SIMPLE PURELY INFINITE C*-ALGEBRAS AND n-FILLING ACTIONS

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ABSTRACT. Let n be a positive integer. We introduce a concept, which we call the *n*-filling property, for an action of a group on a separable unital C^* -algebra A. If $A = C(\Omega)$ is a commutative unital C^* -algebra and the action is induced by a group of homeomorphisms of Ω then the *n*-filling property reduces to a weak version of hyperbolicity. The *n*-filling property is used to prove that certain crossed product C^* -algebras are purely infinite and simple. A variety of group actions on boundaries of symmetric spaces and buildings have the *n*-filling property. An explicit example is the action of $\Gamma = SL_n(\mathbb{Z})$ on the projective *n*-space.

INTRODUCTION

Consider a C^* -dynamical system (A, α, Γ) where A is a separable unital C^* -algebra on which a discrete group Γ acts by *-automorphisms.

Definition 0.1. Let $n \geq 2$ be a positive integer. We say that an action $\alpha : g \mapsto \alpha_g$ of Γ on A is *n*-filling if, for all $b_1, b_2, \ldots, b_n \in A^+$, with $||b_j|| = 1, 1 \leq j \leq n$, and for all $\epsilon > 0$, there exist $g_1, g_2, \ldots, g_n \in \Gamma$ such that $\sum_{j=1}^n \alpha_{g_j}(b_j) \geq 1 - \epsilon$.

If A is a commutative unital C^* -algebra and α is induced by a group of homeomorphisms of the spectrum Ω of A, then the *n*-filling property is equivalent to a generalized global version of hyperbolicity (Proposition 0.3 below). In this setting, the definition was motivated by ideas from [A-D1, LS] and [BCH]. The present article applies the *n*-filling property to give a proof that certain crossed product C^* -algebras are purely infinite and simple (Theorem 1.2). In the commutative case, similar results were obtained in [A-D1, LS] using local properties of

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the action. The paper [A-D1] also considers more general groupoid C^* -algebras. Simple crossed product algebras have been constructed using the related concept of a strongly hyperbolic action in [H, Appendix 2].

Remark 0.2. In order to prove the *n*-filling condition as stated in Definition 0.1 it is sufficient to verify it for all b_1, b_2, \ldots, b_n in a dense subset C of A^+ . For then if $b_1, b_2, \ldots, b_n \in A^+$, with $||b_j|| = 1, 1 \le j \le n$, and if $\epsilon > 0$, choose $c_1, c_2, \ldots, c_n \in C$ such that $||b_j - c_j|| < \frac{\epsilon}{2n}$ for all j and $\sum_{j=1}^n \alpha_{g_j}(c_j) \ge 1 - \epsilon/2$. Write

$$\sum_{j=1}^{n} \alpha_{g_j}(b_j - c_j) = x = x_+ - x_-$$

where $x_+, x_- \in A^+$ and $x_+x_- = 0$. We have $x \ge -\epsilon/2$ and therefore

$$\sum_{j=1}^{n} \alpha_{g_j}(b_j) = \sum_{j=1}^{n} \alpha_{g_j}(c_j) + x \ge 1 - \epsilon/2 - \epsilon/2 = 1 - \epsilon.$$

Suppose that $A = C(\Omega)$, the algebra of continuous complex valued functions on a compact Hausdorff space Ω . If the action arises from an action of Γ on Ω by homeomorphisms, then the *n*-filling condition can be expressed in the following way, which explains its name.

Proposition 0.3. Let Ω be an infinite compact Hausdorff space and let Γ be a group which acts on Ω by homeomorphisms. The induced action α of Γ on $C(\Omega)$ is n-filling if and only if the following condition is satisfied: for any nonempty open subsets U_1, \ldots, U_n of Ω , there exist $g_1, \ldots, g_n \in \Gamma$ such that $g_1U_1 \cup \cdots \cup g_nU_n = \Omega$.

Proof. If the action is *n*-filling, let U_1, \ldots, U_n be nonempty open subsets of Ω . There exist elements $b_1, b_2, \ldots, b_n \in A^+$, with $||b_j|| = 1$, such that $\operatorname{supp}(b_j) \subset U_j, 1 \leq j \leq n$. By hypothesis there exist $g_1, g_2, \ldots, g_n \in$ Γ such that $\sum_{j=1}^n \alpha_{g_j}(b_j) \geq 1/2$. Then if $\omega \in \Omega$ there exists $i \in$ $\{1, 2, \ldots, n\}$ such that $\alpha_{g_i}(b_i)(\omega) > 0$. Therefore $g_i^{-1}\omega \in U_i$, i.e. $\omega \in$ $g_i U_i$. Thus $g_1 U_1 \cup \cdots \cup g_n U_n = \Omega$.

Conversely, suppose the stated assertion holds. Fix $b_1, b_2, \ldots, b_n \in A^+$, with $||b_j|| = 1, 1 \leq j \leq n$, and let $\epsilon > 0$. For each j, the set $U_j = \{\omega \in \Omega; b_j(\omega) > 1 - \epsilon\}$ is a nonempty and open. Choose $g_1, \ldots, g_n \in \Gamma$ such that $g_1U_1 \cup \cdots \cup g_nU_n = \Omega$. If $\omega \in \Omega$, then $g_i^{-1}\omega \in U_i$ for some i and so $\alpha_{g_i}(b_i)(\omega) > 1 - \epsilon$. Therefore $\sum_{j=1}^n \alpha_{g_j}(b_j) \geq 1 - \epsilon$. \Box

Remark 0.4. If the action of the group Γ on the space Ω is topologically transitive (in particular, if it is minimal) then the *n*-filling

condition is equivalent to the following apparently weaker condition: for each nonempty open subset U of Ω , there exist $t_1, \ldots, t_n \in \Gamma$ such that $t_1U \cup \cdots \cup t_nU = \Omega$.

In order to see this, suppose that U_1, \ldots, U_n are nonempty open subsets of Ω . There exists $g_2 \in \Gamma$ such that $U_1 \cap g_2 U_2 \neq \emptyset$. Then there exists $g_3 \in \Gamma$ such that $U_1 \cap g_2 U_2 \cap g_3 U_3 \neq \emptyset$. Finally, there exists $g_n \in \Gamma$ such that $U = U_1 \cap g_2 U_2 \cdots \cap g_n U_n \neq \emptyset$. Then there exist $t_1, \ldots, t_n \in \Gamma$ such that $t_1 U \cup \cdots \cup t_n U = \Omega$ and so $t_1 U_1 \cup t_2 g_2 U_2 \cdots \cup t_n g_n U_n = \Omega$.

Definition 0.5. Let $\phi(\Gamma, \Omega)$ be the smallest integer *n* for which the conclusion of Proposition 0.3 holds. Set $\phi(\Gamma, \Omega) = \infty$ if no such *n* exists; that is, if the action is not *n*-filling for any integer *n*.

Topologically conjugate actions have the same value of $\phi(\Gamma, \Omega)$. It is easy to see that the notion of a 2-filling action is equivalent to what is called a *strong boundary action* in [LS] and an *extremely proximal flow* in [G]. The action of a word hyperbolic group on its Gromov boundary is 2-filling [LS, Example 2.1]. In our first example below (Example 2.1) we show that the canonical action of $\Gamma = SL_n(\mathbb{Z})$ on the projective space $\Pi = \mathbb{P}^{n-1}(\mathbb{R})$ satisfies $\phi(\Gamma, \Pi) = n$.

The final part of the paper is devoted to estimating $\phi(\Gamma, \Omega)$ for some group actions on the boundaries of affine buildings. These estimates show that $\phi(\Gamma, \Omega)$ is not a stable isomorphism invariant for the algebra $C(\Omega) \rtimes_r \Gamma$ (Example 4.3).

1. Purely infinite C^* -algebras from *n*-filling actions

Definition 1.1. An automorphism α of a C^* -algebra A is said to be *properly outer* if for each nonzero α -invariant ideal I of A and for each inner automorphism β of I we have $||\alpha|I - \beta|| = 2$.

We shall say that an action $\alpha : g \mapsto \alpha_g$ is properly outer if for all $g \in \Gamma \setminus \{e\}, \alpha_g$ is properly outer.

The purpose of this section is to prove the following result.

Theorem 1.2. Let (A, α, Γ) be a C^* -dynamical system, where Γ is a discrete group and A is a separable unital C^* -algebra. Suppose that for every nonzero projection $e \in A$ the hereditary C^* -subalgebra eAe is infinite dimensional. Suppose also that the action α is n-filling and properly outer. Then the reduced crossed product algebra $B = A \rtimes_{\alpha,r} \Gamma$ is a purely infinite simple C^* -algebra.

Remark 1.3. If $A = C(\Omega)$, with Ω a compact Hausdorff space, the condition that eAe is infinite dimensional for every nonzero projection $e \in A$ says simply that the space Ω has no isolated points.

It was shown in [AS, Proposition 1] that if the action α is topologically free then α is properly outer.

Proof. (Inspired by [LS, Theorem 5].) Denote by $E : B \to A$ the canonical conditional expectation. Fix $x \in B$, $x \neq 0$. In order to prove the result it is enough to show that there exist $y, z \in B$ such that yxz = 1. Put $a = \frac{x^*x}{\|E(x^*x)\|}$. Let $0 < \epsilon < \frac{1}{2(2n+1)}$. There exists $b \in C_c(\Gamma, A)^+$ such that $\|a - b\| < \epsilon$. Write $b = b_e + \sum_{g \in F} b_g u_g$, where $b_e = E(b) \ge 0$ and $F \subset \Gamma \setminus \{e\}$ is finite. Note that $\epsilon > \|E(a - b)\| = \|E(a) - b_e\| \ge |1 - \|b_e\||$, and so $\|b_e\|^{-1} < 1 + 2\epsilon$. It follows that $\|a - \frac{b}{\|b_e\|}\| = \|b_e\|^{-1}\|(\|b_e\| - 1)a + a - b\| < (1 + 2\epsilon)(\epsilon\|a\| + \epsilon) = \epsilon(1 + 2\epsilon)(1 + \|a\|)$.

Choosing b so that $||a - b|| < \frac{\epsilon}{3(1+||a||)}$ then replacing b by $\frac{b}{||b_e||}$ shows that we can assume that $||b_e|| = 1$.

Since α_g is properly outer for each $g \in F$, it follows from [OP, Lemma 7.1] that there exists $y \in A^+$, ||y|| = 1 such that $||b_e|| \ge ||yb_e y|| > ||b_e|| - \epsilon/|F|$ and $||yb_g\alpha_g(y)|| < \epsilon/|F|$ for all $g \in F$. Using Lemma 1.5 below, we see that there exists $c \in B$ such that $||c|| \le \sqrt{n}$ and $c^*yb_eyc \ge 1 - 3\epsilon$.

Then

$$\begin{aligned} \|c^*yayc - c^*yb_eyc\| &\leq \|c^*yayc - c^*ybyc\| + \|c^*ybyc - c^*yb_eyc\| \\ &\leq n\|a - b\| + n\|yby - yb_ey\| \\ &\leq n\epsilon + n\sum_{g\in F} \|yb_gu_gyu_g^{-1}u_g\| \leq 2n\epsilon \end{aligned}$$

Therefore c^*yayc is invertible since $||(c^*yb_eyc)^{-1}|| \leq \frac{1}{1-3\epsilon}$ and

$$\|1 - (c^*yb_eyc)^{-1}(c^*yayc)\| \le \frac{2n\epsilon}{1 - 3\epsilon} < \frac{n}{2n - 1} < 1.$$

$$z = (c^*yayc)^{-1} \text{ we have } \|E(x^*x)\|^{-1}c^*yx^* \cdot x \cdot ycz = 1$$

Setting $z = (c^* y a y c)^{-1}$ we have $||E(x^* x)||^{-1} c^* y x^* \cdot x \cdot y c z = 1$.

It remains to prove Lemma 1.5. A preliminary observation is necessary.

Lemma 1.4. Let A be a unital C*-algebra such that for every nonzero projection $e \in A$ the hereditary C*-subalgebra eAe is infinite dimensional. Let $b \in A^+$, ||b|| = 1 and let $\epsilon > 0$. For every integer $n \ge 1$ there exist elements $b_1, b_2, \ldots, b_n \in A^+$, with $||b_j|| = 1$, $bb_j = b_j b$, $||bb_j|| \ge 1 - \epsilon$ and $b_i b_j = 0$, for $i \ne j$.

Proof. There are two cases to consider.

Case 1. Suppose that 1 is not an isolated point of Sp(b). Then there exist pairwise disjoint nonempty open sets U_1, \ldots, U_n contained in Sp $(b) \cap [1 - \epsilon, 1]$. Let *C* be the *C*^{*}-subalgebra of *A* generated by $\{b, 1\}$. By functional calculus, there exist $b_1, b_2, \ldots, b_n \in C^+$, $||b_j|| = 1 (1 \le j \le n)$ with $||bb_j|| \ge 1 - \epsilon$ and $b_i b_j = 0, i \ne j$.

Case 2. Suppose that 1 is an isolated point of Sp(b). Then there exists a nonzero projection $e \in A$ such that be = eb = e. By hypothesis the hereditary C^* -subalgebra eAe is infinite dimensional. Therefore every masa of eAe is infinite dimensional [KR, p. 288]. Inside such an infinite dimensional masa of eAe we can find positive elements b_1, b_2, \ldots, b_n , $\|b_j\| = 1 (1 \le j \le n)$ with $b_i b_j = 0, i \ne j$. Then $bb_j = b(eb_j) = eb_j =$ $b_j = b_j b$ and $\|bb_j\| = \|b_j\| = 1$ for $1 \le j \le n$.

Lemma 1.5. Let (A, α, Γ) be as in the statement of Theorem 1.2, let $0 < \epsilon < 1/3$ and let $b \in A^+$, with $1 - \epsilon \le ||b|| \le 1$. Then there exists $c \in B$ such that $||c|| \le \sqrt{n}$ and $c^*bc \ge 1 - 3\epsilon$.

Proof. By Lemma 1.4, there exist $b_1, b_2, \ldots, b_n \in A^+$, with $||b_j|| = 1$, $bb_j = b_j b, \ b_i b_j = 0$ for $i \neq j$, and $||bb_j|| \ge 1 - 2\epsilon$. Since the action is *n*-filling, there exist $g_1, g_2, \ldots, g_n \in \Gamma$ such that $\sum_{i=1}^n \frac{1}{||bb_i||} \alpha_{g_i}(bb_i) \ge 1 - \epsilon$. Therefore $\sum_{i=1}^n \alpha_{g_i}(bb_i) \ge (1 - \epsilon)(1 - 2\epsilon) \ge 1 - 3\epsilon$. Put $c = \sum_{j=1}^n \sqrt{b_j} u_{g_j}^{-1} \in B$.

Now $c^*c = \sum_{i,j} u_{g_i} \sqrt{b_i} \sqrt{b_j} u_{g_j}^{-1} = \sum_{i=1}^n \alpha_{g_i}(b_i) \le n$ and so $||c|| \le \sqrt{n}$. Finally, we have $c^*bc = \sum_{i,j} u_{g_i} \sqrt{b_i} b \sqrt{b_j} u_{g_j}^{-1} = \sum_{i=1}^n \alpha_{g_i}(bb_i) \ge 1 - 3\epsilon$.

2. EXAMPLES

We now give some explicit examples of *n*-filling actions.

Example 2.1. For the canonical action of $\Gamma = SL_n(\mathbb{Z})$ on the projective space $\Pi = \mathbb{P}^{n-1}(\mathbb{R})$, we have $\phi(\Gamma, \Pi) = n$.

Proof. Denote by $u \mapsto [u]$ the canonical map from \mathbb{R}^n onto Π .

We first show that the action of Γ on Π is not (n-1)-filling. Choose a linear subspace E of \mathbb{R}^n of dimension n-1. Let $U = \Pi \setminus [E]$, which is a nonempty open subset of Π . If $t_j \in \Gamma$ $(1 \leq j \leq n-1)$ then $t_1U \cup \cdots \cup t_{n-1}U \neq \Pi$. For the subspace $t_1E \cap \cdots \cap t_{n-1}E$ of \mathbb{R}^n has dimension at least one, and so contains a nonzero vector v. Then $[v] \notin \bigcup_{j=1}^{n-1} t_j U$. Thus the action (Γ, Π) is not (n-1)-filling. It remains to show that it is *n*-filling. For this we use ideas from [BCH, Example 1].

We claim that there exists a basis $\{u_1, u_2, \ldots, u_n\}$ for \mathbb{R}^n , elements $g_1, g_2, \ldots, g_n \in \Gamma$, and (compact) sets $K_1, K_2, \ldots, K_n \subset \Pi$ with $K_1 \cup$

 $K_2 \cup \cdots \cup K_n = \Pi$, and with the following property: for any open neighbourhood U_j of $[u_j]$ $(1 \leq j \leq n)$ there exists a positive integer N_j such that $g_j^n K_j \subset U_j$ for all $n \geq N_j$. It follows that the action is *n*-filling. For let U_1, \ldots, U_n be nonempty open subsets of Π . Since the action of Γ on Π is minimal, we may assume that $[u_j] \in U_j$ $(1 \leq j \leq n)$. Let $t_j = g_j^{-N_j}$, so that $K_j \subset t_j U_j$ $(1 \leq j \leq n)$. Then $t_1 U_1 \cup \cdots \cup t_n U_n =$ Π .

It remains to verify our claim. Fix a positive integer $k \ge 4$ and let $a = \frac{2}{\sqrt{k^2 + 4k} + k}, b = \frac{\sqrt{k^2 + 4k} - k}{2}$. Consider the matrices $A = \begin{pmatrix} k+1 & k \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ k & k+1 \end{pmatrix}$ in $SL_2(\mathbb{Z})$. These matrices have eigenvalues $\lambda_+ = 1 + \frac{1}{a}, \lambda_- = 1 - b$, which satisfy $0 < \lambda_- < 1 < \lambda_+$. The corresponding eigenvectors for A are $\begin{pmatrix} 1 \\ a \end{pmatrix}$ and $\begin{pmatrix} -b \\ 1 \end{pmatrix}$; for B they are $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -b \end{pmatrix}$. If $1 \le j \le n-1$ let

$$g_{j} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & A \\ 0 & 0 & \dots & 1 \end{pmatrix} , \quad u_{j} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ a \\ 0 \end{pmatrix} , \quad v_{j} = \begin{pmatrix} 0 \\ 0 \\ -b \\ 1 \\ 0 \end{pmatrix}$$

where A occupies the j and j + 1 rows and columns and the nonzero entries of the vectors are in rows j and j + 1. Also let

$$g_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & & B \end{pmatrix} \quad , \quad u_n = \begin{pmatrix} 0 \\ 0 \\ \\ \\ \\ \\ \\ \\ 1 \end{pmatrix} \quad , \quad v_n = \begin{pmatrix} 0 \\ 0 \\ \\ \\ \\ \\ \\ \\ 1 \\ -b \end{pmatrix}.$$

Let $R = \max(\frac{1+a}{1-b}, \frac{1+ab}{1-b}) = \frac{1+a}{1-b}$. For $1 \le j \le n-1$ let

$$K_{j} = \{ [\xi_{j}u_{j} + \eta_{j}v_{j} + \sum_{l \neq j, j+1} \xi_{l}e_{l}] ; \xi_{j} \neq 0, \left|\frac{\eta_{j}}{\xi_{j}}\right| \le R, \left|\frac{\xi_{l}}{\xi_{j}}\right| \le R, l \neq j, j+1 \},$$

$$K_n = \{ \left[\xi_n u_n + \eta_n v_n + \sum_{l \neq n-1, n} \xi_l e_l \right]; \ \xi_n \neq 0, \ \left| \frac{\eta_n}{\xi_n} \right| \le R, \ \left| \frac{\xi_l}{\xi_n} \right| \le R, \ l \neq n-1, n \}.$$

Direct computation shows if $[x] \in \Pi$ then $[x] \in K_j$, where $|x_j| = \max_{1 \le l \le n} |x_l|$. Therefore $\Pi = \bigcup_{j=1}^n K_j$.

Let $\epsilon > 0$ and consider the basic open neighborhood U_j of $[u_j]$ defined by

$$U_{j} = \{ [\xi_{j}u_{j} + \eta_{j}v_{j} + \sum_{l \neq j, j+1} \xi_{l}e_{l}]; \xi_{j} \neq 0, \left|\frac{\eta_{j}}{\xi_{j}}\right| < \epsilon, \left|\frac{\xi_{l}}{\xi_{j}}\right| < \epsilon, l \neq j, j+1 \}.$$

Let $N > \frac{\log(R/\epsilon)}{\log(\lambda_+)}$. Recall that $0 < \lambda_- < 1 < \lambda_+$. Therefore $\frac{R}{\lambda_+^N} < \epsilon$. For $m \ge N$ and $[\xi_j u_j + \eta_j v_j + \sum_{l \ne j, j+1} \xi_l e_l] \in K_j$, we have

$$g^{m}[\xi_{j}u_{j} + \eta_{j}v_{j} + \sum_{l \neq j, j+1} \xi_{l}e_{l}] = [\lambda_{+}^{m}\xi_{j}u_{j} + \lambda_{-}^{m}\eta_{j}v_{j} + \sum_{l \neq j, j+1} \xi_{l}e_{l}].$$

Now $\left|\frac{\lambda_{-}^{m}\eta_{j}}{\lambda_{+}^{m}\xi_{j}}\right| \leq \frac{1}{\lambda_{+}^{m}}\left|\frac{\eta_{j}}{\xi_{j}}\right| \leq \frac{R}{\lambda_{+}^{m}} < \epsilon$, and for $l \neq j, j+1$, $\left|\frac{\xi_{l}}{\lambda_{+}^{m}\xi_{j}}\right| \leq \frac{1}{\lambda_{+}^{m}}\left|\frac{\xi_{l}}{\xi_{j}}\right| \leq$ $\frac{R}{\lambda_{+}^{m}} < \epsilon.$ This means that $g_{j}^{m}K_{j} \subset U_{j}$ for all $m \ge N$.

Remark 2.2. The fact that the action of $SL_3(\mathbb{Z})$ on the projective plane $\mathbb{P}^2(\mathbb{R})$ is not 2-filling can also be seen in a different way. More generally the action of a group Γ on a non-orientable compact surface Ω cannot be 2-filling. For let M be a closed subset of Ω homeomorphic to a Möbius band, let $U_1 = M^c$ and let $U_2 \subset \Omega$ be homeomorphic to an open disc in \mathbb{R}^2 . Then it is impossible to have $t_1U_1 \cup t_2U_2 = \Omega$ for $t_1, t_2 \in \Gamma$. For $t_2^{-1}t_1(M)$ would be a homeomorphic copy of a Möbius band embedded in the disc U_2 . To see that this is impossible note that a Mobius band is not disconnected by its centre circle, and apply the Jordan curve theorem.

Definition 2.3. Let the group Γ act on the topological space Ω . An element $q \in \Gamma$ is said to have an attracting fixed point $x \in \Omega$ if qx = xand there exists a neighbourhood V_x of x such that $\lim_{n \to \infty} g^n(V_x) = \{x\}.$

Remark 2.4. Let G be a noncompact semisimple real algebraic group and let Γ be a Zariski-dense subgroup of G. Consider the action of G on its Furstenberg boundary G/P, where P is a minimal parabolic subgroup of G. It follows from [BeL, Appendice] that there exist elements $q \in \Gamma$ which have attracting fixed points in G/P. In fact the set H of all such elements $g \in \Gamma$ is Zariski-dense in G: the elements of H are called *h*-regular in [BeL] and maximally hyperbolic in [BCH].

It follows from a result of H. FURSTENBERG [Fur, Theorem 5.5, Corollary] that if G is a semisimple group with finite centre which acts minimally on a locally compact Hausdorff space Ω with an attracting fixed point, then Ω is necessarily a compact homogeneous space of G.

The following result shows that many of the actions considered in [A-D1, LS] are *n*-filling for some integer n.

Proposition 2.5. Let Ω be a compact Hausdorff space and let (Ω, Γ) be a minimal action. Suppose that there exists an element $g \in \Gamma$ which has an attracting fixed point in Ω . Then the action (Ω, Γ) is n-filling for some integer n.

Proof. Choose $x \in \Omega$ with gx = x and an open neighbourhood V_x of x such that $\lim_{n\to\infty} g^n(V_x) = \{x\}$. Since the action is minimal, the family $\{hV_x; h \in \Gamma\}$ forms an open covering of Ω . By compactness, there exists a finite subcovering $\{h_1V_x, h_2V_x, \ldots, h_nV_x\}$.

Let U_1, \ldots, U_n be nonempty open subsets of Ω . Since the action of Γ on Ω is minimal, we may choose elements $s_j \in \Gamma$ such that $h_j x \in s_j U_j$ $(1 \leq j \leq n)$. For $1 \leq j \leq n$, choose an integer N_j such that $g^{N_j} V_x \subset h_j^{-1} s_j U_j$. Then $h_j V_x \subset t_j U_j$, where $t_j = h_j g^{-N_j} h_j^{-1} s_j$. Therefore $t_1 U_1 \cup \cdots \cup t_n U_n = \Omega$.

Remark 2.6. Consider the action of a noncompact semisimple real algebraic group G on its Furstenberg boundary G/P. Let Γ be a Zariskidense subgroup of G and let n(W) be the order of the Weyl group. In this case one can be more precise: the action $(G/P, \Gamma)$ is n(W)-filling. The proof follows from the remarks in [BCH, page 127]. In the next section we prove an analogue of this result for groups acting on affine buildings.

Recall that an action (Ω', Γ) is said to be a *factor* of the action (Ω, Γ) if there is a continuous equivariant surjection from Ω onto Ω' .

Proposition 2.7. Suppose that the action (Ω, Γ) is n-filling and that (Ω', Γ) is a factor of (Ω, Γ) . Then (Ω', Γ) is an n-filling action.

Proof. This is an easy consequence of the definitions.

3. Group actions on boundaries of affine buildings

We now turn to some examples which motivated our definition of an *n*-filling action. They are discrete analogues of those referred to Remark 2.6. We show that if a group Γ acts properly and cocompactly on an affine building Δ with boundary Ω , then the induced action on Ω is a *n*-filling, where *n* is the number of boundary points of an apartment in Δ . If Δ is the affine Bruhat-Tits building of a linear group then *n* is the order of the associated spherical Weyl group.

An apartment in Δ is a subcomplex of Δ isomorphic to an affine Coxeter complex. Each apartment inherits a natural metric from the Coxeter complex, which gives rise to a well-defined metric on the whole building [Br, Chapter IV.3]. Every geodesic of Δ is a straight line in

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some apartment. A sector (or Weyl chamber) is a sector based at a special vertex in some apartment [Ron]. Two sectors are equivalent (or parallel) if their intersection contains a sector. The boundary Ω is defined to be the set of equivalence classes of sectors in Δ . Fix a special vertex x. For any $\omega \in \Omega$ there is a unique sector $[x, \omega)$ in the class ω having base vertex x [Ron, Theorem 9.6, Lemma 9.7]. In the terminology of [Br, Chapter VI.9] Ω is the set of chambers of the building at infinity Δ^{∞} . Topologically, Ω is a totally disconnected compact Hausdorff space and a basis for the topology is given by sets of the form

$$\Omega_x(v) = \{\omega \in \Omega : [x, \omega) \text{ contains } v\}$$

where v is a vertex of \triangle . See [CMS, §2] for the \widetilde{A}_2 case, which generalizes directly.

We will need to use the fact that Ω also has the structure of a spherical building [Ron, Theorem 9.6], and its apartments are topological spheres.

Definition 3.1. Two boundary points ω, ϖ in Ω are said to be *opposite* [Br, IV.5] if the distance between them is the diameter of the spherical building Ω . Opposite boundary points are opposite in a spherical apartment of Ω which contains them; this apartment is necessarily unique. Two subsets of Ω are opposite if each point in one set is opposite each point in the other.

We define $\mathcal{O}(\omega)$ to be the set of all $\omega' \in \Omega$ such that ω' is opposite to ω . It is easy to see that $\mathcal{O}(\omega)$ is an open set.

Lemma 3.2. If $\omega \in \Omega$ and \mathcal{A} is an apartment in Δ , then there exists a boundary point ϖ of \mathcal{A} such that ϖ is opposite ω .

PROOF: Consider the geometric realization of the spherical building Ω . By [Ron, Theorem (A.19)], the subcomplex Ω' obtained from Ω by deleting all chambers opposite ω is geodesically contractible. However this is impossible if Ω' contains the spherical apartment of Ω made up of the boundary points of \mathcal{A} .

Corollary 3.3. If $\omega_1, \ldots, \omega_n$ are the boundary points of an apartment then

$$\Omega = \mathcal{O}(\omega_1) \cup \cdots \cup \mathcal{O}(\omega_n)$$

Remark 3.4. The union is not disjoint in general, as is seen by considering the example of a tree.

Lemma 3.5. Two chambers ω_1, ω_2 in Ω are opposite if and only if they are represented by opposite sectors S_1, S_2 with the same base vertex in some apartment of Δ . Moreover if two sectors S_1, S_2 in an apartment \mathcal{A} with the same base vertex represent opposite elements ω_1, ω_2 in Ω , then S_1, S_2 are opposite sectors and \mathcal{A} is the unique apartment containing them.

PROOF: Suppose that ω_1, ω_2 in Ω are opposite. There exists an apartment \mathcal{A} containing sectors S_1, S_2 representing ω_1, ω_2 respectively [Ron, Proposition 9.5] or [Br, VI.8, Theorem]. By taking parallel sectors, we may assume that S_1, S_2 have the same base vertex $x \in \mathcal{A}$. The sectors of \mathcal{A} based at x correspond to the chambers of an apartment in Ω containing ω_1, ω_2 [Ron, Theorem 9.8]. Therefore S_1, S_2 are opposite sectors. The converse is clear.

The final assertion follows from [Br, VI.9, Lemma 2 and IV.5 Theorem 1]. $\hfill \Box$

Remark 3.6.

(a) It is not necessarily true that if ω_1, ω_2 in Ω are opposite then the sectors $[z, \omega_1), [z, \omega_2)$ based at any vertex z are opposite sectors in some apartment.

(b) If C_1, C_2 are opposite chambers with a common vertex x in an apartment, then $\Omega_x(C_1)$ and $\Omega_x(C_2)$ are opposite sets in Ω .

Suppose that a group Γ acts properly and cocompactly on an affine building Δ of dimension n. An apartment \mathcal{A} in Δ is said to be *periodic* if there is a subgroup $\Gamma_0 < \Gamma$ preserving \mathcal{A} such that $\Gamma_0 \setminus \mathcal{A}$ is compact [Gr, 6.B₃]. Note that Γ_0 is commensurable with \mathbb{Z}^n , and this concept coincides with the notion of periodicity described in [MZ],[RR] for buildings of type $\tilde{\mathcal{A}}_2$. In [BB], a periodic apartment is called Γ closed. This terminology makes it clear that periodicity depends upon the choice of the group Γ acting on the building.

It is important to observe that there are many periodic apartments. In fact, according to [BB, Theorem 8.9], any compact subset of an apartment is contained in some periodic apartment

Now let \mathcal{A}_0 be a periodic apartment, and fix a special vertex z in \mathcal{A}_0 . Choose a pair of opposite sectors W^+ , W^- in \mathcal{A}_0 based at z. Denote by ω^{\pm} the boundary points represented by W^{\pm} , respectively. By periodicity of the apartment there is a periodic direction represented by a line L in any of the sector directions of \mathcal{A}_0 . For definiteness choose this direction to be that of the sector W^+ . This means that there is an element $u \in \Gamma$ which leaves L invariant and translates the apartment \mathcal{A}_0 in the direction of L. (In the terminology of [BB, Moz], L is said to be an *axis* of u.) Then $u^n \omega^+ = \omega^+$, $u^n \omega^- = \omega^-$ for all $n \in \mathbb{Z}$. Moreover $u^n z$ is in the interior of W^+ for n > 0 and in the interior of W^- for n < 0. (See Figure 1. Here and in what follows, the figures illustrate the case of a building Δ of type A_2 , where each apartment contains precisely six sectors based at a given vertex.) The element u above is the analogue of the maximally hyperbolic elements in [BCH].



FIGURE 1. The periodic apartment \mathcal{A}_0 .

The following crucial result shows that ω^{-} is an attracting fixed point for u^{-1} .

Proposition 3.7. Let \mathcal{A}_0 be a periodic apartment and choose a pair of opposite boundary points ω^{\pm} . Let $u \in \Gamma$ be an element which translates the apartment \mathcal{A}_0 in the direction of ω^+ . Then u^{-1} attracts $\mathcal{O}(\omega^+)$ towards ω^- , that is: for each compact subset G of $\mathcal{O}(\omega^+)$ we have $\lim_{n\to\infty} u^{-n}(G) = \{\omega^-\}.$

PROOF: We use the notation introduced above. Let $\omega \in \mathcal{O}(\omega^+)$. By considering a retraction of \triangle centered at ω^+ [Br, p.170, VI.8, Theorem], we see that \triangle is a union of apartments which contain a subsector of W^+ . Moreover for any sector W representing ω there are subsectors $V^+ \subset W^+$ and $V \subset W$ which lie in a common apartment \mathcal{A} . Replacing V^+ by a subsector, we may assume that V^+ has base vertex $u^N z$ for some N, that is $V^+ = [u^N z, \omega^+)$. Replacing V by a parallel sector in \mathcal{A} we may also assume that V has base vertex $u^N z$. By Lemma 3.5, Vlies in the apartment \mathcal{A} as shown in Figure 2.

For each $N \geq 0$ let G_N denote the set of all boundary points $\omega \in \mathcal{O}(\omega^+)$ such that $[u^N z, \omega)$ and $[u^N z, \omega^+)$ are opposite sectors in some apartment $\mathcal{A}^{(N)}$. Then $G_0 \subset G_1 \subset G_2 \subset \ldots$ is an increasing family of compact open sets and we have observed above that $\bigcup_{N=0}^{\infty} G_N =$



FIGURE 2. The apartment \mathcal{A} .

 $\mathcal{O}(\omega^+)$. The result will follow if we show that $\lim_{n\to\infty} u^{-n}(G_N) = \{\omega^-\}$ for each $N \ge 0$. It is clearly enough to consider the case N = 0.

Consider a basic open neighbourhood of ω^- of the form $\Omega_z(v)$, where $v \in [z, \omega^-) \subset \mathcal{A}_0$. Choose an integer $p \ge 0$ such that $u^n v \in [z, \omega^+)$ for all $n \ge p$. If $\omega \in G_0$ then $u^n v \in [u^n z, \omega)$ (that is $v \in [z, u^{-n}\omega)$) for all $n \ge p$. (See Figure 3.) This means that $u^{-n}\omega \in \Omega_z(v)$ for all $n \ge p$. Thus $u^{-n}(G_0) \subset \Omega_z(v)$ for all $n \ge p$. This proves the result. \Box



FIGURE 3. Sectors in the apartment $u^{-n}\mathcal{A}^{(0)}$.

Theorem 3.8. Suppose that a group Γ acts properly and cocompactly on the vertices of an affine building Δ with boundary Ω . Let k denote the number of boundary points of an apartment of Δ . Then the action (Ω, Γ) is k-filling.

PROOF: Let U_1, \ldots, U_k be nonempty open subsets of Ω . Let \mathcal{A}_0 be a periodic apartment with boundary points $\omega_j, 1 \leq j \leq k$. By minimality of the action we can assume that $\omega_j \in U_j, 1 \leq j \leq k$. By Corollary 3.3, we have $\Omega = \mathcal{O}(\omega_1) \cup \cdots \cup \mathcal{O}(\omega_n)$. It follows from the existence of a partition of unity that there exist compact sets $K_j \subset \mathcal{O}(\omega_j), 1 \leq j \leq k$ such that $\Omega = K_1 \cup \cdots \cup K_k$.

Let $u_j \in \Gamma$ translate the apartment \mathcal{A}_0 in the direction of ω_j , $1 \leq j \leq k$. Then by Proposition 3.7, there exists $N_j \geq 0$ such that $u_j^{-n}K_j \subset U_j$ whenever $n \geq N_j$, $1 \leq j \leq k$. In other words, $K_j \subset u_j^n U_j$ whenever $n \geq N_j$, $1 \leq j \leq k$. Let $t_j = u_j^{N_j}$. Then

$$\Omega = K_1 \cup \cdots \cup K_k \subset t_1 U_1 \cup \cdots \cup t_k U_k$$

as required.

Remark 3.9. The action of an A_2 group Γ on the boundary Ω of the associated building is 6-filling. We do not know the precise value of $\phi(\Gamma, \Omega)$, but it is certainly greater than 2. To see this, fix a point $\omega_0 \in \Omega$ and choose U to be a nonempty open set opposite ω_0 . If $t_1, t_2 \in \Gamma$ then t_1U and t_2U are opposite the boundary points $t_1\omega_0$ and $t_2\omega_0$ respectively and therefore cannot cover Ω . To see this, choose a hexagonal apartment of Ω which contains $t_1\omega_0$ and $t_2\omega_0$ and choose a chamber ϖ in this apartment which is not opposite $t_1\omega_0$ or $t_2\omega_0$. Then ϖ cannot lie in $t_1U \cup t_2U$. Therefore $2 < \phi(\Gamma, \Omega) \leq 6$.

4. Purely infinite simple C^* -algebras

Throughout this section we consider only affine buildings of type A_2 . The \tilde{A}_2 buildings are a particularly natural setting for our investigation. They are the simplest two-dimensional buildings, but they do not necessarily arise from linear groups. Crossed product C^* -algebras associated with them have been studied in [RS1, RS2]. In this case the building Δ is a simplicial complex whose maximal simplices (*chambers*) are triangles. An apartment of Δ is a subcomplex isomorphic to the Euclidean plane tessellated by equilateral triangles.

The boundary Ω may be identified with the flag complex of a projective plane (P, L) [Br, page 81]. Flags will be denoted (x_1, x_2) where $x_1 \in x_2$. If we identify chambers of Ω with sectors based at a fixed

vertex v_0 of type 0, then a sector wall whose base panel is of type 1 corresponds to an element of P and a sector wall whose base panel is of type 2 corresponds to an element of L [Ron, Section 9.3]. P is the minimal boundary of Δ and has been studied in [CMS], where it is denoted Ω^l . The topology on P comes from the natural quotient map $\Omega \to P$. Moreover the action of Γ on Ω induces an action on P. Similar statements apply to L, and there is a homeomorphism $P \cong L$.



FIGURE 4. Sector walls w_1, w_2, w_3 corresponding to points in P.

From now on assume that the group Γ is an \widetilde{A}_2 group: that is Γ acts simply transitively in a type rotating manner on the vertices of an affine building Δ of type \widetilde{A}_2 .

Proposition 4.1. The actions (Ω, Γ) , (P, Γ) are topologically free. That is, if $g \in \Gamma \setminus \{e\}$ then

$$Int\{\omega \in \Omega : g\omega = \omega\} = \emptyset$$
$$Int\{w \in P : gw = w\} = \emptyset$$

PROOF: The statement for the action on Ω is proved in [RS1, Theorem 4.3.2].

Suppose that the result fails for the action on P. Then there exists an open set $V \subset P$ such that gw = w for all $w \in V$. Let $\widetilde{V} = \pi^{-1}(V)$, where $\pi : \Omega \to P$ is the quotient map. Then \widetilde{V} is a nonempty open subset of Ω . By [RS1, Proposition 4.3.1], \widetilde{V} contains all six boundary points of some apartment \mathcal{A} of Δ . These boundary points are the six

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chambers of an apartment \mathcal{A}_0 in Ω , as illustrated in Figure 5. The apartment \mathcal{A}_0 contains three points $w_1, w_2, w_3 \in P$. These three points lie in V and hence are fixed by g. It follows that the lines $l_1, l_2, l_3 \in L$ are also fixed by g. Therefore each boundary point of \mathcal{A}_0 is fixed by g. By the proof of [RS1, Theorem 4.3.2], it follows that $g\mathcal{A} = \mathcal{A}$ and g acts by translation on \mathcal{A} . The same is true for all nearby apartments \mathcal{A}' , since the corresponding walls $w'_1, w'_2, w'_3 \in P$ will also be fixed by g, if they belong to V. The argument of [RS1, Theorem 4.3.2] now gives a contradiction.



FIGURE 5. The apartment \mathcal{A}_0

Proposition 4.2. If Γ is an \widetilde{A}_2 group, then the algebras $C(\Omega) \rtimes \Gamma$, $C(P) \rtimes \Gamma$ are simple purely infinite C^* -algebras.

PROOF: The actions are topologically free by Proposition 4.1 and hence properly outer [AS, Proposition 1]. Moreover they are 6-filling by Theorem 3.8. The result follows from Theorem 1.2. \Box

We now give examples of properly outer actions (Ω_i, Γ_i) , i = 1, 2, with $\phi(\Gamma_1, \Omega_1) = 2$ and $\phi(\Gamma_2, \Omega_2) > 2$ but for which $C(\Omega_1) \rtimes \Gamma_1$ is stably isomorphic to $C(\Omega_2) \rtimes \Gamma_2$.

Example 4.3. Let $\Gamma_1 \subset PSL(2, \mathbb{R})$ be a non-cocompact Fuchsian group isomorphic to \mathbb{F}_3 , the free group on three generators. Consider the action of Γ_1 on the boundary S^1 of the Poincaré disc. This action is 2-filling and the algebra $\mathcal{A}_1 = C(S^1) \rtimes \Gamma_1$ is p.i.s.u.n. with K-theory given by $K_0(\mathcal{A}_1) = K_1(\mathcal{A}_1) = \mathbb{Z}^4$, $[\mathbf{1}] = (1,0,0,0)$ [A-D2]. (The K-theory is independent of the embedding $\Gamma_1 \subset PSL(2, \mathbb{R})$.)

Let Γ_2 be the A_2 group B.3 of [CMSZ]. This group is a lattice subgroup of PGL₃(\mathbb{Q}_2) and acts naturally on the corresponding building of type \widetilde{A}_2 and its boundary Ω . By Remark 3.9, $2 < \phi(\Gamma, \Omega) \leq 6$. By [RS2], the algebra $\mathcal{A}_2 = C(\Omega) \rtimes \Gamma_2$ is p.i.s.u.n. and satisfies the Universal Coefficient Theorem. By [RS3] the K-theory of \mathcal{A}_2 is given by $K_0(\mathcal{A}_2) = K_1(\mathcal{A}_2) = \mathbb{Z}^4$, $[\mathbf{1}] = 0$.

It follows from the classification theorem of [Kir] that \mathcal{A}_1 , \mathcal{A}_2 are stably isomorphic (but not isomorphic, since the classes [1] do not correspond).

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