded. There are precisely q^2 parallelograms π of shape $(3, 2)$ such that $t(\pi) = \tau$. For once τ is chosen, there are q choices for the chamber δ_1 . Once δ_1 is chosen, there are q choices for δ_2 and the whole parallelogram of shape $(3, 2)$ is then completely determined. It follows that there are q^2 possibilities, as claimed. This proves that for each $b \in A$, there are q^2 choices for $a \in A$ such that $M_1(b, a) = 1$. That is, each column of the matrix M_1 has precisely q^2 nonzero entries. A similar argument applies to rows. \square

6. K-THEORY

The main result of [12], later extended to a more general class of C^* -algebras in [4], is that the K-theory of a rank 2 Cuntz-Krieger algebra can be expressed

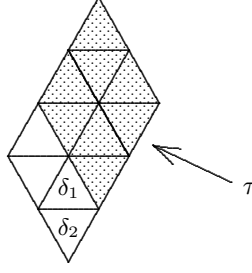


FIGURE 12

in terms of the transition matrices M_1, M_2 . The matrix $(I - M_1, I - M_2)$ defines a homomorphism $\mathbb{Z}^A \oplus \mathbb{Z}^A \rightarrow \mathbb{Z}^A$. Let r be the rank, and T the torsion part, of the finitely generated abelian group $C(\Gamma) = \text{coker } (I - M_1, I - M_2)$. Thus $C(\Gamma) \cong \mathbb{Z}^r \oplus T$.

Theorem 6.1. *Let Δ be a building of type \tilde{A}_2 and order q , with maximal boundary Ω . Let Γ be a group of type preserving automorphisms of Δ which acts regularly on the chambers of Δ . Then*

$$(6) \quad K_0(\mathfrak{A}_\Gamma) = K_1(\mathfrak{A}_\Gamma) = \mathbb{Z}^{2r} \oplus T.$$

Proof. The equation (6) is the result of combining [12, Proposition 4.13] with the analogue of [12, Lemma 5.1], whose proof carries over without change to the larger tiles considered in the present article. \square

In order to compute the K-theory of \mathfrak{A}_Γ it is convenient to reduce the size of the tiles. A ball of radius one in an apartment is a hexagon containing six chambers. A *reduced tile* is a hexagon together with a choice of one of its six chambers, which determines the direction “up”. The reduced tile ξ is said to be of type $i \in \mathbb{Z}_3$, if its central vertex $v_{11}(\xi)$ is of type i . Denote by X_i the set of reduced tiles of type i . If $\xi \in X_i$ then its chambers are denoted $\xi_k, \bar{\xi}_k, k \in \mathbb{Z}_3$ as in Figure 13, with the “up” chamber denoted by ξ_i . The vertices $v_{kl}(\xi)$ of type $k - l + i$ are labelled as in Figure 13. This is consistent with previous notation, in the sense that if τ is any tile containing the hexagon ξ then $v_{kl}(\xi) = v_{kl}(\tau)$ whenever both sides are defined. Note that there are q^2 tiles τ containing ξ , since once the hexagon is fixed there are q choices of $v_{00}(\tau)$ and q choices of $v_{22}(\tau)$.

Remark 6.2. The link of a vertex v in Δ is the spherical building of a projective plane of order q . The hexagonal boundary of a reduced tile ξ with central vertex v is an apartment in this spherical building. The vertex $v_{ij}(\xi)$ is a point [line] in the projective plane if $i - j = 1$ [$i - j = 2$].

Define a new alphabet Λ to be the set of Γ -orbits of reduced tiles. Since the action of Γ is type preserving, we may write $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ where $\Lambda_i = \{\Gamma\xi : \xi \in X_i\}$. For each fixed chamber δ , there are q^3 reduced tiles $\xi \in X_i$ such that $\xi_i = \delta$ [9, Lemma 4.7]. Since Γ acts regularly on the set of chambers of Δ , $\#\Lambda_i = q^3$ and $\#\Lambda = 3q^3$. Each reduced tile is contained in q^2 tiles, and so $\#A = 3q^5$.

The new transition matrices N_1, N_2 are defined by the diagrams in Figure 14. In each of the diagrams, the region is a union of two hexagons (one shaded, one partly shaded), which are representatives of the elements a, b respectively. Once the shaded hexagon is chosen, there are q^2 choices for the hexagon at the higher level which contains two shaded triangles. For there are q choices for the unshaded triangle meeting the shaded hexagon in its upper horizontal edge. Once this unshaded triangle is chosen there are q choices for the unshaded triangle adjacent to it. The remaining two unshaded triangles are then completely determined, since

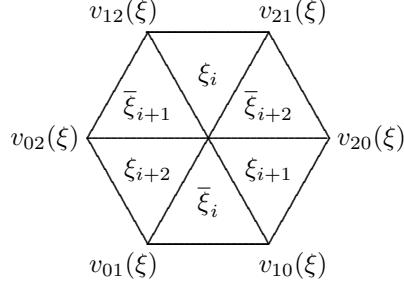
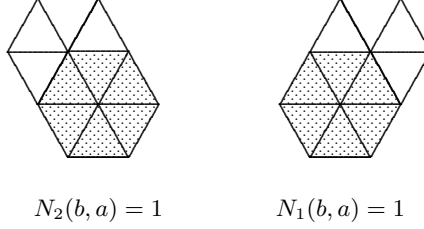
FIGURE 13. A reduced tile of type i .

FIGURE 14. New transition matrices.

they lie in the convex hull of the region consisting of the shaded hexagon and the two chosen triangles.

Lemma 6.3. $C(\Gamma) = \text{coker } (I - N_1, I - N_2)$.

Proof. This follows by applying successively parts (i) and (ii) of [12, Lemma 6.1] as indicated in Figure 15. The resulting $\{0, 1\}$ matrices are precisely N_1 and N_2 . \square

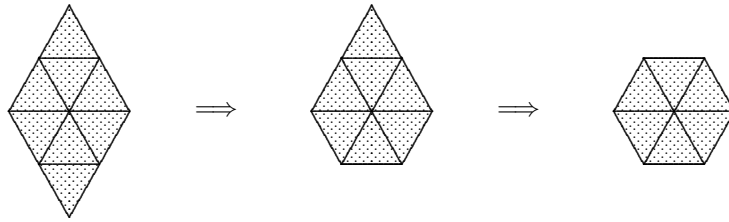


FIGURE 15. Two-step reduction of a tile to a hexagon

Remark 6.4. The $\{0, 1\}$ -matrices N_1, N_2 cannot be used to define the rank 2 Cuntz-Krieger algebra, although they can be used to compute its K-theory. This is because the analogue of [11, Lemma 7.4] would fail.

Proposition 6.5. For $i = 1, 2$, we have $K_i(\mathfrak{A}_\Gamma) = \mathbb{Z}_{q^2-1}$.

Proof. Lemma 6.3 implies that, as a finitely presented abelian group, we have

$$(7) \quad C(\Gamma) = \langle \Lambda \mid a = \sum_{b \in \Lambda} N_j(b, a)b, \quad a \in \Lambda, j = 1, 2 \rangle.$$

Fix $i \in \mathbb{Z}_3$ and $\xi \in X_i$, so that $a = \Gamma\xi \in \Lambda_i$. Then $N_1(b, a) = 1$ if and only if $b = \Gamma\eta$ where $\eta \in X_{i+1}$, $\eta_i = \xi_i$ and $\bar{\eta}_{i+1} = \bar{\xi}_{i+2}$. Since Γ acts regularly on the set of chambers,

$$(8) \quad a = \sum_{\substack{\eta \in X_{i+1} \\ \eta_i = \xi_i \\ \bar{\eta}_{i+1} = \bar{\xi}_{i+2}}} \Gamma\eta.$$

Similarly

$$(9) \quad a = \sum_{\substack{\zeta \in X_{i+2} \\ \zeta_i = \xi_i \\ \bar{\zeta}_{i+2} = \bar{\xi}_{i+1}}} \Gamma\zeta.$$

Equation (8) implies that a depends only on the chambers ξ_i and $\bar{\xi}_{i+2}$ of ξ . Equation (9) implies that a depends only on the chambers ξ_i and $\bar{\xi}_{i+1}$ of ξ . Therefore a depends only on ξ_i . More precisely, if $\vartheta \in X_i$ is any hexagon with $\vartheta_i = \xi_i$, then $\Gamma\vartheta = \Gamma\xi$ in $C(\Gamma)$. To see this, choose a hexagon $\varphi \in X_i$ with $\varphi_i = \xi_i$, $v_{02}(\varphi) = v_{02}(\vartheta)$ and $v_{20}(\varphi) = v_{20}(\xi)$. Then $\Gamma\varphi = \Gamma\xi$, by (8), and $\Gamma\varphi = \Gamma\vartheta$, by (9), so that $\Gamma\vartheta = \Gamma\xi$, as claimed. However, Γ acts transitively on chambers. Therefore all elements of Λ_i are equal, to a_i , say. Thus

$$\begin{aligned} C(\Gamma) &= \langle a_0, a_1, a_2 \mid a_i = q^2 a_{i+1}, a_i = q^2 a_{i-1}, i \in \mathbb{Z}_3 \rangle \\ &= \langle a_0 \mid a_0 = q^4 a_0, a_0 = q^6 a_0 \rangle \\ &= \langle a_0 \mid a_0 = q^2 a_0 \rangle = \mathbb{Z}_{q^2-1}. \end{aligned}$$

In particular, since $C(\Gamma)$ is finite, it follows from Theorem 6.1 that for $i = 0, 1$,

$$(10) \quad K_i(\mathfrak{A}_\Gamma) = \mathbb{Z}_{q^2-1}.$$

□

Recall that the classical Cuntz algebra \mathcal{O}_n is generated by n isometries whose range projections sum to the identity operator.

Corollary 6.6. *The algebra \mathfrak{A}_Γ is stably isomorphic to $\mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$.*

Proof. $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n) = 0$ [3]. The Künneth Theorem for tensor products [2, Theorem 23.1.3] shows that $K_i(\mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}) = \mathbb{Z}_{q^2-1}$, $i = 0, 1$. Since the algebras involved are p.i.s.u.n. and satisfy the U.C.T. [11, Remark 6.5], the result follows from the classification theorem for such algebras [6]. □

Remark 6.7. In order to determine the isomorphism class of $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$ we need to use that fact that it is classified (in the class of p.i.s.u.n. C^* -algebras satisfying the U.C.T.) by the invariants $(K_0(\mathfrak{A}_\Gamma), [\mathbf{1}], K_1(\mathfrak{A}_\Gamma))$ as abelian groups with distinguished element in K_0 [6]. It turns out that \mathfrak{A}_Γ is not isomorphic to $\mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$, but to $M_r \otimes \mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$, for some r . In order to determine r , we use the fact that, for any C^* -algebra \mathfrak{C} , the identity matrix $I_{M_r \otimes \mathfrak{C}}$ is a direct sum of r copies of $I_{\mathfrak{C}}$, so that the class of the identity element in K_0 satisfies $[\mathbf{1}_{M_r \otimes \mathfrak{C}}] = r \cdot [\mathbf{1}_{\mathfrak{C}}]$. Now, for any decorating set E , the C^* -algebra \mathcal{A}_E is stably isomorphic to \mathcal{A}_D [11, Corollary 5.15] and so $K_i(\mathcal{A}_E) = \mathbb{Z}_{q^2-1}$. Since $q^2 \cdot [\mathbf{1}] = [\mathbf{1}]$ in K_0 , it follows from the classification theorem that $M_{q^2} \otimes \mathcal{A}_E \cong \mathcal{A}_E$.

Lemma 6.8. *Let $k = (q+1)(q^2 + q + 1)$, the number of chambers of Δ which contain the vertex O .*

- *The decoration $\delta : D \rightarrow A$, defined by $\delta(\tau) = \Gamma\tau$, is equivalent to the decoration $\delta'_0 : A_0 \times \{1, 2, \dots, k\} \rightarrow A$, defined by $\delta'_0((a_0, i)) = a_0$.*

- $\mathcal{A}_D \cong \mathcal{A}_{A_0 \times \{1,2,\dots,k\}}$.

Proof. Let c_1, \dots, c_k be the chambers in Δ based at O . For each i , let D_i denote the set of tiles $\tau \in D$ which have initial chamber c_i . Since Γ acts freely and transitively on the set of chambers of Δ , the restriction δ_i of δ to D_i is a bijection onto A_0 . Define $\eta : D \rightarrow A_0 \times \{1, 2, \dots, k\}$ by $\eta(\tau) = (\delta_i(\tau), i)$ for $\tau \in D_i$, $i = 1, 2, \dots, k$. Then η is bijective and $\delta'_0 \eta = \delta$. Thus δ and δ'_0 are equivalent decorations. \square

The fact that $\mathcal{A}_D \cong \mathcal{A}_{A_0 \times \{1,2,\dots,k\}}$ follows immediately from Remark 4.3. \square

Lemma 6.9. $\mathfrak{A}_\Gamma \cong M_{3(q+1)} \otimes \mathcal{A}_{A_0}$.

Proof. By Proposition 5.3, Lemma 6.8, Lemma 4.4, and Remark 6.7,

$$\mathfrak{A}_\Gamma \cong \mathcal{A}_D \cong \mathcal{A}_{A_0 \times \{1,2,\dots,k\}} \cong M_k \otimes \mathcal{A}_{A_0} \cong M_{3(q+1)} \otimes \mathcal{A}_{A_0},$$

since $k \equiv 3(q+1) \pmod{q^2-1}$. \square

Lemma 6.10. $\mathcal{A}_A \cong M_3 \otimes \mathcal{A}_{A_0}$.

Proof. For $i = 0, 1, 2$, let $e_i = \sum_{a \in A_i} s_{a,a}$, so that $e_0 + e_1 + e_2 = \mathbf{1}$. It is an immediate consequence of relations (5b), (5d) that the generators of \mathcal{A}_A satisfy

$$e_i s_{u,v} e_i = \begin{cases} s_{u,v} & \text{if } o(u), o(v) \in A_i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows immediately from the uniqueness statement of Theorem 4.2 that

$$(11) \quad e_i \mathcal{A}_A e_i \cong \mathcal{A}_{A_i}.$$

The relations (5) show that if $M_j(b, a) = 1$ then the partial isometry $s_{b,ab}$ has initial projection $s_{ab,ab}$ and final projection $s_{b,b}$. Thus $[s_{ab,ab}] = [s_{b,b}]$. It follows from (5c) and Lemma 5.4 that, if $a \in A_i$ then $[s_{a,a}] = \sum_{b \in A_{i+1}} M_1(b, a) [s_{b,b}]$. Therefore

$$\begin{aligned} [e_i] &= \sum_{a \in A_i} [s_{a,a}] = \sum_{a \in A_i} \sum_{b \in A_{i+1}} M_1(b, a) [s_{b,b}] \\ &= \sum_{b \in A_{i+1}} \left(\sum_{a \in A_i} M_1(b, a) \right) [s_{b,b}] = q^2 [e_{i+1}] \end{aligned}$$

since $\sum_{a \in A_i} M_1(b, a) = q^2$ for each $b \in A_{i+1}$, by Lemma 5.4. Similarly (replacing M_1 by M_2), $[e_i] = q^2 [e_{i-1}]$. It follows that $[e_i] = q^4 [e_i] = q^6 [e_i]$. Therefore $[e_i] = q^2 [e_i]$ and $[e_i] = [e_{i+1}]$, for all $i \in \mathbb{Z}/3\mathbb{Z}$. Thus $[e_1] = [e_2] = [e_3]$ in $K_0(\mathcal{A}_A)$. In view of the description of the K-theory of purely infinite C^* -algebras in [3, Section 1], this implies that the projections e_0, e_1, e_2 are Murray-von Neumann equivalent in \mathcal{A}_A . It follows from (11) that $\mathcal{A}_{A_0} \cong \mathcal{A}_{A_1} \cong \mathcal{A}_{A_2}$ and that $\mathcal{A}_A \cong M_3 \otimes \mathcal{A}_{A_0}$. \square

Lemma 6.11. $(K_0(\mathcal{A}_A), [1]) \cong (\mathbb{Z}_{q^2-1}, 3q)$.

Proof. Since \mathcal{A}_A is stably isomorphic to \mathfrak{A}_Γ , it follows from Proposition 6.5 that $K_0(\mathcal{A}_A) \cong K_0(\mathfrak{A}_\Gamma) \cong C(\Gamma) \cong \mathbb{Z}_{q^2-1}$. By the proof of [12, Proposition 8.3], the isomorphism between $K_0(\mathcal{A}_A)$ and the group

$$(12) \quad \langle A \mid a = \sum_{b \in A} M_j(b, a)b, \quad a \in A, j = 1, 2 \rangle.$$

maps $[1]$ to the element $\sum_{a \in A} a$. However $\#A = 3q^5 \equiv 3q \pmod{q^2-1}$. The result follows. \square

Theorem 6.12. $(K_0(\mathfrak{A}_\Gamma), [1], K_1(\mathfrak{A}_\Gamma)) \cong (\mathbb{Z}_{q^2-1}, 3(q+1), \mathbb{Z}_{q^2-1})$.

Proof. By Lemmas 6.9 and 6.10,

$$\mathfrak{A}_\Gamma \cong M_{3(q+1)} \otimes \mathcal{A}_{A_0} \cong M_{q+1} \otimes M_3 \otimes \mathcal{A}_{A_0} \cong M_{q+1} \otimes \mathcal{A}_A.$$

It follows from Lemma 6.11 that $[\mathbf{1}_\Gamma] = (q+1) \cdot 3q = 3(q+1)$ in \mathbb{Z}_{q^2-1} . \square

In view of the classification theorem [6] and Remark 6.7, this completes the proof of Theorem 1.1.

Remark 6.13. The order of $[\mathbf{1}]$ in $K_0(\mathfrak{A}_\Gamma)$ is

$$\begin{aligned} \frac{q^2 - 1}{(q^2 - 1, 3(q+1))} &= \frac{q - 1}{(q - 1, 3)} \\ &= \begin{cases} q - 1 & \text{if } q \not\equiv 1 \pmod{3}, \\ \frac{q-1}{3} & \text{if } q \equiv 1 \pmod{3}. \end{cases} \end{aligned}$$

Compare this with the conjecture in [12, Remark 8.4].

There are other ways of expressing \mathfrak{A}_Γ in terms of rank-1 Cuntz-Krieger algebras, in addition to that of Theorem 1.1. Here is an example. Let $\mathcal{A}_0 = \mathcal{O}_A$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then \mathcal{A}_0 is a simple rank-1 Cuntz-Krieger algebra such that $K_*(\mathcal{A}_0) = (\mathbb{Z}, \mathbb{Z})$.

Proposition 6.14. *Under the hypotheses of Theorem 1.1, \mathfrak{A}_Γ is stably isomorphic to $\mathcal{A}_0 \otimes \mathcal{O}_{q^2}$.*

Proof. The Künneth Theorem for tensor products [2, Theorem 23.1.3] shows that $K_*(\mathcal{A}_1 \otimes \mathcal{A}_2) = (\mathbb{Z}_{q^2-1}, \mathbb{Z}_{q^2-1})$. Since the algebras involved are all p.i.s.u.n. and satisfy the U.C.T., the result follows from the Classification Theorem [1]. \square

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