# ABELIAN SUBALGEBRAS OF VON NEUMANN ALGEBRAS FROM FLAT TORI IN LOCALLY SYMMETRIC SPACES

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ABSTRACT. Consider a compact locally symmetric space M of rank r, with fundamental group  $\Gamma$ . The von Neumann algebra  $\operatorname{VN}(\Gamma)$  is the convolution algebra of functions  $f \in \ell_2(\Gamma)$  which act by left convolution on  $\ell_2(\Gamma)$ . Let  $T^r$  be a totally geodesic flat torus of dimension r in M and let  $\Gamma_0 \cong \mathbb{Z}^r$  be the image of the fundamental group of  $T^r$  in  $\Gamma$ . Then  $\operatorname{VN}(\Gamma_0)$  is a maximal abelian  $\star$ -subalgebra of  $\operatorname{VN}(\Gamma)$  and its unitary normalizer is as small as possible. If M has constant negative curvature then the Pukánszky invariant of  $\operatorname{VN}(\Gamma_0)$  is  $\infty$ .

## 1. INTRODUCTION

If  $\Gamma$  is a group, then the von Neumann algebra  $VN(\Gamma)$  is the convolution algebra

$$VN(\Gamma) = \{ f \in \ell^2(\Gamma) : f \star \ell^2(\Gamma) \subseteq \ell^2(\Gamma) \}.$$

It is well known that if  $\Gamma$  is an infinite conjugacy class [ICC] group then  $VN(\Gamma)$  is a factor of type II<sub>1</sub>. If  $\Gamma_0$  is a subgroup of  $\Gamma$ , then  $VN(\Gamma_0)$  embeds as a subalgebra of  $VN(\Gamma)$  via  $f \mapsto \overline{f}$ , where

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

This article is concerned with examples where  $\Gamma_0$  is an abelian subgroup of  $\Gamma$  and  $VN(\Gamma_0)$  is a maximal abelian  $\star$ -subalgebra (masa) of  $VN(\Gamma)$ .

If  $\mathcal{A}$  is a von Neumann subalgebra of a von Neumann algebra  $\mathcal{M}$ then the *unitary normalizer*  $N(\mathcal{A})$  is the set of unitaries u in  $\mathcal{M}$  such that  $u\mathcal{A}u^{-1} = \mathcal{A}$ . The subalgebra  $\mathcal{A}$  is said to be *singular* if  $N(\mathcal{A}) \subseteq \mathcal{A}$ , so that the only normalizing unitaries already belong to  $\mathcal{A}$ .

Let  $\Gamma$  be a torsion free cocompact lattice in a semisimple Lie group G of rank r with no centre and no compact factors. Consider the Riemannian symmetric space X = G/K and the compact locally symmetric space  $M = \Gamma \setminus X$ . A flat in X is an isometrically embedded euclidean space in X. The rank r of X is the dimension of a maximal flat in X.

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Suppose that  $T^r$  is a totally geodesic flat torus of dimension r in M. Let  $\Gamma_0 \cong \mathbb{Z}^r$  be the image of the fundamental group  $\pi(T^r)$  under the natural monomorphism from  $\pi(T^r)$  into  $\Gamma = \pi(M)$ . We show that  $VN(\Gamma_0)$  is a singular masa of  $VN(\Gamma)$ . In fact we prove two stronger results : Theorems 1.1 and 3.5 below. Fix  $g_1 \in \Gamma_0$  and let  $\Gamma_1 \cong \mathbb{Z}$  be the subgroup of  $\Gamma_0$  generated by  $g_1$ . Then we have inclusions

$$\operatorname{VN}(\Gamma_1) \subseteq \operatorname{VN}(\Gamma_0) \subseteq \operatorname{VN}(\Gamma).$$

The result below implies that if  $g_1$  is the homotopy class of a regular geodesic (as defined subsequently) then  $VN(\Gamma_0)$  is the *unique* masa of  $VN(\Gamma)$  containing  $VN(\Gamma_1)$ .

**Theorem 1.1.** Let  $g_1 \in \Gamma_0$  be the class of a regular closed geodesic cin  $T^r$ , and let  $\Gamma_1 \cong \mathbb{Z}$  be the subgroup of  $\Gamma_0$  generated by  $g_1$ . Let ube a unitary operator in  $\mathcal{M}$  such that  $u \mathrm{VN}(\Gamma_1) u^{-1} \subseteq \mathrm{VN}(\Gamma_0)$ . Then  $u \in \mathrm{VN}(\Gamma_0)$ .

The second main result (Theorem 3.5) implies that  $VN(\Gamma_0)$  is a strongly singular masa in the sense of [SS]. This improves a result of [RSS, Theorem 4.9], which proved strong singularity under the additional hypothesis that the diameter of  $T^r$  is small. The two new ideas leading to this improvement are the use of the Amenable Subgroup Theorem (Lemma 2.1) and the replacement of the Furstenberg Boundary by the Tits Boundary in the subsequent arguments.

If M has constant negative curvature then it is proved in Theorem 4.6 that the Pukánszky invariant of  $VN(\Gamma_0)$  is  $\infty$ .

### 2. Preliminaries

We first recall some concepts which are needed for the statements and proofs of the results. Let G be a semisimple Lie group with no centre and no compact factors. Let X = G/K be the associated symmetric space, where K is a maximal compact subgroup of G. Then X is a contractible space of nonpositive curvature. A geodesic L in X is called *regular* if it lies in only one maximal flat; it is called *singular* if it is not regular. Let F be a maximal flat in X and let  $p \in F$ . Let  $S_p$  denote the union of all the singular geodesics through p. Then  $F - S_p$  has finitely many connected components, called *Weyl chambers* with origin p.

If A, B are subsets of X, and  $\delta > 0$ , then the notation  $A \subset B$  means that  $d(a, B) \leq \delta$ , for all  $a \in A$ . Define the Hausdorff distance between A and B to be

(1) 
$$\operatorname{hd}(A, B) = \inf \{ \delta \le \infty : A \subset_{\delta} B \text{ and } B \subset_{\delta} A \}.$$

Any complete geodesic L in X is the union of two geodesic rays which intersect at their common origin. Define an equivalence relation  $\sim$  on the set of geodesic rays in X by

(2) 
$$L_1 \sim L_2 \iff \operatorname{hd}(L_1, L_2) < \infty$$
.

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The sphere at infinity  $X(\infty)$  is the set of equivalence classes of geodesic rays in X [GJT, Chapter III]. Denote by  $L(\infty)$  the class in  $X(\infty)$ of a geodesic ray L. The set  $X(\infty)$  may be given the structure of a spherical building whose maximal simplices are called Weyl chambers at infinity, and there is a natural action of G on  $X(\infty)$ . The parabolic subgroups of G are the stabilizers  $G_z = \{x \in G : x(z) = z\}$ , for some  $z \in X(\infty)$  [GJT, Proposition 3.8]. Moreover,  $G_z$  is a minimal parabolic subgroup of G if and only if  $z = L(\infty)$ , where L is a geodesic ray in a Weyl chamber [BGS, pp 248–9]. Two such minimal parabolic subgroups  $G_{z_1}$ ,  $G_{z_2}$  coincide if and only if  $z_1, z_2$  belong to the same Weyl chamber at infinity [GJT, Proposition 3.16].

If F is a maximal flat in X, then the restriction of the equivalence relation  $\sim$  to rays in F allows one to define the sphere at infinity  $F(\infty)$ . There is a natural embedding of  $F(\infty)$  into  $X(\infty)$ , and it is convenient to identify  $F(\infty)$  with its image in  $X(\infty)$ .

Now let  $\Gamma$  be a torsion free cocompact lattice in G. Then  $\Gamma$  acts freely on the symmetric space X = G/K and the quotient manifold M = $\Gamma \setminus X$  has universal covering space X. The manifold M is a compact locally symmetric space of nonpositive curvature, with fundamental group  $\Gamma$ , and  $\Gamma$  acts freely on X.

Let  $T^r \subset M$  be a totally geodesic embedding of a flat r-torus in M. Choose and fix a point  $\xi \in T^r$ . Consider the fundamental groups  $\Gamma = \pi(M,\xi)$  and  $\Gamma_0 = \pi(T^r,\xi) \cong \mathbb{Z}^r$ . Since no geodesic loop in M is null-homotopic, the inclusion  $i: T^r \to M$  induces an injective homomorphism  $i_*: \Gamma_0 \to \Gamma$ . We identify  $\Gamma_0$  with its image in  $\Gamma$ . There is a maximal flat  $F_0 \cong \mathbb{R}^r$  in X such that  $\Gamma_0$  acts cocompactly by translations upon  $F_0$ , and  $p(F_0) = T^r$  [BH, Theorem II.7.1].

The flat  $F_0$  is the unique  $\Gamma_0$ -invariant flat in X. For if  $F_1$  is another such, then since  $\Gamma_0$  acts isometrically and the action on  $F_0$  is cocompact, we have  $F_0 \subset_{\delta} F_1$ , for some  $\delta > 0$ . Therefore  $F_0 = F_1$ , by [Mos, Lemma 7.3 (iv)], applied to a maximal flat containing  $F_1$ .

Choose  $\tilde{\xi} \in F_0$  such that  $p(\tilde{\xi}) = \xi$ . For any element  $x \in \Gamma$ , there is a unique geodesic loop based at  $\xi$  which represents x. This is the loop c of shortest length in the class x and it is the projection of the geodesic segment  $[\tilde{\xi}, x\tilde{\xi}]$  in X.

The following technical lemma will play a crucial role later, in our improvement to [RSS, Theorem 4.9].

**Lemma 2.1.** Let  $z \in F_0(\infty)$  lie in a Weyl chamber at infinity. Then

$$G_z \cap \Gamma = \Gamma_0.$$

*Proof.* The group  $G_z = \{x \in G : x(z) = z\}$  is a minimal parabolic subgroup of G, and so has a cocompact solvable normal subgroup. Therefore  $G_z$  is amenable, as is the discrete subgroup  $\Gamma_z = G_z \cap \Gamma$ .

Also  $\Gamma_z \supseteq \Gamma_0$ , since  $\Gamma_0$  acts by translation upon  $F_0$  and so stabilizes each point of  $F_0(\infty)$ .

By the Amenable Subgroup Theorem [AB, Corollary B], there is a  $\Gamma_z$ -invariant flat  $F_z$  in X. Since  $\Gamma_z \supseteq \Gamma_0$ ,  $F_z$  is  $\Gamma_0$ -invariant and so  $F_z = F_0$ , by the remarks preceding this lemma. Thus  $\Gamma_z F_0 = F_0$ . If  $x \in \Gamma_z$  then  $x\tilde{\xi} \in F_0$ , since  $\tilde{\xi} \in F_0$ . Therefore the geodesic segment

 $[\tilde{\xi}, x\tilde{\xi}]$  in  $F_0$  projects to a closed geodesic in  $T^r$ , whose class in  $\Gamma$  is precisely x. Therefore  $x \in \Gamma_0$ .

## 3. SINGULARITY RESULTS FOR ABELIAN SUBALGEBRAS

The terminology and notation introduced in the previous section will be used without further comment. A *regular* geodesic in M is, by definition, the image of a regular geodesic under the covering projection  $p: X \to M$ . It follows from [Mos, §11] that  $T^r$  contains a closed regular geodesic. We now prove Theorem 1.1, which we restate here, for convenience.

**Theorem 3.1.** Let  $g_1 \in \Gamma_0$  be the class of a regular closed geodesic cin  $T^r$ , and let  $\Gamma_1 \cong \mathbb{Z}$  be the subgroup of  $\Gamma_0$  generated by  $g_1$ . Let ube a unitary operator in  $\mathcal{M}$  such that  $u \mathrm{VN}(\Gamma_1) u^{-1} \subseteq \mathrm{VN}(\Gamma_0)$ . Then  $u \in \mathrm{VN}(\Gamma_0)$ .

*Proof.* Suppose that  $x_0 \in \text{supp } u$ . We must prove that  $x_0 \in \Gamma_0$ .

Since  $u \in \ell^2(\Gamma)$ , there are only a finite number of cosets  $\Gamma_0 y$ , with  $y \in \Gamma$ , such that  $||u|_{\Gamma_0 y}||_2 \geq |u(x_0)|$ . Call these cosets  $\Gamma_0 y_1, \ldots, \Gamma_0 y_n$ . We claim that

(3) 
$$x_0\Gamma_1 \subset \Gamma_0 y_1 \cup \cdots \cup \Gamma_0 y_n.$$

To prove this, note that if  $z \in \Gamma_1$  then  $u \star \delta_z \star u^{-1} = f$  is a unitary operator which lies in VN( $\Gamma_0$ ), by hypothesis. Therefore

$$|u(x_0)| = |(u \star \delta_z)(x_0 z)|$$
  
=  $|(f \star u)(x_0 z)|$   
=  $\left|\sum_{t \in \Gamma_0} f(t)u(t^{-1}x_0 z)\right|$  (since supp  $f \subseteq \Gamma_0$ )  
 $\leq ||u|_{\Gamma_0 x_0 z}||_2$  (since  $f$  is unitary).

This shows that  $x_0 z \in \Gamma_0 y_1 \cup \cdots \cup \Gamma_0 y_n$ , which proves (3).

We now show that (3) implies that  $x_0 \in \Gamma_0$ . Lift c to a regular geodesic L in X through a point  $\tilde{\xi} \in X$ . Regularity means that Llies in a *unique* maximal flat  $F_0$  and  $p(F_0) = T^r$ . The diagram below illustrates the case  $X = SL_3(\mathbb{R})/SO_3(\mathbb{R})$ , where r = 2 and there are six Weyl chambers in the flat  $F_0$  with origin  $\tilde{\xi}$ .

Let  $P = [\tilde{\xi}, g_1 \tilde{\xi}] \subseteq F_0$ , so that  $\Gamma_1 P = L$ , since  $g_1$  acts on L by translation. Let  $\delta = \max\{d(y_j p, p) : 1 \leq j \leq n, p \in P\}$ . Then for



 $1 \leq j \leq n$ , we have  $y_j P \subset P$ , and so

(4) 
$$\Gamma_0 y_j P \underset{\delta}{\subset} \Gamma_0 P \subset F_0.$$

It follows from (3) that

(5) 
$$x_0 L = x_0 \Gamma_1 P \underset{\delta}{\subset} F_0.$$

Since  $x_0L$  is a regular geodesic, [Mos, Lemma 7.3(iii)] implies that  $d(x_0L, F_0) = 0$ . Consequently  $x_0L \subset F_0$ , by [Mos, Lemma 3.7]. In particular,  $x_0\tilde{\xi} \in F_0$ . Therefore the geodesic segment  $[\tilde{\xi}, x_0\tilde{\xi}]$  projects to a closed geodesic in  $T^r$  whose class in  $\Gamma$  is precisely  $x_0$ . Hence  $x_0 \in \Gamma_0$ .

**Remark 3.2.** Theorem 3.1 implies that  $VN(\Gamma_0)$  is a singular masa of  $VN(\Gamma)$  and that it is also the unique masa of  $VN(\Gamma)$  containing  $VN(\Gamma_1)$ . Since closed geodesics are dense in the set of all geodesics of  $T^r$ , we can choose regular geodesics  $c_1, c_2, \ldots, c_r$  in  $T^r$  which lift to regular geodesics in linearly independent directions in  $F_0$ . Applying Theorem 3.1 to each of the geodesics  $c_j$  shows that  $VN(\Gamma)$  contains a masa  $\mathcal{A} = VN(\Gamma_0)$  with the following property :

 $\mathcal{A}$  contains abelian subalgebras  $\mathcal{B}_j$ ,  $1 \leq j \leq r$  such that

- (a)  $\mathcal{A}$  is the *unique* mass containing  $\mathcal{B}_j$ ;
- (b)  $\mathcal{A}$  is generated by  $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_r$ ;
- (c)  $\mathcal{B}_i, \mathcal{B}_j$  are orthogonal for  $i \neq j$ , in the sense that  $Tr(b_i b_j) = 0$ whenever  $Tr(b_i) = Tr(b_j) = 0$  [Po1, Definition 2.2].

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This construction shows how one can see something of the rank of the locally symmetric space M in the group von Neumann algebra of the fundamental group  $\Gamma = \pi(M)$ . This is of interest in connection with a conjecture of A. Connes.

**Rigidity Conjecture**. If ICC groups  $\Gamma_1, \Gamma_2$  have Property (T) of Kazhdan, then

$$VN(\Gamma_1) \cong VN(\Gamma_2) \Rightarrow \Gamma_1 \cong \Gamma_2.$$

In our setup, one consequence of the truth of Connes' conjecture would be that the rank of M is determined by  $VN(\Gamma)$ .

Now we define a relative version of the notion of a strongly singular masa defined in [SS]. Let  $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{M}$  be von Neumann subalgebras of a type II<sub>1</sub> factor  $\mathcal{M}$ , and let  $\mathbb{E}_{\mathcal{N}}$  denote the unique trace preserving conditional expectation onto any von Neumann subalgebra  $\mathcal{N}$  of  $\mathcal{M}$ . Say that  $\mathcal{A} \subseteq \mathcal{C}$  is a strongly singular pair of von Neumann subalgebras of  $\mathcal{M}$  if for all von Neumann subalgebras  $\mathcal{B}$  with  $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$  the inequality

(6) 
$$\|\mathbb{E}_{\mathcal{B}} - \mathbb{E}_{u\mathcal{B}u^*}\|_{\infty,2} \ge \|(I - \mathbb{E}_{\mathcal{C}})(u)\|_2$$

holds for all unitaries  $u \in \mathcal{M}$ . [As in [RSS], the notation  $||T||_{\infty,2}$  means that the norm of the linear map T is taken relative to operator norm on its domain and the  $\ell^2$  norm on its range. If  $\mathcal{A} = \mathcal{C}$  then this reduces to the definition of a strongly singular subalgebra given in [SS].

The next two results are mild generalizations of [RSS, Lemma 2.1] and [RSS, Lemma 4.1]. The proofs are included for completeness.

**Lemma 3.3.** Let  $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{M}$  be von Neumann subalgebras of a type II<sub>1</sub> factor  $\mathcal{M}$ . Suppose that, given  $\varepsilon > 0$  and a unitary operator  $u \in \mathcal{M}$ , there exists a unitary  $v \in \mathcal{A}$ , such that

(7) 
$$\|\mathbb{E}_{\mathcal{C}}(u^*vu) - \mathbb{E}_{\mathcal{C}}(u^*)v\mathbb{E}_{\mathcal{C}}(u)\|_2 < \varepsilon.$$

Then  $\mathcal{A} \subseteq \mathcal{C}$  is a strongly singular pair. That is, (6) holds, whenever  $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ .

*Proof.* We have

(8) 
$$\|\mathbb{E}_{\mathcal{B}} - \mathbb{E}_{u\mathcal{B}u^*}\|_{\infty,2}^2 \ge \|v - \mathbb{E}_{u\mathcal{B}u^*}(v)\|_2^2 \qquad \text{since } v \in \mathcal{B}$$
$$= \|v - u\mathbb{E}_{\mathcal{B}}(u^*vu)u^*\|_2^2$$
$$= \|u^*vu - \mathbb{E}_{\mathcal{B}}(u^*vu)\|_2^2 \qquad \text{by orthogonality}$$
$$\ge 1 - \|\mathbb{E}_{\mathcal{C}}(u^*vu)\|_2^2 \qquad \text{by orthogonality}$$
$$\ge 1 - (\|\mathbb{E}_{\mathcal{C}}(u^*)v\mathbb{E}_{\mathcal{C}}(u)\|_2 + \varepsilon)^2 \qquad \text{by (7)}$$
$$\ge 1 - (\|\mathbb{E}_{\mathcal{C}}(u)\|_2 + \varepsilon)^2 \qquad \text{by (7)}$$
$$\ge 1 - (\|\mathbb{E}_{\mathcal{C}}(u)\|_2 + \varepsilon)^2 = \|(I - \mathbb{E}_{\mathcal{C}})(u)\|_2^2 - \varepsilon^2 - 2\varepsilon \|\mathbb{E}_{\mathcal{C}}(u)\|_2.$$

Since  $\varepsilon > 0$  was arbitrary, the result follows.

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**Lemma 3.4.** Let  $\Gamma_1 < \Gamma_0 < \Gamma$  be subgroups of a discrete I.C.C. group. The following condition implies that  $VN(\Gamma_1) \subseteq VN(\Gamma_0)$  is a strongly singular pair.

If  $x_1, \ldots, x_m \in \Gamma$  and

(10) 
$$\Gamma_1 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,,$$

then  $x_i \in \Gamma_0$  for some *i*.

*Proof.* The condition in question is equivalent to the following: If  $x_1, \ldots, x_n, y_1, \ldots, y_n \in \Gamma \setminus \Gamma_0$ , then there exists  $\gamma_0 \in \Gamma_1$  such that

(11)  $x_i \gamma_0 y_j \notin \Gamma_0, \quad 1 \le i, j \le n.$ 

To see this replace  $x_i$  by  $x_i^{-1}$ , replace  $y_j$  by  $y_j^{-1}$  and replace each of the sets  $\{x_1, \ldots, x_n\}$ ,  $\{y_1, \ldots, y_n\}$  by their union, which is renamed  $\{x_1, \ldots, x_m\}$ .

Let  $\mathcal{A} = \text{VN}(\Gamma_1)$  and  $\mathcal{C} = \text{VN}(\Gamma_0)$ . Given  $\varepsilon > 0$  and a unitary operator  $u \in \text{VN}(\Gamma)$ , we must show that there exists a unitary  $v \in \mathcal{A}$ , such that (7) is satisfied. To do this, approximate u by a finite linear combination of group elements  $y_1, \ldots, y_n, y_{n+1}, \ldots, y_p$ , where  $y_1, \ldots, y_n \notin$  $\Gamma_0$  and  $y_{n+1}, \ldots, y_p \in \Gamma_0$ . Let  $x_i = y_i^{-1}, 1 \leq i \leq p$  and choose  $\gamma_0 \in \Gamma_1$ satisfying (11). Now

$$\mathbb{E}_{\mathcal{C}}(x_i\gamma_0 y_j) = \mathbb{E}_{\mathcal{C}}(x_i)\gamma_0 \mathbb{E}_{\mathcal{C}}(y_j), \qquad 1 \le i, j \le n,$$

both sides being zero if  $i \leq n$  or  $j \leq n$ , since  $x_i \gamma_0 y_j$  is then orthogonal to  $\mathcal{B}$ . The equation (7) follows by taking a close enough approximation.

The next result implies that  $VN(\Gamma_0)$  is a strongly singular masa of  $VN(\Gamma)$  in the sense of [SS]. It improves [RSS, Theorem 4.9], by removing a superfluous hypothesis on the diameter of the embedded torus  $T^r$ .

**Theorem 3.5.** Let  $g_1$  be the class of a regular closed geodesic c in  $T^r$ , and let  $\Gamma_1 \cong \mathbb{Z}$  be the subgroup of  $\Gamma_0 = \pi(T^r)$  generated by  $g_1$ . Then  $VN(\Gamma_1) \subseteq VN(\Gamma_0)$  is a strongly singular pair.

**Remark 3.6.** Theorem 3.5 is clearly closely related to Theorem 3.1, but neither result appears to contain the other.

*Proof.* Lift c to a regular geodesic L in X through  $\tilde{\xi}$ , where  $p(\tilde{\xi}) = \xi$ and  $L = L^+ \cup L^-$  is a union of two geodesic rays with common origin  $\tilde{\xi}$ . Regularity means that L lies in a *unique* maximal flat  $F_0$  and  $p(F_0) = T^r$ .

Suppose that (10) holds. That is, we have elements  $x_1, \ldots, x_m \in \Gamma$  such that

(12) 
$$\Gamma_1 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,.$$

Now  $g_1$  acts on L by translation. Let  $P = [\tilde{\xi}, g_1 \tilde{\xi}]$ , so that  $L = \Gamma_1 P$ . Let  $\delta = \max\{d(x_j p, p) : 1 \leq j \leq n, p \in P\}$ . For  $1 \leq j \leq n$ , this implies that  $x_j P \subset P$  and so

(13) 
$$\Gamma_0 x_j P \underset{\delta}{\subset} \Gamma_0 P \subset F_0$$

Hence, for each i, j, we have  $x_i \Gamma_0 x_j P \subset x_i F_0$ . It follows from (12) that

(14) 
$$L = \Gamma_1 P \underset{\delta}{\subset} x_1 F_0 \cup x_2 F_0 \cup \dots \cup x_m F_0.$$

Let  $z = L(\infty) \in F_0(\infty)$ . Now for  $1 \leq j \leq m$ , the element  $x_j g_1 x_j^{-1}$  acts by translation upon the maximal flat  $x_j F_0$  and hence preserves each boundary point of  $x_j F_0$ . Express each  $x_j F_0$  as a (finite) union of Weyl chambers  $W_{\alpha,j}$  with base vertex  $x_j \tilde{\xi}$ . Thus

(15) 
$$L^+ \subset_{\delta} \bigcup_{\alpha,j} W_{\alpha,j}$$

Now for each  $\alpha$ , j the function  $p \mapsto d(p, W_{\alpha,j})$  is convex on L by [Mos, Lemma 3.6]. According to (15), we have

(16) 
$$\min_{\alpha,j} d(p, W_{\alpha,j}) \le \delta$$

for all  $p \in L$ . This implies that for some  $\alpha, j$ , the function  $p \mapsto d(p, W_{\alpha,j})$  is monotonically decreasing on  $L^+$ . Choose such  $\alpha, j$ . Then

$$L^+ \subset_{\epsilon} W_{\alpha,j}$$

for some  $\epsilon > 0$ . By [Mos, Lemma 7.3(i)], there is a geodesic ray  $L' \in x_j F_0$  which is asymptotic to L. Thus  $z = L^+(\infty) = L'(\infty) \in x_j F_0(\infty)$ and  $x_j g_1 x_j^{-1} z = z$ .

Since L is a regular geodesic, z lies in a Weyl chamber at infinity and it follows from Lemma 2.1 that  $x_j g_1 x_j^{-1} = h_0 \in \Gamma_0$ . Since  $x_j g_1^n x_j^{-1} = h_0^n$ , we have  $x_j g_1^n = h_0^n x_j$ . Therefore

$$d(x_j g_1^n \tilde{\xi}, h_0^n \tilde{\xi}) = d(h_0^n x_j \tilde{\xi}, h_0^n \tilde{\xi}) = d(x_j \tilde{\xi}, \tilde{\xi}) \,.$$

Thus  $d(x_jL, F_0) < \infty$ , from which it follows that  $x_jL \subset F_0$ , by [Mos, Lemma 7.3(iii) and Lemma 3.7], applied to  $x_jL^{\pm}$ . In particular,  $x_j\tilde{\xi} \in F_0$ . Hence the geodesic segment  $[\tilde{\xi}, x_j\tilde{\xi}]$  projects to a closed geodesic in  $T^r$  whose class in  $\Gamma$  is precisely  $x_j$ . It follows that  $x_j \in \Gamma_0$ .  $\Box$ 

# 4. The Pukánszky invariant in constant negative Curvature

Let  $\Gamma_0$  be an abelian subgroup of a countable group  $\Gamma$  such that  $\mathcal{A} = VN(\Gamma_0)$  is a mass of  $\mathcal{M} = VN(\Gamma)$ . Recall that  $\mathcal{A}$  is the von Neumann subalgebra of  $B(\ell^2(\Gamma))$  defined by the left convolution operators

$$\lambda(f):\phi\mapsto f\star\phi$$

where  $f \in \ell^2(\Gamma_0)$  and  $f \star \ell^2(\Gamma) \subseteq \ell^2(\Gamma)$ . The algebra  $\mathcal{A}$  also acts on  $\ell^2(\Gamma)$  by right convolution

$$\rho(f):\phi\mapsto\phi\star f,$$

where  $f \in \mathcal{A}$ . Let  $\mathcal{A}^{opp}$  be the von Neumann subalgebra of  $B(\ell^2(\Gamma))$  defined by this right action of  $\mathcal{A}$ .

Let  $\mathcal{B}$  be the von Neumann algebra generated by  $\mathcal{A} \cup \mathcal{A}^{opp}$  and let p denote the orthogonal projection of  $\ell^2(\Gamma)$  onto the closed subspace generated by  $\mathcal{A}$ . Then p is in the centre of  $\mathcal{B}'$  and  $\mathcal{B}'p$  is abelian. The von Neumann algebra  $\mathcal{B}'(1-p)$  is of type I and may therefore be expressed as a direct sum  $\mathcal{B}_{n_1} \oplus \mathcal{B}_{n_2} \oplus \ldots$  of algebras  $\mathcal{B}_{n_i}$  of type  $I_{n_i}$ , where  $1 \leq n_1 < n_2 < \cdots \leq \infty$ . The Pukánszky invariant [Po2] is the set  $\{n_1, n_2, \ldots\}$ . It is an isomorphism invariant of the pair  $(\mathcal{A}, \mathcal{M})$ , since any automorphism of  $\mathcal{M}$  is implemented by a unitary in  $\mathcal{B}(\ell^2(\Gamma))$ . It has been shown [NS, Corollary 3.3] that all subsets of the natural numbers can be realized as the Pukánszky invariant of some masa of the hyperfinite  $II_1$  factor.

A subgroup  $\Gamma_0$  of a group  $\Gamma$  is malnormal if  $g^{-1}\Gamma_0 g \cap \Gamma_0 = \{1\}$  for all  $g \in \Gamma - \Gamma_0$ . Recall the following result from [RS, Proposition 3.6], which we shall apply in proving Theorem 4.6.

**Proposition 4.1.** Suppose that  $\Gamma_0$  is an abelian subgroup of a countable group  $\Gamma$  such that  $\mathcal{A} = \text{VN}(\Gamma_0)$  is a mass of  $\text{VN}(\Gamma)$ . If  $\Gamma_0$  is malnormal then the Pukánszky invariant of  $\mathcal{A}$  is  $n = \#(\Gamma_0 \setminus \Gamma / \Gamma_0 - \{\Gamma_0\})$ .

Return now to the setup of Theorem 3.1. Thus  $T^r$  is a totally geodesic flat torus of dimension r in the compact locally symmetric space M and  $\Gamma_0 \cong \mathbb{Z}^r$  is the image of the fundamental group  $\pi(T^r, \xi)$  in  $\Gamma$ . The maximal flat  $F_0$  in X covers  $T^r$  and the element  $\tilde{\xi} \in F_0$  projects to  $\xi \in M$ .

It is not always true that  $\Gamma_0$  is malnormal in  $\Gamma$ . Nevertheless, there is a weaker result.

**Lemma 4.2.** Suppose that  $g \in \Gamma$  and that  $g^{-1}\Gamma_0 g \cap \Gamma_0$  contains an element  $x_0 \neq 1$  which is the class of a regular closed geodesic c in  $T^r$ . Then  $g \in \Gamma_0$ .

*Proof.* This follows immediately from Theorem 3.1.

**Corollary 4.3.** Suppose that  $g \in \Gamma$  and that  $g^{-1}\Gamma_0 g \cap \Gamma_0$  contains a free abelian group of rank r. Then  $g \in \Gamma_0$ .

*Proof.* Combine Lemma 4.2 and [Mos, Lemma 11.1].

**Corollary 4.4.** Suppose that M has strictly negative curvature and that  $x_0$  is the class of a simple closed geodesic in M. Then  $\Gamma_0 = \langle x_0 \rangle$  is malnormal in  $\Gamma$ .

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In order to apply Proposition 4.1 to find the Pukánszky invariant of  $VN(\Gamma_0)$ , we must determine the size of  $\Gamma_0 \backslash \Gamma / \Gamma_0$ . This is done geometrically by considering the diagonal action of  $\Gamma$  on the set

$$\mathcal{F} = \{ (g_1 F_0, g_2 F_0) : g_1, g_2 \in \Gamma \}$$

**Lemma 4.5.** There is a bijection between  $\Gamma_0 \setminus \Gamma / \Gamma_0$  and the set of  $\Gamma$ -orbits of elements of  $\mathcal{F}$ , under the diagonal action.

*Proof.* The required bijection is the composition of the bijections :

$$\Gamma(F_0, gF_0) \mapsto \Gamma(\Gamma_0 \tilde{\xi}, g\Gamma_0 \tilde{\xi}) = \Gamma(\tilde{\xi}, \Gamma_0 g\Gamma_0 \tilde{\xi}) \mapsto \Gamma_0 g\Gamma_0 \,,$$

where  $g \in \Gamma$ . The verification of bijectivity is easy, given that  $\Gamma$  acts freely on X and that the stabilizer of  $F_0$  in  $\Gamma$  is  $\Gamma_0$ .

**Theorem 4.6.** Suppose that M has constant negative curvature and that  $x_0$  is the class of a simple closed geodesic in M. If  $\Gamma_0 = \langle x_0 \rangle$ , then

- (a)  $\#(\Gamma_0 \setminus \Gamma / \Gamma_0) = \infty$ ;
- (b) the Pukánszky invariant of  $VN(\Gamma_0)$  is  $\infty$ .

Proof. Assuming the curvature is -1, the symmetric space X which covers M is a real hyperbolic space of dimension  $n \geq 2$ , and the maximal flats in X are geodesics. By Proposition 4.1 and Corollary 4.4, it suffices to prove part (a). By Lemma 4.5, this is equivalent to the existence of infinitely many  $\Gamma$ -orbits of pairs of geodesics in  $\mathcal{F} = \{(g_1F_0, g_2F_0) : g_1, g_2 \in \Gamma\}$ , where the geodesic  $F_0$  is the axis of  $x_0$ . Now the distance in X between the geodesics  $gg_1F_0$  and  $gg_2F_0$  is independent of  $g \in \Gamma$ . It is therefore enough to prove that there are elements  $g \in \Gamma$  for which  $d(F_0, gF_0)$  is arbitrarily large.

The unit ball  $\{x \in \mathbb{R}^n : |x| < 1\}$ , with the appropriate metric will be used as a model for X [Ni, 1.1], and its boundary sphere S will have its usual metric. Choose  $g \in \Gamma \setminus \Gamma_0$ . We show that  $d(F_0, g^m F_0) \to \infty$ as  $m \to \infty$ . The element g is hyperbolic [BH, II.6.3]. For it cannot be elliptic ( $\Gamma$  is torsion free) and it cannot be parabolic ( $\Gamma$  is co-compact). Therefore g has attracting and repelling fixed points  $z^+, z^- \in S$ . See, for example [Ba, Lemma III.3.3]. Now  $\{z^+, z^-\} \cap \{z_1, z_2\} = \emptyset$ , by Lemma 2.1. It follows that, for  $j = 1, 2, g^m z_j \to z^+$  as  $m \to \infty$ .

Let  $K = \min\{|z^+ - a| : a \in F_0\} > 0$ , where  $|\cdot|$  is the euclidean norm on  $\mathbb{R}^n$ . If  $0 < \epsilon < K$ , there exists an integer m such that  $|g^m z_j - z^+| < \epsilon, j = 1, 2$ . There is a unique point  $a_m \in F_0$  which is closest (in the hyperbolic metric) to  $g^m F_0$ . Let

$$s = d(a_m, q^m F_0) = d(F_0, q^m F_0).$$

The explicit formula in [Ni, Theorem 1.2.1] gives

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$$\cosh s = \frac{2|g^{m}z_{1} - a_{m}||g^{m}z_{2} - a_{m}|}{|g^{m}z_{1} - g^{m}z_{2}|(1 - |a_{m}|^{2})}$$
$$\geq \frac{2(|z^{+} - a_{m}| - \epsilon)^{2}}{2\epsilon}$$
$$\geq \frac{(K - \epsilon)^{2}}{\epsilon}.$$

Letting  $\epsilon \to 0$  proves the result.

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