Operator algebras associated with groups acting on buildings and their K-theory

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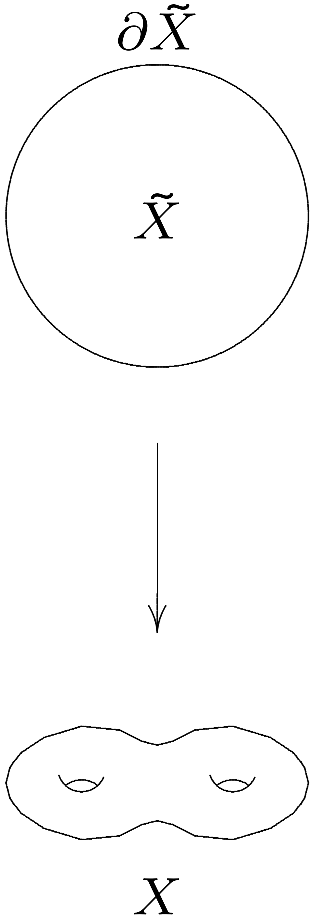
MOTIVATION



Γ acts on $\partial\tilde{X}$

ANALYSIS

$\Gamma{=}\pi(X)$

GEOMETRY

What does $(\Gamma, \partial\tilde{X})$ reveal about X?

Example

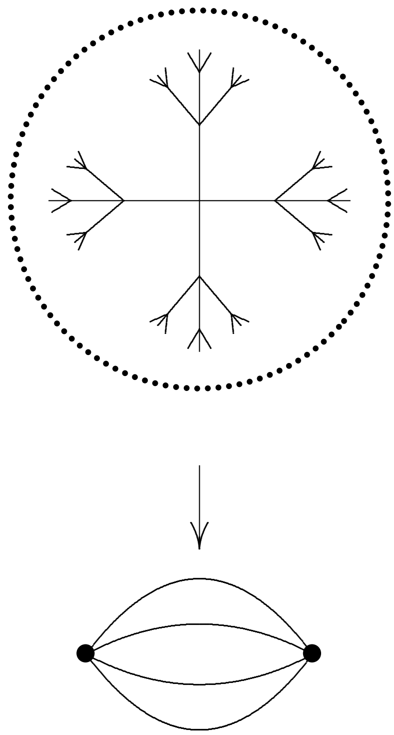
X : A finite connected graph.

Δ : The universal covering space (a tree).

$\partial\Delta$: The boundary of Δ.

Δ and $\partial\Delta$

X



Let $\Gamma = \pi(X)$, the fundamental group of X. Γ is a free group which acts freely on $\widetilde{X}$ and

$$\Gamma \backslash \Delta = X.$$

Let $q \geq 2$, a prime power.

Let $\mathbb{K}$ be a nonarchimedean local field with residue field of order q.

Example. The field $\mathbb{F}_q((t))$ of formal sums

$$x = a_j t^j + \cdots + a_0 + a_1 t + a_2 t^2 + \cdots ,$$

where each $a_i \in \mathbb{F}_q$ and $a_j \neq 0$.

$$|x| = q^{-j}$$ *discrete valuation*

The ring of integers

$$\begin{aligned} \mathcal{O} &= \{x \in \mathbb{F}_q((t)) : |x| \leq 1\} \\ &= \text{ set of sums with } j \geq 0 \\ &= \overline{\{\text{polynomials}\}} \end{aligned}$$ *a compact open subring*

The tree of $\mathsf{PGL}_2(\mathbb{K})$

Let

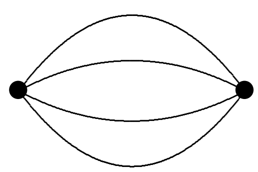
$$G = \mathsf{PGL}_2(\mathbb{K}) = \mathsf{GL}_2(\mathbb{K})/\mathbb{K}^\times.$$

G acts on a homogeneous tree Δ of degree $q+1$. (A **building** of type $\tilde{A}_1$.)

If $\Gamma < G$ is a torsion free cocompact lattice, then Γ is a free group which acts on Δ.

The graph $X = \Gamma\backslash\Delta$ has universal cover Δ and fundamental group Γ.

Can choose $\Gamma < \mathsf{PGL}_2(\mathbb{F}_3((t)))$ with quotient graph

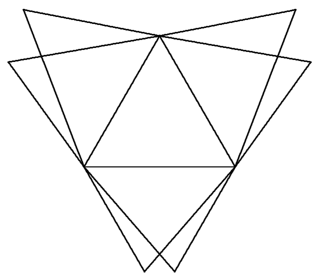


The building of $\mathrm{PGL}_3(\mathbb{K})$

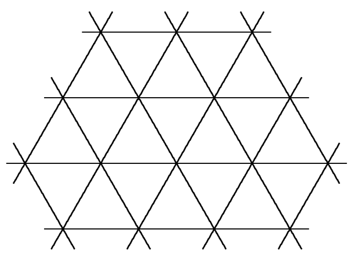
$G = \mathrm{PGL}_3(\mathbb{K})$ acts on its **building** of type $\widetilde{A}_2$, which is a topologically contractible 2-dimensional complex Δ.

Each edge of Δ lies on $q+1$ triangles.

$q = 2$:

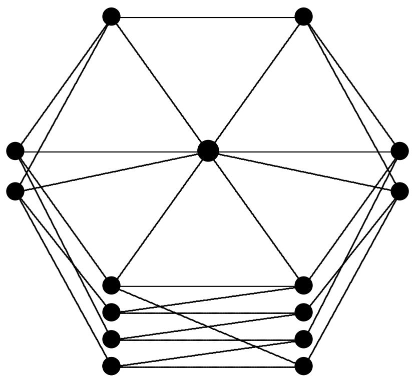


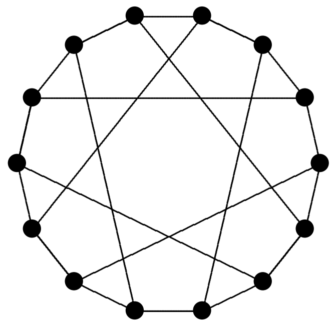
Δ is a union of **apartments**: flat subcomplexes isomorphic to a tessellation of $\mathbb{R}^2$ by equilateral triangles.



$\tilde{A}_2$ buildings are 2-dimensional analogues of trees, but have a more rigid structure.

The neighbours of a vertex ($q = 2$).





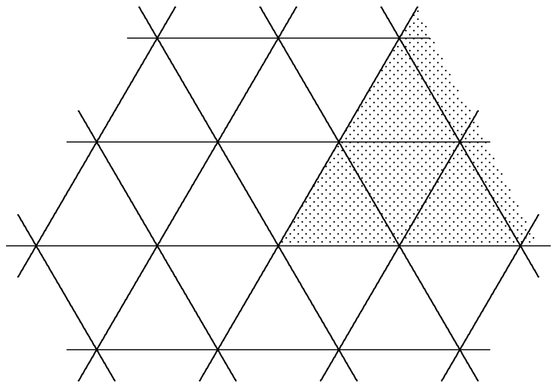
7 point
projective plane

On the left: a ball of radius one in an $\tilde{A}_2$ building (of order 2).

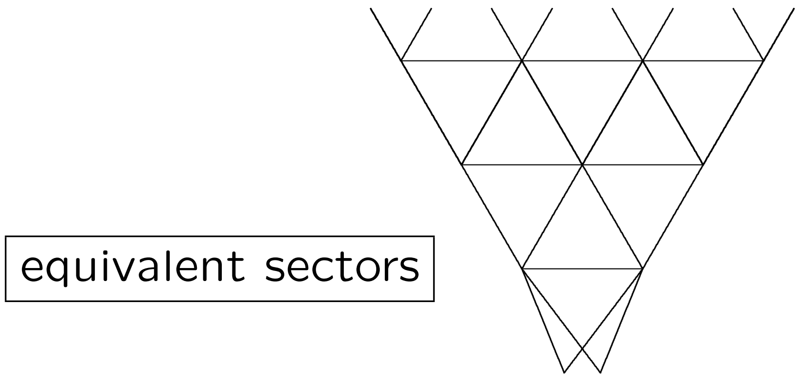
On the right: the graph obtained by deleting the centre of the ball and all the edges connected to it.

The boundary $\partial\Delta$

A *sector* is a simplicial cone in some apartment.



Two sectors are *equivalent* if they contain a common subsector.



Boundary points are equivalence classes of sectors.

The building Δ of $\mathsf{PGL}_n(\mathbb{K})$

A lattice $L \subseteq \mathbb{K}^n$ is a free $\mathcal{O}$-submodule of rank n.

$$L = \mathcal{O}v_1 + \mathcal{O}v_2 + \cdots + \mathcal{O}v_n,$$

where $(v_1, \ldots, v_n)$ is a basis of $\mathbb{K}^n$.

Equivalence relation :

$$L_1 \sim L_2 \quad \Longleftrightarrow \quad L_1 = aL_2, \qquad a \in \mathbb{K}^\times$$

An equivalence class $[L]$ is a **vertex** of Δ.

An **edge** of Δ is $([L_1], [L_2])$ where

$$L_1 \supset L_2 \supset qL_1.$$

$\mathrm{GL}_n(\mathbb{K})$ acts transitively on the set of bases $(v_1, \ldots, v_n)$ of $\mathbb{K}^n$.

$\therefore G = \mathrm{PGL}_n(\mathbb{K})$ acts transitively on vertices.

The vertex set of Δ is G/K where

$$K = \mathrm{PGL}_n(\mathcal{O}) < G$$

is maximal compact.

The boundary is $\partial\Delta \cong G/B$ where

$$B = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} \cap G.$$

... a Borel subgroup of G.

The boundary action

Let $\Gamma < \mathrm{PGL}_n(\mathbb{K})$ be a torsion free cocompact lattice.

Study the action of Γ on $\partial\Delta$ by forming the **crossed product C^*-algebra**:

$$\begin{aligned}\mathcal{A}(\Gamma) &= C(\partial\Delta) \rtimes \Gamma \\ &= C^*\langle \Gamma \cup C(\partial\Delta)\ ;\ \gamma(f) = \gamma f \gamma^{-1}\rangle\end{aligned}$$

where $\gamma(f)(t) = f(\gamma^{-1}t)$.

Here

$C(\partial\Delta) \subset \mathcal{A}(\Gamma)$	an abelian subalgebra
$\Gamma \subset \mathcal{A}(\Gamma)$	a group of unitaries

$\mathcal{A}(\Gamma)$ is simple, purely infinite and classified by the abelian group $K_0(\mathcal{A}(\Gamma))$ and [1], where

$$K_0(\mathcal{A}) = \{[p] : p \text{ is a nonzero idempotent in } \mathcal{A}\} .$$

$\mathbf{C_r^*(\Gamma)}$

Let $\Gamma < G$ be a torsion free discrete group, and

$$C_r^*\Gamma = \overline{(\mathbb{C}\Gamma)} \subseteq B(\ell^2\Gamma)\,.$$

The mapping

$$C_r^*\Gamma \hookrightarrow C(\partial\Delta) \rtimes \Gamma = \mathcal{A}(\Gamma)\,,$$

induces a mapping

$$K_*(C_r^*\Gamma) \to K_*(\mathcal{A}(\Gamma))\,,$$

which is **not injective**,

$$\underset{\text{not torsion}}{[1]} \in K_0(C_r^*\Gamma) \qquad \mapsto \underset{\text{torsion}}{[1]} \in K_0(\mathcal{A}(\Gamma))\,.$$

Conjecture : This is the **only** reason for failure of injectivity.

$\tilde{A}_1$ **buildings**

If Γ is a torsion free lattice in $\mathrm{PGL}_2(\mathbb{K})$ then Γ acts freely on a homogenous tree Δ of degree $q+1$. Γ is a free group of rank m, where

$$\chi(\Gamma) := \chi(\Gamma\backslash\Delta) = 1 - m$$

and

$$K_0(C_r^*\Gamma) = \mathbb{Z}, \qquad K_1(C_r^*\Gamma) = \mathbb{Z}^m .$$

[Pimsner-Voiculescu, 1982]

$\mathcal{A}(\Gamma)$ is a **Cuntz-Krieger algebra** and is classified by its K_0 group:

$$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^m \oplus \mathbb{Z}/n\mathbb{Z}$$

where $n = m - 1 = -\chi(\Gamma)$ and $\langle[\mathbf{1}]\rangle = \mathbb{Z}/n\mathbb{Z}$.

Note that $K_0(\mathcal{A}(\Gamma))$ determines Γ : the free group of rank m.

$\tilde{A}_2$ **buildings**

If Γ is a torsion free lattice in $\mathrm{PGL}_3(\mathbb{K})$ then

$$K_0(C_r^*\Gamma) = \mathbb{Z}^{\chi(\Gamma)}, \qquad K_1(C_r^*\Gamma) = \Gamma/[\Gamma,\Gamma]\,,$$

where

$$\begin{aligned}\chi(\Gamma) &:= \chi(\Gamma\backslash\Delta) \qquad \text{(Euler-Poincaré characteristic)}\\ &= \frac{1}{3}(q-1)(q^2-1)\cdot|\Gamma\backslash G/K|\,.\end{aligned}$$

Proof. The Baum-Connes assembly map

$$K_*^\Gamma(\Delta) \to K_*(C_r^*\Gamma)$$

is an isomorphism [V. Lafforgue, 1998]. So

$$K_*(C_r^*\Gamma) = H_*(\Gamma,\mathbb{Z})\,.$$

Since Γ has homological dimension 2,

$$K_1(C_r^*\Gamma) = H_1(\Gamma,\mathbb{Z})\,,$$

where $H_1(\Gamma,\mathbb{Z})$ is finite, since Γ has property T of Kazhdan.

$$K_0(C_r^*\Gamma) = H_0(\Gamma,\mathbb{Z}) \oplus H_2(\Gamma,\mathbb{Z}) = \mathbb{Z}^{\chi(\Gamma)}\,.$$

The algebras $\mathcal{A}(\Gamma) = C(\partial\Delta) \rtimes \Gamma$ are examples of **higher rank Cuntz-Krieger algebras** whose structure theory has been developed by G. Robertson and T. Steger (1998-2001).

Theorem. Let $\Gamma < \mathrm{PGL}_3(\mathbb{K})$ be a torsion free cocompact lattice. Then $m.[1] = 0$ in $K_0(\mathcal{A}(\Gamma))$, where

$$m = \frac{1}{3}\gcd(3, q-1) \cdot (q^2 - 1) \cdot |\Gamma \backslash G/K| \,.$$

Strong numerical evidence suggests that the order of $[1]$ is actually :

$$\frac{(q-1)}{\gcd(3, q-1)} \cdot |\Gamma \backslash G/K|$$

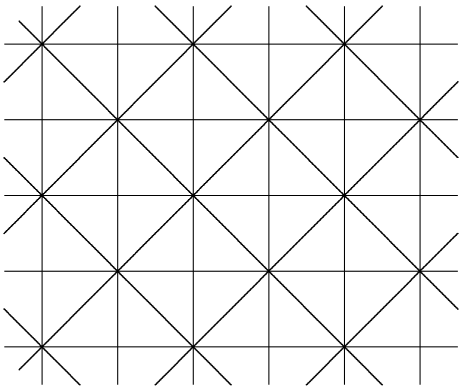
The corresponding formula for $\Gamma < \mathrm{PGL}_2(\mathbb{K})$ is **true**: in that case,

$$\frac{(q-1)}{\gcd(2, q-1)} \cdot |\Gamma \backslash G/K| = -\chi(\Gamma).$$

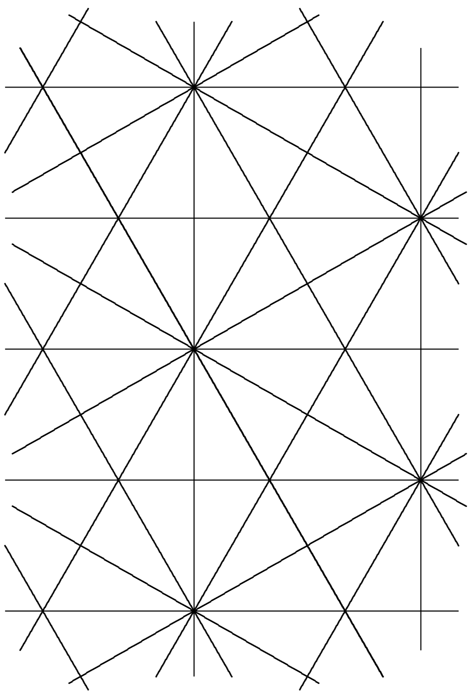
Other affine buildings: the boundary algebras $\mathcal{A}(\Gamma)$ are again simple and purely infinite, but $K_0(\mathcal{A}(\Gamma))$ is harder to compute. However

Theorem. Let G be the group of $\mathbb{K}$ rational points of an absolutely almost simple, simply connected linear algebraic $\mathbb{K}$-group. Let Γ be a lattice in G. Then $[\mathbf{1}]$ has finite order in $K_0(\mathcal{A}(\Gamma))$.

Other 2-dimensional affine buildings



$\widetilde{B}_2$ apartment



$\widetilde{G}_2$ apartment

Example: $\mathrm{q} = 4$

The "regular" $\tilde{A}_2$ group $\Gamma < \mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K} = \mathbb{F}_4\,((t))$. [Cartwright, Mantero, Steger, Zappa, 1993]

Γ has generators x_j, $0 \le j \le 20$,
and relations

$$\begin{cases} x_j x_{j+3} x_{j-6} = 1 & (0 \le j \le 20), \\ x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & (0 \le j \le 6)\,. \end{cases}$$

$|\Gamma \backslash G/K| = 1$ and $q = 4 \quad \Rightarrow \quad \chi(\Gamma) = 15\,.$

[1] is a generator for $K_0(C_r^*\Gamma) = \mathbb{Z}^{15}$. The other 14 generators are in $M_n(C_r^*\Gamma)$, $n > 1$.

$$K_0(C_r^*\Gamma) = \mathbb{Z}^{14} \oplus \langle [1] \rangle = \mathbb{Z}^{15}$$

$$\swarrow$$

$$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^{28} \oplus 0 \oplus (\mathbb{Z}/2\mathbb{Z})^{12} \oplus (\mathbb{Z}/3\mathbb{Z})^6$$

$$K_1(C_r^*\Gamma) = \Gamma/[\Gamma,\Gamma] = (\mathbb{Z}/2\mathbb{Z})^6 \oplus (\mathbb{Z}/3\mathbb{Z})^2$$

Example: $\mathbf{q = 5}$

The "regular" $\tilde{A}_2$ group $\Gamma < \mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K} = \mathbb{F}_5((t))$, has 31 generators, 62 relations.

$$|\Gamma\backslash G/K| = 1, \qquad \chi(\Gamma) = 32\,.$$

$$\begin{array}{l} K_0(C_r^*\Gamma) = \mathbb{Z}^{31} \oplus \langle[1]\rangle \\ \qquad\qquad\qquad\qquad\quad \downarrow \\ K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^{62} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z})^6 \end{array}$$

$$K_1(C_r^*\Gamma) = (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z})^3$$

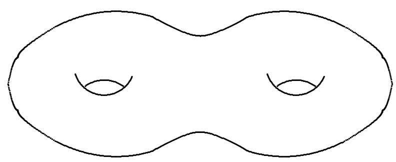
Unlike the previous example, this $K_0(\mathcal{A}(\Gamma))$ does not contain 2 copies of $K_1(C_r^*\Gamma)$.

Continuous analogues

Fuchsian groups

The tree is a combinatorial analogue of the Poincaré disc $\mathbb{D}$.

A torsion free cocompact Fuchsian group Γ acts on $\mathbb{D}$ and is the fundamental group of the Riemann surface $X = \Gamma \backslash \mathbb{D}$.



Γ act on $\partial\mathbb{D} = \mathbb{S}^1$ and the boundary algebra $\mathcal{A}(\Gamma)$ is a **Cuntz-Krieger algebra**.

$$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^{n+3} \oplus \mathbb{Z}/n\mathbb{Z}$$

where $n = -\chi(X)$ is the order of $[\mathbf{1}]$.

(C. Anantharaman)

Kleinian groups

$\mathrm{PGL}_2(\mathbb{C})$ acts on hyperbolic 3-space and its boundary S^2.

Let $\Gamma < \mathrm{PGL}_2(\mathbb{C})$ be a countable group.

$[\mathbf{1}]$ is not torsion in $K_0(\mathcal{A}(\Gamma))$ (A.Connes).

Note that $\chi(\Gamma) = 0$ if Γ is torsion free and cocompact.

$\mathrm{PGL}_3(\mathbb{R})$

Let Γ be a torsion free cocompact lattice in $\mathrm{PGL}_3(\mathbb{R})$.

$\dim(G/K) = 5$ (odd) $\Rightarrow \chi(\Gamma) = 0$.

$K_i(\mathcal{A}(\Gamma)) = K^{i+1}(\Gamma \backslash G) = ?$

(Uses A.Connes' Thom Isomorphism.)